On global exponential stability for impulsive cellular neural networks with time-varying delays

Ivanka M. Stamova\textsuperscript{a,\ast}, Rajcho Ilarionov\textsuperscript{b}

\textsuperscript{a} Bourgas Free University, 8000 Bourgas, Bulgaria
\textsuperscript{b} Technical University of Gabrovo, 5300 Gabrovo, Bulgaria

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\section*{Abstract}

In this paper, the problem of global exponential stability for cellular neural networks (CNNs) with time-varying delays and fixed moments of impulsive effect is studied. A new sufficient condition has been presented ensuring the global exponential stability of the equilibrium points by using piecewise continuous Lyapunov functions and the Razumikhin technique combined with Young's inequality. The results established here extend those given previously in the literature. Compared with the method of Lyapunov functionals as in most previous studies, our method is simpler and more effective for stability analysis.

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1. Introduction

Cellular neural networks (CNNs) were introduced by Chua and Yang in 1988 [1,2]. Impressive applications of CNNs have been proposed for various fields such as optimization, linear and nonlinear programming, associative memory, pattern recognition and computer vision [3–7]. However, it is necessary to solve some dynamic image processing and pattern recognition problems by using Delayed Cellular Neural Networks (DCNNs). The study of the stability of CNNs and DCNNs is known to be an important problem in theory and applications.

On the other hand, the state of electronic networks is often subject to instantaneous perturbations and the networks experience abrupt change at certain instants which may be caused by a switching phenomenon, frequency change or other sudden noise; that is, the networks exhibit impulsive effects [8,9]. For instance, according to Arbib [10] and Haykin [11], when a stimulus from the body or the external environment is received by receptors the electrical impulses will be conveyed to the neural net and impulsive effects arise naturally in the net.

Therefore, a neural network model with delay and impulsive effects should be more accurate to describe the evolutionary process of the systems. Since delays and impulses can affect the dynamical behavior of the system, it is necessary to investigate the effects of both delay and impulsive effects on the stability of neural networks. Such a generalization of the DCNN notion should enable us to study different types of classical problems as well as to “control” the solvability of the DCNN (without impulses).

A large number of criteria on the stability of DCNNs without impulses have been derived (see, e.g., [12,3–5,13–15]). Correspondingly, there is not much work dedicated to investigating the stability of impulsive neural networks. Recently, Akca et al. [16], Stamov [17], and Stamov and Stamova [18] have formulated some mathematical models of impulsive neural networks described by measure differential equations or general impulsive differential equations. Impulses can make unstable systems stable, so they have been widely used in many fields such as physics, chemistry, biology, population Errata
The problems of the existence and uniqueness of equilibrium states of neural networks with impulses have been investigated by many authors. Efficient sufficient conditions for the existence and uniqueness of an equilibrium of time-varying delays. The main results are obtained by using piecewise continuous Lyapunov functions [8] and the Razumikhin technique [21,9,22,7] combined with Young's inequality [23]. Examples are given to demonstrate the effectiveness of the results. The results in the earlier references emerge as special cases of the main results given here.

2. Statement of the problem. Preliminary notes and definitions

Let $R^n$ denote the $n$-dimensional Euclidean space, $R_+ = [0, \infty)$ and $t_0 \in R_+$.

We consider the following impulsive CNNs with time-varying delays:

$$
\begin{align*}
\dot{x}_i(t) &= -c_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} f_j(x_j(t-\tau_j(t))) + I_i, \quad t \neq t_k, \; t > t_0, \\
\Delta x_i(t_k) &= x_i(t_k + 0) - x_i(t_k) = P_i(x_i(t_k)), \quad k = 1, 2, \ldots,
\end{align*}
$$

(2.1)

$i = 1, 2, \ldots, n$, where $n$ corresponds to the numbers of units in a neural network; $x_i(t)$ corresponds to the state of the $i$th unit at time $t$; $f_j(x_j(t))$ denotes the output of the $j$th unit at time $t$; $a_{ij}, b_{ij}, I_i, c_i$ are constants: $a_{ij}$ denotes the strength of the $j$th unit on the $i$th unit at time $t$, $b_{ij}$ denotes the strength of the $j$th unit on the $i$th unit at time $t - \tau_j(t)$, $I_i$ denotes the external bias on the $i$th unit, where $\tau_j(t)$ corresponds to the transmission delay along the axon of the $j$th unit and satisfies $0 \leq \tau_j(t) \leq \tau$ ($\tau = \text{const}$), and $c_i$ represents the rate with which the $i$th unit will reset its potential to the resting state in isolation when disconnected from the network and external inputs; and $t_k, k = 1, 2, \ldots$, are the moments of impulsive perturbations and satisfy $t_0 < t_1 < t_2 < \cdots$ and $\lim_{k \to \infty} t_k = \infty$. The numbers $x_i(t_k)$ and $x_i(t_k + 0)$ are, respectively, the states of the $i$th unit before and after impulse perturbation at the moment $t_k$; and $P_i(x_i(t_k))$ represents the abrupt change of the state $x_i(t)$ at the impulsive moment $t_k$.

Let $J \subset R$ be an interval. Define the following classes of functions.

$PC[J, R^n] = \{ \sigma : J \to R^n : \sigma(t)$ is continuous everywhere except at some points $t_k \in J$ at which $\sigma(t_k - 0)$ and $\sigma(t_k + 0)$ exist and $\sigma(t_k - 0) = \sigma(t_k) \}$.

$PCB[J, R^n] = \{ \sigma \in PC[J, R^n] : \sigma(t)$ is bounded on $J \}$.

Let $\varphi \in PCB[[-\tau, 0], R^n]$. Denote by $x(t) = x(t; t_0, \varphi), x \in R^n$, the solution of system (2.1), satisfying the initial condition

$$
\begin{align*}
\begin{cases}
x(t; t_0, \varphi) &= \varphi(t - t_0), \quad t_0 - \tau \leq t \leq t_0, \\
x(t_0 + 0; t_0, \varphi) &= \varphi(0).
\end{cases}
\end{align*}
$$

(2.2)

The solution $x(t) = x(t; t_0, \varphi) = (x_1(t; t_0, \varphi), \ldots, x_n(t; t_0, \varphi))^T$ of problem (2.1) and (2.2) is a piecewise continuous function with points of discontinuity of the first kind $t_k, k = 1, 2, \ldots$, where it is continuous from the left, i.e. the following relations are valid:

$$
\begin{align*}
x_i(t_k - 0) &= x_i(t_k), \quad k = 1, 2, \ldots,
\end{align*}
$$

$$
\begin{align*}
x_i(t_k + 0) &= x_i(t_k) + P_i(x_i(t_k)), \quad t_k > 0.
\end{align*}
$$

In particular, a constant point $x^* \in R^n, x^* = (x_1^*, x_2^*, \ldots, x_n^*)^T$, is called an equilibrium point of (2.1), if $x^* = x^*(t; t_0, x^*)$ is a solution of (2.1).

We introduce the following conditions.

H2.1 There exist constants $L_i > 0$ such that

$$\begin{align*}
|f_i(u) - f_i(v)| \leq L_i |u - v|
\end{align*}$$

for all $u, v \in R, u \neq v, i = 1, 2, \ldots, n$.

H2.2 The functions $P_{i_k}$ are continuous on $R, i = 1, 2, \ldots, n, k = 1, 2, \ldots$.

H2.3 $t_0 < t_1 < t_2 < \cdots < t_k < t_{k+1} < \cdots$ and $t_k \to \infty$ as $k \to \infty$.

H2.4 There exists a unique equilibrium

$$
\begin{align*}
x^* = \text{col}(x_1^*, x_2^*, \ldots, x_n^*)
\end{align*}
$$

of system (2.1) such that

$$
\begin{align*}
c_i x_i^* &= \sum_{j=1}^{n} a_{ij} f_j(x_j^*) + \sum_{j=1}^{n} b_{ij} f_j(x_j^*) + I_i,
\end{align*}
$$

and

$$
\begin{align*}
P_{i_k}(x_i^*) &= 0, \quad i = 1, 2, \ldots, n, \; k = 1, 2, \ldots.
\end{align*}
$$

Remark 2.1. The problems of the existence and uniqueness of equilibrium states of neural networks without impulses have been investigated by many authors. Efficient sufficient conditions for the existence and uniqueness of an equilibrium of
systems of type (2.1) are given in [19]. Note that, when a neural network is designed to function as an associative memory, it is required that there exist many stable equilibrium points, whereas in the case of solving optimization problems it is necessary that the designed neural network must have a unique equilibrium point that is globally asymptotically stable. Therefore, it is of great interest to establish conditions that ensure the global asymptotic stability of a unique equilibrium point of a neural network.

We introduce the following notations:

$$ G_k = (t_{k-1}, t_k) \times \mathbb{R}^n, \quad k = 1, 2, \ldots; \quad G = \bigcup_{k=1}^{\infty} G_k. $$

Let \( \| \varphi - x^* \| = \sup_{t \in [t_0 - \tau, t_0] \cap \mathbb{Z}} \left[ \sum_{i=1}^{n} |\psi_i(t - t_0) - x_i^*|^p \right]^{1/p}, \quad p \geq 1 \), be the distance between the function \( \varphi \in PCB([-\tau, 0], \mathbb{R}^n) \) and the equilibrium \( x^* \in \mathbb{R}^n \).

**Definition 2.1.** The equilibrium \( x^* = col(x_1^*, x_2^*, \ldots, x_n^*) \) of system (2.1) is said to be **globally exponentially stable** if there exists constants \( \alpha > 0 \) and \( M \geq 1 \) such that

$$ \| x(t; t_0, \varphi) - x^* \| \leq M \| \varphi - x^* \| e^{-\alpha(t-t_0)}, \quad t \geq t_0. $$

Later, we shall use piecewise continuous Lyapunov functions \( V : [t_0, \infty) \times \mathbb{R}^n \rightarrow R_+ \).

**Definition 2.2.** We say that the function \( V : [t_0, \infty) \times \mathbb{R}^n \rightarrow R_+ \) belongs to the class \( V_0 \) if the following conditions are fulfilled:

1. The function \( V \) is continuous in \( \bigcup_{k \in \mathbb{Z}} G_k \) and \( V(t, 0) = 0 \) for \( t \in [t_0, \infty) \).
2. The function \( V \) satisfies locally the Lipschitz condition with respect to \( x \) on each of the sets \( G_k \).
3. For each \( k = 1, 2, \ldots \) there exist the finite limits

$$ V(t_k - 0, x) = \lim_{t \rightarrow t_k^-} V(t, x), \quad V(t_k + 0, x) = \lim_{t \rightarrow t_k^+} V(t, x). $$

4. For each \( k = 1, 2, \ldots \) the following equalities are valid:

$$ V(t_k - 0, x) = V(t_k, x). $$

Let \( V \in V_0 \). For any \( (t, x) \in [t_{k-1}, t_k) \times \mathbb{R}^n, k = 1, 2, \ldots \), the upper right-hand derivative \( D^+ V(t, x(t)) \) with respect to system (2.1) is defined by

$$ D^+ V(t, x(t)) = \lim_{h \rightarrow 0^+} \inf_{R} \frac{1}{h} \left[ V(t + h, x(t + h)) - V(t, x(t)) \right]. $$

In the proof of the main results we shall use the following lemma.

**Lemma 2.1** (Young’s Inequality [23]). Assume that \( a > 0, b > 0, p > 1, \frac{1}{p} + \frac{1}{q} = 1 \); then the inequality

$$ ab \leq \frac{1}{p} a^p + \frac{1}{q} b^q $$

holds.

### 3. Main results

**Theorem 3.1.** Assume that:

1. Conditions H2.1–H2.4 hold.
2. There exist real constants \( \zeta_{ij}, \eta_{ij} \gamma_i > 0 \), \( i, j = 1, 2, \ldots, n \) and \( p \geq 1 \) such that the following inequalities hold:

$$ \min_{1 \leq \ell \leq n} \left\{ p c_i - \sum_{j=1}^{n} \left( \frac{\eta_{ij} \gamma_i}{\gamma_j} L_{ij} |a_{ij}|^{p(1-\eta_j)} + (p-1)L_{ij} |a_{ij}|^{p \eta_i} + (p-1) L_{ij} |b_{ij}|^{p \eta_j} \right) \right\} > \max_{1 \leq \ell \leq n} \left\{ \sum_{j=1}^{n} \frac{\eta_{ij}}{\gamma_j} L_{ij} |b_{ij}|^{p(1-\eta_j)} \right\} > 0. $$

3. There exist constants \( \sigma_k, 0 < \sigma_k < 2 \), \( i = 1, 2, \ldots, n, k = 1, 2, \ldots \), such that the following equalities are satisfied:

$$ P_k(x_i(t_k)) = -\sigma_k (x_i(t_k) - x_i^*). $$

Then the equilibrium \( x^* \) of (2.1) is globally exponentially stable.

**Proof.** Let \( y_i(t) = x_i(t) - x_i^* \) and define a Lyapunov function

$$ V(t, y) = \sum_{i=1}^{n} y_i |y_i(t)|^p. $$
Then, for \( t = t_k \), from condition 3 of Theorem 3.1, we obtain

\[
V(t_k + 0, y(t_k) + \Delta y(t_k)) = \sum_{i=1}^{n} \gamma_i|x_i(t_k) - x^*_i - \sigma_{ik}(x_i(t_k) - x^*_i)|^p = \sum_{i=1}^{n} \gamma_i|1 - \sigma_{ik}|^p|x_i(t_k) - x^*_i|^p < \sum_{i=1}^{n} \gamma_i|x_i(t_k) - x^*_i|^p = V(t_k, y(t_k)). \quad k = 1, 2, \ldots.
\] (3.1)

Let \( t \geq t_0 \) and \( t \neq t_k, k = 1, 2, \ldots \). Then, for the upper right-hand derivative \( D^+V(t, y(t)) \) of \( V \) with respect to system (2.1) we get

\[
D^+V(t, y(t)) = \sum_{i=1}^{n} \gamma_i|y_i(t)|^{p-1}\text{sgn}(y_i(t))\dot{y}_i(t)
= \sum_{i=1}^{n} \gamma_i|y_i(t)|^{p-1}\text{sgn}(y_i(t)) \left[-c_iy_i(t) + \sum_{j=1}^{n} a_{ij}(f_j(y_j(t) + x^*_j) - f_j(x^*_j)) + \sum_{j=1}^{n} b_{ij}(f_j(y_j(t) - \tau_j(t)) + x^*_j) - f_j(x^*_j))\right]
\leq \sum_{i=1}^{n} \gamma_i\left[-c_i|y_i(t)|^p + \sum_{j=1}^{n} |a_{ij}|L_j|y_j(t)|^{p-1}|y_j(t)| + \sum_{j=1}^{n} |b_{ij}|L_j|y_j(t)|^{p-1}|y_j(t) - \tau_j(t)|\right]
= \sum_{i=1}^{n} \gamma_i\left[-c_i|y_i(t)|^p + \sum_{j=1}^{n} L_j|y_j(t)||a_{ij}|^{1-\zeta_i}\left(|a_{ij}|^{\frac{\rho_j}{\rho_i-1}}|y_j(t)|\right)^{p-1} + \sum_{j=1}^{n} L_j|y_j(t)-\tau_j(t)||b_{ij}|^{1-\eta_j}\left(|b_{ij}|^{\frac{\rho_j}{\rho_i-1}}|y_j(t)|\right)^{p-1}\right].
\] (3.2)

Let \( a = |y_j(t)||a_{ij}|^{1-\zeta_i}, b = \left(|a_{ij}|^{\frac{\rho_j}{\rho_i-1}}|y_j(t)|\right)^{p-1} \); by Lemma 2.1, we have

\[
|y_j(t)||a_{ij}|^{1-\zeta_i}\left(|a_{ij}|^{\frac{\rho_j}{\rho_i-1}}|y_j(t)|\right)^{p-1} \leq \frac{1}{p}|a_{ij}|^{p(1-\zeta_i)}|y_j(t)|^p + \frac{p-1}{p}|a_{ij}|^{\frac{\rho_j}{\rho_i-1}}|y_j(t)|^p.
\] (3.3)

Similarly, using Lemma 2.1 for

\[
a = |y_j(t) - \tau_j(t)||b_{ij}|^{1-\eta_j} \quad \text{and} \quad b = \left(|b_{ij}|^{\frac{\rho_j}{\rho_i-1}}|y_j(t)|\right)^{p-1},
\]
we get

\[
|y_j(t) - \tau_j(t)||b_{ij}|^{1-\eta_j}\left(|b_{ij}|^{\frac{\rho_j}{\rho_i-1}}|y_j(t)|\right)^{p-1} \leq \frac{1}{p}|b_{ij}|^{p(1-\eta_j)}|y_j(t) - \tau_j(t)|^p + \frac{p-1}{p}|b_{ij}|^{\frac{\rho_j}{\rho_i-1}}|y_j(t)|^p.
\] (3.4)

Substituting (3.3) and (3.4) into (3.2), we obtain

\[
D^+V(t, y(t)) \leq \sum_{i=1}^{n} \gamma_i\left[-c_i|y_i(t)|^p + \sum_{j=1}^{n} \frac{1}{p}L_j|a_{ij}|^{p(1-\zeta_i)}|y_j(t)|^p + \sum_{j=1}^{n} \frac{1}{p}L_j|b_{ij}|^{p(1-\eta_j)}|y_j(t) - \tau_j(t)|^p + \sum_{j=1}^{n} \frac{1}{p}L_j|a_{ij}|^{\frac{\rho_j}{\rho_i-1}}|y_j(t)|^p + \sum_{j=1}^{n} \frac{1}{p}L_j|b_{ij}|^{\frac{\rho_j}{\rho_i-1}}|y_j(t)|^p\right]
\leq -k_1V(t, y(t)) + k_2 \sup_{t-\tau \leq \tau \leq t} V(s, y(s)),
\]
where

\[
\begin{aligned}
k_1 &= \min_{1 \leq i \leq n} \left\{ pc_i - \sum_{j=1}^{n} \left( \frac{\gamma_j}{\gamma_i} L_j a_{ji} |p^{(1-\eta_i)} + (p-1) L_j a_{ji} \right) \right. \\
&\quad \left. + (p-1) L_j b_{ji} |p^{\eta_j} \right\} > 0, \\
k_2 &= \max_{1 \leq i \leq n} \left\{ \sum_{j=1}^{n} \frac{\gamma_j}{\gamma_i} L_j b_{ji} |p^{(1-\eta_j)} \right\} > 0.
\end{aligned}
\]

From the above estimate for any \( y(t) \) which satisfy the Razumikhin condition

\[
V(s, y(s)) \leq V(t, y(t)), \quad t - \tau \leq s \leq t,
\]

we have

\[
D^+ V(t, y(t)) \leq -(k_1 - k_2) V(t, y(t)), \quad t \neq t_k, \quad k = 1, 2, \ldots.
\]

By virtue of condition 2 of Theorem 3.1 there exit a real number \( k > 0 \) such that

\[
k_1 - k_2 \geq k,
\]

and it follows that

\[
D^+ V(t, y(t)) \leq -k V(t, y(t)), \quad t \neq t_k, \quad t \geq t_0.
\]  

Then, using (3.1) and (3.5), we get

\[
\gamma_\min \|x(t) - x^*\|^p \leq V(t, y(t)) \leq e^{-\lambda(t-t_0)} V(t_0 + 0, y(t_0 + 0)), \quad t \geq t_0.
\]

So,

\[
\|x(t) - x^*\| \leq \frac{\gamma^{1/p}}{\gamma_\min} \|\varphi - x^*\| e^{-\lambda(t-t_0)}, \quad t \geq t_0.
\]

and this completes the proof of the theorem. \( \square \)

**Corollary 3.1.** If in Theorem 3.1 condition 2 is replaced by the condition

\[
\min_{1 \leq i \leq n} \left\{ c_i - \sum_{j=1}^{n} \left( \frac{\gamma_j}{\gamma_i} L_j a_{ji} \right) \right\} > \max_{1 \leq i \leq n} \left\{ \sum_{j=1}^{n} \frac{\gamma_j}{\gamma_i} L_j b_{ji} \right\} > 0,
\]

where \( \gamma_i > 0, \quad i = 1, 2, \ldots, n \), then the equilibrium \( x^* \) of (2.1) is globally exponentially stable.

**Proof.** Taking \( p = 1 \) in theorem above, then we can easily obtain Corollary 3.1. \( \square \)

In the case when \( p = 2, \quad \xi_{ij} = 0.5, \quad \eta_{ij} = 0.5, \quad i, j = 1, 2, \ldots, n \), we deduce the following corollary of Theorem 3.1.

**Corollary 3.2.** If in Theorem 3.1 condition 2 is replaced by the condition

\[
\min_{1 \leq i \leq n} \left\{ c_i - \sum_{j=1}^{n} \left( \frac{\gamma_j}{\gamma_i} L_j a_{ji} + L_j (|a_{ji}| + |b_{ji}|) \right) \right\} > \max_{1 \leq i \leq n} \left\{ \sum_{j=1}^{n} \frac{\gamma_j}{\gamma_i} L_j b_{ji} \right\} > 0,
\]

where \( \gamma_i > 0, \quad i = 1, 2, \ldots, n \), then the equilibrium \( x^* \) of (2.1) is globally exponentially stable.

**Remark 3.1.** If we let \( \gamma_i = 1, \quad i = 1, 2, \ldots, n \), then Corollaries 1 and 2 correspond to Theorems 1 and 2 in [19], respectively. That is, our theorem includes the main results in [19] as a special cases.

**Remark 3.2.** In some papers (see for example [20] and the references therein), by constructing Lyapunov functions, results on the global exponential stability of impulsive high-order Hopfield-type neural networks with time-varying delays are presented. Such systems include system (2.1) as a special case. In contrast to our results, all of those results require that the impulsive moments depend on the upper bound of the delay function. Thus, the conditions given in [20] are more restrictive and conservative.

**Remark 3.3.** Young’s inequality was first used by Cao [3]. However, the results in [3] only hold for constant delays.
4. Examples

In the following, we will give three examples to show our results.

Example 4.1. Consider the impulsive neural differential equations with time-varying delays

\[
\begin{aligned}
\dot{x}_i(t) &= -c_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} f_j(x_j(t - \tau_j(t))) + l_i, \quad t \neq t_k, \quad t > 0, \\
\end{aligned}
\]

where \( n = 2, l_1 = 0.22727272, l_2 = 0.2424243, c_1 = c_2 = 1.5, f_i(x_i) = \frac{1}{2}(|x_i + 1| - |x_i - 1|) \) \((i = 1, 2), 0 \leq \tau_i(t) \leq \tau (\tau = 1),

\[
\begin{aligned}
(a_{ij})_{2 \times 2} &= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 0.1 & -0.1 \\ 0.2 & 0.1 \end{pmatrix}, \\
(b_{ij})_{2 \times 2} &= \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} 1 & -0.3 \\ -0.3 & 0.2 \end{pmatrix},
\end{aligned}
\]

with

\[
\begin{aligned}
x_1(t_k + 0) &= \frac{1.8181818 - x_1(t_k)}{2}, \quad k = 1, 2, \ldots, \\
x_2(t_k + 0) &= \frac{1.0606064 - x_2(t_k)}{6}, \quad k = 1, 2, \ldots,
\end{aligned}
\]

where the impulsive moments are such that \( 0 < t_1 < t_2 < \cdots, \) and \( \lim_{k \to \infty} t_k = \infty. \)

Clearly, \( f_i \) satisfies assumption H2.1 with \( L_1 = L_2 = 1. \) Also, we have that

\[
0 < \sigma_{1k} = \frac{3}{2} < 2, \quad 0 < \sigma_{2k} = \frac{7}{6} < 2,
\]

and therefore condition 3 of Theorem 3.1 is satisfied.

One can check that

\[
\min_{1 \leq i \leq n} \left\{ c_i - \sum_{j=1}^{n} |a_{ij}| \right\} = 1.2 < \max_{1 \leq i \leq n} \left\{ \sum_{j=1}^{n} |b_{ij}| \right\} = 1.3,
\]

and

\[
\min_{1 \leq i \leq n} \left\{ 2c_i - \sum_{j=1}^{n} (|a_{ij}| + (|a_{ij}| + |b_{ij}|)) \right\} = \max_{1 \leq i \leq n} \left\{ \sum_{j=1}^{n} |b_{ij}| \right\} = 1.3.
\]

Therefore, the main results in [19] do not hold for this example. On the other hand, if we let \( p = 3, \) \( \gamma_i = 1.5, \gamma_i = 1, i, j = 1, 2 \) in Theorem 3.1, one can verify that

\[
\min_{1 \leq i \leq n} \left\{ 3c_i - \sum_{j=1}^{n} (|a_{ij}| + 2(|a_{ij}| + |b_{ij}|)) \right\} = 1.4 > \max_{1 \leq i \leq n} \left\{ \sum_{j=1}^{n} |b_{ij}| \right\} = 1.3.
\]

Hence, the unique equilibrium

\[
x^* = (x_1^*, x_2^*)^T = (0, 6060606, 0, 1515152)^T
\]

of (4.1), (4.2) is globally exponential stable.

If we consider again the system (4.1) but with impulsive perturbations of the form

\[
\begin{aligned}
x_1(t_k + 0) &= 6x_1(t_k) - 3, \quad k = 1, 2, \ldots, \\
x_2(t_k + 0) &= \frac{1.0606064 - x_2(t_k)}{6}, \quad k = 1, 2, \ldots,
\end{aligned}
\]

the point (4.3) will be again an equilibrium of (4.1), (4.4) but there is nothing we can say about its exponential stability, because \( \sigma_{1k} = -5 < 0. \)

The example shows that (1) our theorem includes the main results in [19] as a special cases; (2) by means of appropriate impulsive perturbations we can control the neural network system's dynamics.
Example 4.2. Consider the impulsive neural differential equations with time-varying delays (4.1), where $n = 2$, $l_1 = 0.38235294, l_2 = 0.02941178$, $c_1 = c_2 = 1, f_i(x_i) = \frac{1}{2}(|x_i + 1| - |x_i - 1|) (i = 1, 2), 0 \leq \tau_i(t) \leq \tau (\tau = 1),$

\[
\begin{pmatrix} a_{ij} \\ b_{ij} \end{pmatrix}_{2 \times 2} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} 0.1 & -0.1 \\ 0.1 & 0.1 \\ 0.5 & -0.1 \\ -0.1 & 0.5 \end{pmatrix},
\]

with

\[
\begin{align*}
 x_1(t_k + 0) &= \frac{0.7352941 + x_1(t_k)}{2}, & k &= 1, 2, \ldots, \\
 x_2(t_k + 0) &= \frac{1.3235295 + 2x_2(t_k)}{5}, & k &= 1, 2, \ldots,
\end{align*}
\]

(4.5)

where the impulsive moments are such that $0 < t_1 < t_2 < \cdots$, and $\lim_{k \to \infty} t_k = \infty$.

Clearly, $f_i$ satisfies the assumption H2.1 with $l_1 = l_2 = 1$. Also we have that

\[ 0 < \sigma_{1k} = \frac{1}{2} < 2, \quad 0 < \sigma_{2k} = \frac{3}{5} < 2. \]

It is easy to verify that

\[ \min_{1 \leq i \leq n} \left\{ c_i - \sum_{j=1}^{n} |a_{ij}| \right\} = 0.8 > \max_{1 \leq i \leq n} \left\{ \sum_{j=1}^{n} |b_{ij}| \right\} = 0.6. \]

According to Corollary 3.1 for $\gamma_i = 1, i = 1, 2$, the unique equilibrium

\[ x^* = (x_1^*, x_2^*) = (0.7352941, 0, 4411765)^T \]

of (4.1), (4.5) is globally exponential stable. By a simple computation, we can see that the matrix $A + A^T = \begin{pmatrix} 0.2 & 0 \\ 0 & 0.2 \end{pmatrix}$ is not negative semidefinite. Thus the condition in [24] does not hold.

If we consider again the system (4.1) but with impulsive perturbations of the form

\[
\begin{align*}
 x_1(t_k + 0) &= \frac{0.7352941 + x_1(t_k)}{2}, & k &= 1, 2, \ldots, \\
 x_2(t_k + 0) &= 5x_2(t_k) - 1.764706, & k &= 1, 2, \ldots,
\end{align*}
\]

(4.7)

the point (4.6) will be again an equilibrium of (4.1), (4.7) but there is nothing we can say about its exponential stability, because $\sigma_{2k} = -4 < 0$.

5. Conclusions

This paper presents a sufficient condition for global exponential stability of the equilibrium point for a neural network model with time-varying delays and impulsive effects. The results impose constraint conditions on the network parameters of neural system independent of the delay parameters. The results are applicable to more general neuron activation functions than both the usual sigmoid activation functions in Hopfield networks and the piecewise linear function in standard cellular networks. The results are also compared with the previously reported results in the literature, implying that the results obtained in this paper provide one more set of criteria for determining the stability of impulsive neural networks with time delays. The examples considered show that by means of appropriate impulsive perturbations we can control the stability properties of neural networks.

References