Three positive solutions of semilinear elliptic equations in exterior cylinder domains

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Abstract

In this paper, assume that \( h \) is nonnegative and \( \|h\|_{L^2} > 0 \), we prove that if \( \|h\|_{L^2} \) is sufficiently small, then there are at least three positive solutions of Eq. (1) in an exterior cylinder domain.

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1. Introduction

Let \( N \) be a positive integer with \( N \geq 3 \). For \( z = (z_1, \ldots, z_N) \in \mathbb{R}^N \), define \( Pz = (z_1, \ldots, z_{N-1}, 0) \). Consider the semilinear elliptic equation

\[
\begin{cases}
-\Delta u + u = |u|^{p-2}u + h(z) & \text{in } \Omega; \\
u \in H^1_0(\Omega),
\end{cases}
\]

where \( \Omega = (\mathbb{R}^{N-1} \setminus \Omega^{N-1}) \times \mathbb{R}, \) \( \Omega^{N-1} \) is a smooth bounded domain in \( \mathbb{R}^{N-1} \), \( 2 < p < 2^* = 2N/(N - 2) \), \( h \in L^2(\Omega) \cap L^{(N+r)/2}(\Omega) \) (\( r > 0 \) if \( N \geq 4 \) and \( r = 0 \) if \( N = 3 \)) and \( h \) is nonnegative. Let

\[
d(p, \alpha) = (p - 2) \left( \frac{1}{p - 1} \right)^{\frac{n-1}{p-2}} \left( \frac{2p}{p - 2} \right)^{\frac{1}{2}} \alpha(\Omega)^{\frac{1}{2}},
\]

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and \( h(z) \geq 0 \) and \( 0 < \| h \|_{L^2} < d(p, \alpha) \). Associated with Eq. (1), we consider the functionals \( a, b, \) and \( J_h \), for \( u \in H^1_0(\Omega) \),

\[
\begin{align*}
a(u) &= \int_{\Omega} (|\nabla u|^2 + u^2); \\
b(u) &= \int_{\Omega} |u|^p; \\
J_h(u) &= \frac{1}{2} a(u) - \frac{1}{p} b(u^+) - \int_{\Omega} hu.
\end{align*}
\]

By Rabinowitz [10, Proposition B.10], \( a, b, \) and \( J_h \) are of \( C^2 \). For \( h = 0 \), we consider the semi-linear elliptic equation

\[
\begin{align*}
-\Delta u + u &= |u|^{p-2}u \quad \text{in } \Omega; \\
u &\in H^1_0(\Omega),
\end{align*}
\]

and the energy functional \( J(u) = \frac{1}{2} a(u) - \frac{1}{p} b(u^+) \). Lien–Tzeng–Wang [8] proved that there is no positive ground state solution of Eq. (2) in a ball up domain or a large domain \( \Omega \). Tzeng–Wang [13] proved that if \( \rho \) is sufficiently small, then Eq. (2) admits a positive higher energy solution in \( \Omega \), where \( \Omega^{N-1} \subset B^{N-1}_\rho = \{ x \in \mathbb{R}^{N-1} | |x| < \rho \} \).

For \( h \geq 0 \), suppose that \( h \) is small and exponential decay, Zhu [16] and Hsu–Wang [6] proved that Eq. (1) admits at least two positive solutions in \( \mathbb{R}^N \) and an exterior strip domain, respectively. Without the condition of exponential decay, Cao–Zhou [5] proved that Eq. (1) admits at least two positive solutions in \( \mathbb{R}^N \). In this paper, we use the techniques (see Lemma 30) of the Bahri–Li’s minimax method [2] to show that there exist at least three positive solutions of Eq. (1) in \( \Omega \).

2. Existence of (PS)-sequences

We define the Palais–Smale (denoted by (PS)) sequences, (PS)-values, and (PS)-conditions in \( H^1_0(\Omega) \) for \( J_h \) as follows.

**Definition 1.**

(i) For \( \beta \in \mathbb{R} \), a sequence \( \{ u_n \} \) is a (PS)\( _\beta \)-sequence in \( H^1_0(\Omega) \) for \( J_h \) if \( J_h(u_n) = \beta + o(1) \) and \( J'_h(u_n) = o(1) \) strongly in \( H^{-1}(\Omega) \) as \( n \to \infty \);

(ii) \( \beta \in \mathbb{R} \) is a (PS)-value in \( H^1_0(\Omega) \) for \( J_h \) if there is a (PS)\( _\beta \)-sequence in \( H^1_0(\Omega) \) for \( J_h \);

(iii) \( J_h \) satisfies the (PS)\( _\beta \)-condition in \( H^1_0(\Omega) \) if every (PS)\( _\beta \)-sequence in \( H^1_0(\Omega) \) for \( J_h \) contains a convergent subsequence.

**Lemma 2.** Let \( u \in H^1_0(\Omega) \) be a critical point of \( J_h \), then \( u \) is a nonnegative solution of Eq. (1). Moreover, if \( u \neq 0 \) or \( h \neq 0 \), then \( u \) is positive in \( \Omega \).

**Proof.** Suppose that \( u \in H^1_0(\Omega) \) satisfies \( \langle J'_h(u), \varphi \rangle = 0 \) for any \( \varphi \in H^1_0(\Omega) \), that is,

\[
\int_{\Omega} (\nabla u \nabla \varphi + u \varphi) = \int_{\Omega} \left( u^{p-1}_+ \varphi + h \varphi \right) \quad \text{for any } \varphi \in H^1_0(\Omega).
\]
Thus, \( u \) is a weak solution of \(-\Delta u + u = u^{p-1}_+ + h(z)\) in \( \Omega \). Since \( h \geq 0 \), by the maximum principle, \( u \) is nonnegative. If \( u \neq 0 \) or \( h \neq 0 \), we have that \( u \) is positive in \( \Omega \).

Let

\[
M_h = \{ u \in H^1_0(\Omega) \setminus \{0\} \mid u \geq 0 \text{ and } \langle J'_h(u), u \rangle = 0 \} \quad \text{and} \quad \alpha_h(\Omega) = \inf_{u \in M_h} J_h(u).
\]

Denote by \( M_0 = M \), \( J_0(u) = J(u) \) and \( \alpha_0(\Omega) = \alpha(\Omega) \).

By Chen–Wang [4], we have the following lemmas.

**Lemma 3.** There is a bijective \( C^{1,1} \) map \( m \) from the unit sphere \( \Sigma \) in \( H^1_0(\Omega) \) to \( M \). Moreover, \( M \) is path-connected and there exists a constant \( c > 0 \) such that for any \( u \in M \), \( \| u \|_{H^1} \geq c \) and \( J(u) \geq c \).

**Lemma 4.**

(i) For each \( u \in H^1_0(\Omega) \setminus \{0\} \), there exists a \( s_u > 0 \) such that \( s_u u \in M \);

(ii) Let \( \beta > 0 \) and \( \{u_n\} \) be a sequence in \( H^1_0(\Omega) \setminus \{0\} \) for \( J \) such that \( J(u_n) = \beta + o(1) \) and \( a(u_n) = b(u^{p+1}_n) + o(1) \). Then there is a sequence \( \{s_n\} \) in \( \mathbb{R}^+ \) such that \( s_n = 1 + o(1) \), \( \{s_n u_n\} \) in \( M \) and \( J(s_n u_n) = \beta + o(1) \).

**Lemma 5.** For each nonnegative \( u \in H^1_0(\Omega) \setminus \{0\} \), we have

\[
\left( \frac{a(u)}{b(u)} \right)^{\frac{1}{p-2}} \geq \left( \frac{2p}{p-2} \right)^{\frac{1}{2}} \alpha(\Omega)^{\frac{1}{2}}.
\]

**Proof.** Applying Lemma 4. \( \square \)

**Lemma 6** (Palais–Smale Decomposition Lemma for \( J_h \)). Let \( \{u_n\} \) be a \((PS)_\beta\)-sequence in \( H^1_0(\Omega) \) for \( J_h \). Then there are a subsequence \( \{u_n\} \), a positive integer \( l \), sequences \( \{z_i^n\}_{n=1}^{\infty} \) in \( \mathbb{R}^N \), functions \( u \in H^1_0(\Omega) \), and \( w^i \neq 0 \) in \( H^1(\mathbb{R}^N) \) for \( 1 \leq i \leq l \) such that

\[
|z^n_i| \to \infty \quad \text{for } 1 \leq i \leq l;
\]

\[-\Delta u + u = |u|^{p-2} u + h(z) \quad \text{in } \Omega;\]

\[-\Delta w^i + w^i = |w^i|^{p-2} w^i \quad \text{in } \mathbb{R}^N;\]

\[u_n = u + \sum_{i=1}^{l} w^i (\cdot - z^n_i) + o(1) \quad \text{strongly in } H^1(\mathbb{R}^N);\]

\[J_h(u_n) = J_h(u) + \sum_{i=1}^{l} J(w^i) + o(1).\]

In addition, if \( u_n \geq 0 \), then \( u \geq 0 \) and \( w^i \geq 0 \) for \( 1 \leq i \leq l \).

**Proof.** See Zhu–Zhou [17]. \( \square \)

Define \( \psi(u) = \langle J'_h(u), u \rangle = a(u) - b(u^+_+) - \int_\Omega hu \). Then
Lemma 7. For each \( u \in M_h \), we have \( \langle \psi'(u), u \rangle = a(u) - (p - 1)b(u) \neq 0 \).

**Proof.** By Tarantello [12, Lemma 2.3] and Cao–Zhou [5]. \( \square \)

By Lemma 7, we write \( M_h = M_h^+ \cup M_h^- \), where

\[
M_h^+ = \{ u \in M_h \mid a(u) - (p - 1)b(u) > 0 \},
\]

\[
M_h^- = \{ u \in M_h \mid a(u) - (p - 1)b(u) < 0 \}.
\]

Define

\[
\alpha^+_h(\Omega) = \inf_{u \in M_h^+} J_h(u); \quad \alpha^-_h(\Omega) = \inf_{u \in M_h^-} J_h(u).
\]

By Wang–Wu [15], we have the following lemma.

**Lemma 8.** \( \{ u_n \} \) is a \((PS)_{\alpha(\Omega)}\)-sequence in \( H_0^1(\Omega) \) for \( J \) if and only if \( J(u_n) = \alpha(\Omega) + o(1) \) and \( a(u_n) = b(u_n^+) + o(1) \). In particular, every minimizing sequence \( \{ u_n \} \) in \( M \) of \( \alpha(\Omega) \) is a \((PS)_{\alpha(\Omega)}\)-sequence in \( H_0^1(\Omega) \) for \( J \).

For each nonnegative \( u \in H_0^1(\Omega) \setminus \{ 0 \} \), we write

\[
t_{\text{max}} = \left( \frac{a(u)}{(p - 1)b(u)} \right)^{\frac{1}{p - 2}} > 0.
\]

**Lemma 9.** For each nonnegative \( u \in H_0^1(\Omega) \setminus \{ 0 \} \), we have the following results:

(i) There is a unique number \( t^- = t^-(u) > t_{\text{max}} > 0 \) such that \( t^- u \in M_h^- \) and \( J_h(t^-u) = \max_{t \geq t_{\text{max}}} J_h(tu) \);
(ii) \( t^-(u) \) is a continuous function;
(iii) \( M_h^- = \left\{ u \in H_0^1(\Omega) \setminus \{ 0 \} \mid u \geq 0 \text{ and } \frac{1}{\| u \|_{H^1}} t^- \left( \frac{u}{\| u \|_{H^1}} \right) = 1 \right\} \);
(iv) If \( \int_\Omega hu > 0 \), then there is a unique number \( 0 < t^+ = t^+(u) < t_{\text{max}} \) such that \( t^+ u \in M_h^+ \) and \( J_h(t^+ u) = \min_{0 \leq t \leq t^-} J_h(tu) \).

**Proof.** See Tarantello [12] and Cao–Zhou [5]. \( \square \)

**Lemma 10.**

(i) For each \( u \in M_h^+ \), we have \( \int_\Omega hu > 0 \) and \( J_h(u) < 0 \). In particular, \( \alpha_h(\Omega) \leq \alpha_h^+(\Omega) < 0 \);
(ii) \( J_h \) is coercive and bounded below on \( M_h \).

**Proof.** (i) For each \( u \in M_h^+ \), \( a(u) - (p - 1)b(u) > 0 \) and \( a(u) = b(u) + \int_\Omega hu \). Then

\[
\int_\Omega hu = a(u) - b(u) > (p - 2)b(u) > 0.
\]
Hence
\[ J_h(u) = \left( \frac{1}{2} - \frac{1}{p} \right) b(u) - \frac{1}{2} \int_\Omega hu < \frac{p - 2}{2p} b(u) - \frac{p - 2}{2} b(u) \]
\[ = -\frac{(p - 1)(p - 2)}{2p} b(u) < 0. \]

(ii) By Tarantello [12, p. 288]. \qed 

Lemma 11. Let \( u \) be in \( M_h \) such that \( J_h(u) = \min_{v \in M_h} J_h(v) = \alpha_h(\Omega) \). Then

(i) \( \int_\Omega hu > 0; \)
(ii) \( u \) is a solution of Eq. (1) in \( \Omega \).

Proof. (i) By Lemma 10(i), we have
\[ 0 > \alpha_h(\Omega) = J_h(u) = \left( \frac{1}{2} - \frac{1}{p} \right) a(u) - \left( 1 - \frac{1}{p} \right) \int_\Omega hu. \]
Thus, \( \int_\Omega hu > 0. \)

(ii) By Lemma 7, \( \langle \psi'(v), v \rangle \neq 0 \) for each \( v \in M_h \). Since \( J_h(u) = \min_{v \in M_h} J_h(v) \), by the Lagrange multiplier theorem, there is a \( \lambda \in \mathbb{R} \) such that \( J_h'(u) = \lambda \psi'(u) \) in \( H^{-1}(\Omega) \). Then we have
\[ 0 = \langle J_h'(u), u \rangle = \lambda \langle \psi'(u), u \rangle. \]
Thus, \( \lambda = 0 \) and \( J_h'(u) = 0 \) in \( H^{-1}(\Omega) \). Therefore, \( u \) is a solution of Eq. (1) in \( \Omega \) with \( J_h(u) = \alpha_h(\Omega) \). \qed 

By Cao–Zhou [5], we have the following lemma.

Lemma 12.

(i) There exists a \( (PS)_{\alpha_h(\Omega)} \)-sequence \( \{u_n\} \) in \( M_h \) for \( J_h \);
(ii) There exists a \( (PS)_{\alpha^+_h(\Omega)} \)-sequence \( \{u_n\} \) in \( M^+_h \) for \( J_h \);
(iii) There exists a \( (PS)_{\alpha^-_h(\Omega)} \)-sequence \( \{u_n\} \) in \( M^-_h \) for \( J_h \).

3. Existence of the first solution

By Lemma 12(i), there is a \( (PS)_{\alpha_h(\Omega)} \)-sequence \( \{u_n\} \) in \( M_h \) for \( J_h \). Then we have the following \( (PS)_{\alpha_h(\Omega)} \)-condition.

Lemma 13. Let \( \{u_n\} \subset M_h \) be a \( (PS)_{\alpha_h(\Omega)} \)-sequence for \( J_h \). Then there exist a subsequence \( \{u_n\} \) and a nonzero \( u_0 \in H^1_0(\Omega) \) such that \( u_n \to u_0 \) strongly in \( H^1_0(\Omega) \). Moreover, \( u_0 \) is a positive solution of Eq. (1) such that \( J_h(u_0) = \alpha_h(\Omega) \).
Proof. Since \( \{u_n\} \subset M_h \) be a \((PS)_{\alpha h(\Omega)}\)-sequence for \( J_h \), then \( \{u_n\} \) is bounded in \( H^1_0(\Omega) \). Thus, there are a subsequence \( \{u_n\} \) and a nonzero \( u^0 \in H^1_0(\Omega) \) such that \( u_n \rightharpoonup u^0 \) weakly in \( H^1_0(\Omega) \). Applying the Palais–Smale Decomposition Lemma 6, we get
\[
0 > \alpha_h(\Omega) + o(1) = J_h(u_n) \geq \alpha_h(\Omega) + l\alpha(\Omega).
\]
Then \( l = 0 \). Hence, \( u_n \rightharpoonup u^0 \) strongly in \( H^1_0(\Omega) \) and \( J_h(u^0) = \alpha_h(\Omega) \). Moreover, \( u^0 \) is a positive solution of Eq. (1) in \( \Omega \). \( \Box \)

We prove that \( u_0 \) is the unique critical point of \( J_h \) in \( B(r_0) \) in the following lemmas.

Lemma 14. Let \( r_0 = \left( \frac{1}{p-1} \right)^{\frac{1}{p-2}} \left( \frac{2p}{p-2} \right)^{\frac{1}{2}} \alpha(\Omega)^{\frac{1}{2}} \). Then

(i) \( M^+_h \subset B(r_0) = \{ u \in H^1_0(\Omega) \mid \|u\|_{H^1} < r_0 \} \);

(ii) \( J_h(u) \) is strictly convex in \( B(r_0) \).

Proof. (i) If \( u \in M^+_h \), then \( a(u) > (p - 1)b(u) \) and \( a(u) = b(u) + \int_{\Omega} hu \). Thus,
\[
a(u) < \frac{1}{p-1}a(u) + \|h\|_{L^2}\|u\|_{H^1}.
\]
This implies
\[
\|u\|_{H^1} < \left( \frac{p-1}{p-2} \right) \|h\|_{L^2} < \left( \frac{p-1}{p-2} \right) (p - 2) \left( \frac{1}{p-1} \right)^{\frac{p-1}{p-2}} \left( \frac{2p}{p-2} \right)^{\frac{1}{2}} \alpha(\Omega)^{\frac{1}{2}}
\]
\[
= \left( \frac{1}{p-1} \right)^{\frac{1}{p-2}} \left( \frac{2p}{p-2} \right)^{\frac{1}{2}} \alpha(\Omega)^{\frac{1}{2}} = r_0.
\]

(ii) We know
\[
J''_h(u)(v, v) = a(v) - (p - 1) \int_{\Omega} |u|^{p-2}v^2 \quad \text{for all } v \in H^1_0(\Omega).
\]
Thus, by Lemma 5, we obtain
\[
J''_h(u)(v, v) \geq a(v) - (p - 1) \|u\|_{L^p}^{-2} \|v\|_{L^p}^2
\]
\[
\geq a(v) - (p - 1) \left[ a(u)^{\frac{p-2}{2}} \left( \frac{p-2}{2p} \right)^{\frac{p-2}{2}} \alpha(\Omega)^{-\frac{(p-2)^2}{2p}} \right]
\]
\[
\times \left[ a(v) \left( \frac{p-2}{2p} \right)^{\frac{p-2}{2}} \alpha(\Omega)^{-\frac{(p-2)^2}{2p}} \right]
\]
\[
\geq a(v) \left[ 1 - (p - 1) \left( \frac{2p}{p-2} \alpha(\Omega) \right)^{\frac{2-p}{2}} \|u\|_{H^1}^{-2} \right]
\]
\[
> 0 \quad \text{for } u \in B(r_0) \setminus \{0\}.
\]
Thus, \( J''_h(u) \) is positive definite for \( u \in B(r_0) \) and \( J_h \) is strictly convex in \( B(r_0) \). \( \Box \)
By Lemma 13, there exists a solution \( u_0 \in \mathbf{M}_h \) of Eq. (1) such that \( J_h(u_0) = \alpha_h(\Omega) \). Furthermore, we have the following lemma.

**Lemma 15.**

(i) \( u_0 \in \mathbf{M}_h^+ \) and \( J_h(u_0) = \alpha_h^+(\Omega) = \alpha_h(\Omega) \);

(ii) \( u_0 \) is the unique critical point of \( J_h(u) \) in \( B(r_0) \), where \( r_0 \) is defined as in Lemma 14;

(iii) \( J_h(u_0) \) is a local minimum in \( H_0^1(\Omega) \).

**Proof.** (i) By Lemma 11(i), \( \int_\Omega h u_0 > 0 \). We claim that \( u_0 \in \mathbf{M}_h^+ \). Otherwise, if \( u_0 \in \mathbf{M}_h^- \), then by Lemma 9, there exists a unique \( t^-(u_0) = 1 > t^+(u_0) > 0 \) such that \( t^+(u_0) u_0 \in \mathbf{M}_h^+ \) and

\[
\alpha_h(\Omega) \leq \alpha_h^+(\Omega) \leq J_h(t^+(u_0) u_0) < J_h(t^-(u_0) u_0) = \alpha_h(\Omega),
\]

which is a contradiction. Since \( u_0 \in \mathbf{M}_h^- \), \( \alpha_h^+(\Omega) \leq J_h(u_0) = \alpha_h(\Omega) \leq \alpha_h^+(\Omega) \), that is, \( J_h(u_0) = \alpha_h^+(\Omega) = \alpha_h(\Omega) \).

(ii) By part (i) and Lemma 14.

(iii) See Cao–Zhou [5, p. 452]. \( \square \)

**Lemma 16.** Let \( u \in H_0^1(\Omega) \) be a critical point of \( J_h \), then either \( u \in \mathbf{M}_h^- \) or \( u = u_0 \).

**Proof.** Let \( u \in H_0^1(\Omega) \) be a critical point of \( J_h \), we get \( u \in \mathbf{M}_h = \mathbf{M}_h^+ \cup \mathbf{M}_h^- \). Since \( \mathbf{M}_h^+ \cap \mathbf{M}_h^- = \emptyset \), \( \mathbf{M}_h^+ \subset B(r_0) \) and \( u_0 \) is the unique critical point of \( J_h(u) \) in \( B(r_0) \), where \( r_0 \) is defined as in Lemma 14, then either \( u \in \mathbf{M}_h^- \) or \( u = u_0 \). \( \square \)

4. Existence of the second solution

Using the arguments of Chen–Chen–Wang [3, Proposition 1] and Zhu–Zhou [17], we have the following lemma.

**Lemma 17.** Assume that \( h \in L^2(\Omega) \cap L^{(N+r)/2}(\Omega) \) (\( r > 0 \) if \( N \geq 4 \) and \( r = 0 \) if \( N = 3 \)). Let \( u \) be a positive solution of Eq. (1) in \( \Omega \). Then for any \( \varepsilon > 0 \), there are positive constants \( c_\varepsilon \) and \( c'_\varepsilon \) and \( R \) such that \( \Omega \subset B_R = \{ x \in \mathbb{R}^{N-1} | |x| < R \} \) and

\[
|u(z)| \geq c_\varepsilon \exp(- (1 + \varepsilon) |Pz|) \quad \text{for } |Pz| \geq R \text{ and } |z_N| < c'_\varepsilon.
\]

We know that there is a positive radially symmetric smooth solution \( w \) of Eq. (2) in \( \mathbb{R}^N \) such that \( J(w) = \alpha(\mathbb{R}^N) \). Recall the facts:

(i) for any \( \varepsilon > 0 \), there exist constants \( C_0, C'_0 > 0 \) such that for all \( z \in \mathbb{R}^N \)

\[
w(z) \leq C_0 \exp(- |z|) \quad \text{and} \quad |\nabla w(z)| \leq C'_0 \exp(- (1 - \varepsilon)|z|);
\]

(ii) for any \( \varepsilon > 0 \), there exists a constant \( C_\varepsilon > 0 \) such that

\[
w(z) \geq C_\varepsilon \exp(- (1 + \varepsilon)|z|) \quad \text{for all } z \in \mathbb{R}^N.
\]

For such \( R \) in Lemma 17, let \( \psi_R \) be a \( C^\infty \)-function on \( \mathbb{R}^N \) such that \( 0 \leq \psi_R \leq 1 \), \(|\nabla \psi_R| \leq c \) and

\[
\psi_R(z) = \begin{cases} 
1 & \text{for } |Pz| \geq R + 1; \\
0 & \text{for } |Pz| \leq R.
\end{cases}
\]
We define

\[ w_n(z) = \psi_R(z)w(z - e_n) \quad \text{for } n \in \mathbb{N}, \]

where \( e_n = (n, 0, \ldots, 0) \in \mathbb{R}^N \). Clearly, \( w_n \in H^1_0(\Omega) \).

In order to prove Lemma 22, we need the following lemmas.

**Lemma 18.**

(i) \( a(w_n) = b(w_n) + o(1) = \frac{2p}{p-2}a(\mathbb{R}^N) + o(1) \) as \( n \to \infty \);

(ii) \( J(w_n) = \alpha(\Omega) + o(1) = \alpha(\mathbb{R}^N) + o(1) \) as \( n \to \infty \);

(iii) \( w_n \rightharpoonup 0 \) weakly in \( H^1_0(\Omega) \) as \( n \to \infty \).

**Proof.** It is similar to the proof of Wang [14, Lemma 30]. \( \square \)

**Lemma 19.** Let \( E \) be a domain in \( \mathbb{R}^N \). If \( f : E \to \mathbb{R} \) satisfies

\[ \int_E |f(z)e^{\sigma|z|}| \, dz < \infty \quad \text{for some } \sigma > 0, \]

then

\[ \left( \int_E f(z)e^{-\sigma|z-e_n|} \, dz \right)e^{\sigma n} = \int_E f(z)e^{\sigma|z|} \, dz + o(1) \quad \text{as } n \to \infty. \]

**Proof.** Since \( \sigma|e_n| \leq \sigma|z| + \sigma|z - e_n| \), we have

\[ |f(z)e^{-\sigma|z-e_n|}e^{\sigma|e_n|}| \leq |f(z)e^{\sigma|z|}|. \]

Since \( -\sigma|z - e_n| + \sigma|e_n| = \sigma\frac{|z-e_n|}{|e_n|} + o(1) \) as \( n \to \infty \), then the lemma follows from the Lebesgue dominated convergence theorem. \( \square \)

**Lemma 20.** For \( t \geq 0 \), we have the following inequalities:

(i) \( (1 + t)^q \geq 1 + t^q + \frac{q}{q-1}t^{q-1} \) where \( q \geq 2 \);

(ii) \( (1 + t)^q \geq 1 + t^q + qt \) where \( q \geq 2 \);

(iii) \( (1 + t)^q \geq 1 + t + qt + \frac{q}{q-2}t^{q-1} \) where \( q \geq 3 \);

(iv) If \( t \leq c \) for some \( c > 0 \), then \( (1 + t)^q \geq 1 + t^q + qt + A(c)t^2 \) where \( 2 < q < 3 \) and \( A(c) > 0 \).

**Proof.**

(i) Let \( f(t) = (1 + t)^q - 1 - t^q - \frac{q}{q-1}t^{q-1} \) for \( t \geq 0 \) and \( q \geq 2 \). Then \( f(0) = 0 \), and

\[ f'(t) = q[(1 + t)^{q-1} - t^{q-1} - t^{-q-2}] \]

Since \( q \geq 2 \), we get \( (1 + t)^{q-1} = (1 + t)^{q-2} + t(1 + t)^{q-2} \geq t^{q-2} + t^{q-1} \). Thus, \( f'(t) \geq 0 \).

(ii) The proof is similar to (i).

(iii) Let \( g(t) = (1 + t)^q - 1 - t^q - qt - \frac{q}{q-2}t^{q-1} \) for \( t \geq 0 \) and \( q \geq 3 \). Then \( g(0) = 0 \), and by (i), we obtain

\[ g'(t) = q[(1 + t)^{q-1} - t^{q-1} - 1 - \frac{q-1}{q-2}t^{q-2}] \geq 0. \]
(iv) Let \( h(t) = (1 + t)^q - t^q \) for \( 0 \leq t \leq c \) and \( 2 < q < 3 \). Then

\[
\begin{align*}
  h'(t) &= q[(1 + t)^{q-1} - t^{q-1}], \quad h'(0) = q, \\
  h''(t) &= q(q-1)(1 + t)^{q-2} - t^{q-2} > 0,
\end{align*}
\]

and

\[
  h'''(t) = q(q-1)(q-2)(1 + t)^{q-3} - t^{q-3} < 0.
\]

Since \( t \leq c \) for some \( c > 0 \), applying the Taylor theorem, we have

\[
(1 + t)^q - t^q - 1 - qt \geq \frac{q(q-1)}{2}(1 + c)^{q-2} - c^{q-2}t^2.
\]

By Lemma 20, we obtain

\[
(a + b)^q \geq a^q + b^q + qa^{q-1}b + \frac{q}{q-2}ab^{q-1} \quad \text{for } q \geq 3 \text{ and } a, b \geq 0,
\]

and

\[
(a + b)^q \geq a^q + b^q + qa^{q-1}b + A(c)a^{q-2}b^2 \quad \text{for } 2 < q < 3 \text{ and } b/a \leq c.
\]

Lemma 21.

(i) There exists a number \( t_0 > 0 \) such that for \( 0 \leq t < t_0 \) and each \( w_n \in H^1_0(\Omega) \), we have

\[
J_h(u_0 + tw_n) < J_h(u_0) + \alpha(\Omega);
\]

(ii) There exist positive numbers \( t_1 \) and \( n_1 \) such that for any \( t > t_1 \) and \( n \geq n_1 \), we have

\[
J_h(tw_n) < 0.
\]

Proof. (i) Since \( J_h \) is continuous in \( H^1_0(\Omega) \) and \( \{w_n\} \) is bounded in \( H^1_0(\Omega) \), there is a \( t_0 > 0 \) such that for \( 0 \leq t < t_0 \) and each \( w_n \in H^1_0(\Omega) \)

\[
J_h(u_0 + tw_n) < J_h(u_0) + \alpha(\Omega).
\]

(ii) By Lemma 18, \( J_h(tw_n) = (\frac{t^2}{2} - \frac{t^p}{p}) \frac{2p}{p-2} \alpha(\Omega) + o(1) \) as \( n \to \infty \). There is an \( n_1 > 0 \) such that for \( n \geq n_1 \)

\[
J_h(tw_n) < \left(\frac{t^2}{2} - \frac{t^p}{p}\right) \frac{2p}{p-2} \alpha(\Omega) + 1.
\]

Thus, there exists a \( t_1 > 0 \) such that

\[
J_h(tw_n) < 0 \quad \text{for any } t > t_1 \text{ and } n \geq n_1.
\]

Lemma 22. There exists a number \( n_0 > 0 \) such that for \( n \geq n_0 \)

\[
\sup_{t \geq 0} J_h(u_0 + tw_n) < \alpha_h(\Omega) + \alpha(\Omega),
\]

where \( u_0 \) is the local minimum in Lemma 15.
Proof. By Lemma 21, we only need to show that there exists an \( n_0 > 0 \) such that for \( n \geq n_0 \)

\[
\sup_{t_0 \leq t \leq t_1} J_h(u_0 + tw_n) < J_h(u_0) + \alpha(\Omega) = \alpha_h(\Omega) + \alpha(\Omega).
\]

Since \( u_0 \) is a positive solution of Eq. (1) in \( \Omega \), then

\[
\langle u_0, tw_n \rangle_{H^1} = \int_{\Omega} (u_0 - tw_n + tw_n) dz.
\]

For \( t_0 \leq t \leq t_1 \) and \( n \geq n_1 \), since \( J(w) = J(w(z - e_n)) \), \( \sup_{t \geq 0} J(tw) = \alpha(\mathbb{R}^N) \) and \( 0 \leq \psi_R \leq 1 \), we obtain

\[
J_h(u_0 + tw_n) = \frac{1}{2} \|u_0 + tw_n\|^2_{H^1} - \frac{1}{p} \int_{\Omega} (u_0 + tw_n)^p - \int_{\Omega} h(u_0 + tw_n)
\]

\[
= J_h(u_0) + J(tw_n) + \langle u_0, tw_n \rangle_{H^1}
\]

\[
+ \frac{1}{p} \int_{\Omega} \left[ u_0^p + (tw_n)^p - (u_0 + tw_n)^p - phw_n \right]
\]

\[
= J_h(u_0) + J(tw_n) - \frac{1}{p} \int_{\Omega} \left[ (u_0 + tw_n)^p - u_0^p - (tw_n)^p - pu_0^{p-1}(tw_n) \right]
\]

\[
\leq J_h(u_0) + \alpha(\mathbb{R}^N) + \frac{t^2}{2} \int_{\mathbb{R}^N} |\nabla \psi_R|^2 \left[ w(z - e_n) \right]^2 dz
\]

\[
+ t^2 \int_{\mathbb{R}^N} |\nabla \psi_R| |\nabla w(z - e_n)| w(z - e_n) dz
\]

\[
+ \frac{t^p}{p} \int_{\mathbb{R}^N} (1 - \psi_R^p) \left[ w(z - e_n) \right]^p dz
\]

\[
- \frac{1}{p} \int_{\mathbb{R}^N} \left[ (u_0 + tw_n)^p - u_0^p - (tw_n)^p - pu_0^{p-1}(tw_n) \right].
\]

For a small \( \varepsilon > 0 \), since \( \text{supp}(1 - \psi_R^p) = \{z \in \mathbb{R}^N \mid |Pz| \leq R + 1\} \) is unbounded, then

\[
\int_{|Pz| \leq R+1} (1 - \psi_R^p) \left[ w(z - e_n) \right]^p dz \leq C_1 \exp(-(p - \varepsilon)n). \tag{5}
\]

Similarly, we have

\[
\int_{\text{supp}(\nabla \psi_R)} |\nabla \psi_R|^2 \left[ w(z - e_n) \right]^2 dz \leq C_2 \exp(-(2 - \varepsilon)n), \tag{6}
\]

and

\[
\int_{\text{supp}(\nabla \psi_R)} |\nabla \psi_R| |\nabla w(z - e_n)| w(z - e_n) dz \leq C_3 \exp(-(2 - 2\varepsilon)n). \tag{7}
\]
Let \( D = \{ z \in \Omega \mid R \leq |Pz| \leq 2R \text{ and } |z_N| < c' \} \). By Lemma 20(ii), we get
\[
(I) = (u_0 + tw_n)^p - u_0^p - (tw_n)^p - pu_0^{p-1}(tw_n) \geq 0.
\]

Then
\[
\int_{\mathbb{R}^N} (I) \, dz \geq \int_{D} (I) \, dz. \tag{8}
\]

(i) For \( 3 \leq p < 2^* \), by (3)
\[
\int_{D} (u_0 + tw_n)^p \geq \int_{D} \left[ u_0^p + (tw_n)^p + pu_0^{p-1}(tw_n) + \frac{p}{p-2}u_0(tw_n)^{p-1} \right].
\]

Thus, by Lemma 19, there is an \( n'_1 \geq n_1 \) such that if \( n \geq n'_1 \), then
\[
\int_{D} u_0w_n^{p-1} \, dz \geq c_1 \exp(-\min\{1, p-1\}(1+\epsilon)n) \geq c_1 \exp(-(1+\epsilon)n). \tag{9}
\]

Using (5)–(9), we choose an \( \epsilon < 1/3 \) and an \( n_0 \geq n'_1 \) such that for \( n \geq n_0 \)
\[
\sup_{n_0 \leq t \leq t_1} J_h(u_0 + tw_n) < J_h(u_0) + \alpha(\mathbb{R}^N).
\]

(ii) For \( 2 < p < 3 \), since \( \max\{w_n(z)/u_0(z) \mid R \leq |Pz| \leq 2R \} \leq c < \infty \) for each \( n \in \mathbb{N} \), by (4)
\[
\int_{D} (u_0 + tw_n)^p \geq \int_{D} \left[ u_0^p + (tw_n)^p + pu_0^{p-1}(tw_n) + A(c)u_0^{p-2}(tw_n)^2 \right].
\]

Thus, by Lemma 19, there is an \( n'_1 \geq n_1 \) such that if \( n \geq n'_1 \), then
\[
\int_{D} u_0^{p-2}w_n^2 \, dz \geq c_2 \exp(-\min\{2, p-2\}(1+\epsilon)n)
\]
\[
\geq c_2 \exp(-(p-2)(1+\epsilon)n). \tag{10}
\]

Using (5)–(8), (10), we choose an \( \epsilon < (4 - p)/p \) and an \( n_0 \geq n'_1 \) such that for \( n \geq n_0 \)
\[
\sup_{n_0 \leq t \leq t_1} J_h(u_0 + tw_n) < J_h(u_0) + \alpha(\mathbb{R}^N) = \alpha_h(\Omega) + \alpha(\Omega).
\]

By (i) and (ii), we complete the proof. \( \square \)

Let
\[
A_1 = \left\{ u \in H_0^1(\Omega) \setminus \{0\} \mid u \geq 0 \text{ and } \frac{1}{\|u\|_{H^1}}t^{-\left(\frac{u}{\|u\|_{H^1}}\right)} > 1 \right\} \cup \{0\},
\]
\[
A_2 = \left\{ u \in H_0^1(\Omega) \setminus \{0\} \mid u \geq 0 \text{ and } \frac{1}{\|u\|_{H^1}}t^{-\left(\frac{u}{\|u\|_{H^1}}\right)} < 1 \right\}.
\]

From Tarantello [12], we have the following results.
Lemma 23.

(i) \( A \setminus M^-_h = A_1 \cup A_2 \), where \( A = \{ u \in H^1_0(\Omega) \mid u \geq 0 \} \);
(ii) \( M^+_h \subset A_1 \);
(iii) There exist \( t_0 > 1 \) and \( n_2 \geq n_0 \) such that \( u_0 + t_0 w_n \in A_2 \) for each \( n \geq n_2 \), where \( n_0 \) is defined as in Lemma 22;
(iv) There exists a sequence \( \{ s_n \} \subset (0, 1) \) such that \( u_0 + s_n t_0 w_n \in M^-_h \) for each \( n \geq n_2 \);
(v) \( \alpha^- h < \alpha h(\Omega) + \alpha(\Omega) \).

Proof. (i) By Lemma 9(iii).

(ii) For each \( u \in M^+_h \), we have
\[
1 < t_{\text{max}}(u) < t^-(u) = \frac{1}{\| u \|_{H^1}} t^-(\frac{u}{\| u \|_{H^1}}),
\]
then \( M^+_h \subset A_1 \). In particular, \( u_0 \in A_1 \), where \( u_0 \) is defined as in Lemma 15.

(iii) There is a constant \( c > 0 \) such that \( 0 < t^- (\frac{u_0 + t w_n}{\| u_0 + t w_n \|_{H^1}}) < c \) for each \( t \geq 0 \) and each \( n \in \mathbb{N} \).

On the contrary, we consider that there exist a sequence \( \{ t_n \} \) and a subsequence \( \{ w_n \} \) such that \( t^- (\frac{u_0 + t_n w_n}{\| u_0 + t_n w_n \|_{H^1}}) \to \infty \) as \( n \to \infty \). Let \( v_n = \frac{u_0 + t_n w_n}{\| u_0 + t_n w_n \|_{H^1}} \). Claim that \( b(v_n) \) is bounded below away from zero.

Case (a): there is a subsequence \( \{ t_n \} \) such that \( t_n = c_0 + o(1) \) as \( n \to \infty \), where \( c_0 > 0 \). By Lemma 18, we have
\[
a(w_n) = b(w_n) + o(1) = \frac{2p}{p-2} \alpha(\Omega) + o(1).
\]

Thus,
\[
b(v_n) \geq b(w_n) \geq \frac{1}{\| u_0 + t_n w_n \|_{H^1}^p} \frac{\| u_0 \|_{H^1}^p}{t_n^p} \int_{\Omega} (\frac{u_0}{t_n} + w_n)^p - \frac{2p}{p-2} \alpha(\Omega) + o(1).
\]

Case (b): \( t_n \to \infty \) as \( n \to \infty \). The proof is similar to case (a).

Case (c): there is a subsequence \( \{ t_n \} \) such that \( t_n = o(1) \) as \( n \to \infty \). By Lemma 18, we have
\[
\| u_0 + t_n w_n \|_{H^1}^2 = \| u_0 \|_{H^1}^2 + t_n^2 \| w_n \|_{H^1}^2 + 2t_n \langle w_n, u_0 \rangle_{H^1} = \| u_0 \|_{H^1}^2 + o(1).
\]

Thus,
\[
b(v_n) \geq \frac{1}{\| u_0 + t_n w_n \|_{H^1}^p} \frac{\| u_0 \|_{H^1}^p}{t_n^p} \int_{\Omega} u_0^p - \frac{1}{\| u_0 \|_{H^1}^p} \int_{\Omega} u_0^p + o(1).
\]
Since \( t^- (v_n) v_n \in M_h^- \subset M_h \), we have
\[
J_h(t^- (v_n) v_n) = \frac{1}{2} \left[ t^- (v_n) \right]^2 - \frac{1}{p} \left[ t^- (v_n) \right]^p b(v_n) - t^- (v_n) \int_\Omega h v_n
\]
\[\rightarrow - \infty \text{ as } n \rightarrow \infty.\]
However, \( J_h \) is bounded below on \( M_h \), which is a contradiction. Let
\[
t_0 = \left( \frac{p - 2}{2 p \alpha(\Omega)}c^2 - a(u_0) \right)^{\frac{1}{2}} + 1,
\]
then
\[
\left\| u_0 + t_0 w_n \right\|_{H^1}^2 = a(u_0) + t_0^2 \left( \frac{2p}{p - 2} \right) \alpha(\Omega) + o(1)
\]
\[> c^2 + o(1) \geq \left[ t^- \left( \frac{u_0 + t_0 w_n}{\|u_0 + t_0 w_n\|_{H^1}} \right) \right]^2 + o(1).
\]
Thus, there is an \( n_2 \geq n_0 \), where \( n_0 \) is defined as in Lemma 22, such that, for \( n \geq n_2 \),
\[
\frac{1}{\left\| u_0 + t_0 w_n \right\|_{H^1}} t^- \left( \frac{u_0 + t_0 w_n}{\|u_0 + t_0 w_n\|_{H^1}} \right) < 1,
\]
or \( u_0 + t_0 w_n \in A_2 \).
(iv) Define a path \( \gamma_n(s) = u_0 + s t_0 w_n \) for \( s \in [0, 1] \) and each \( n \geq n_2 \) where \( t_0 > 1 \), then
\[
\gamma_n(0) = u_0 \in A_1, \quad \gamma_n(1) = u_0 + t_0 w_n \in A_2.
\]
Since \( \frac{1}{\|u\|_{H^1}} t^- \left( \frac{u}{\|u\|_{H^1}} \right) \) is a continuous function for nonzero \( u \) and \( \gamma_n([0, 1]) \) is connected, there exists a sequence \( \{s_n\} \subset (0, 1) \) such that \( u_0 + s_n t_0 w_n \in M_h^- \).
(v) By part (iv) and Lemma 22,
\[
\alpha_h^- \leq J_h(u_0 + s_n t_0 w_n) < J_h(u_0) + \alpha(\Omega) = \alpha_h(\Omega) + \alpha(\Omega). \quad \square
\]
Kwong [7] proved that there is the unique positive solution \( w \) of Eq. (2) in \( \mathbb{R}^N \) such that \( J(w) = \alpha(\mathbb{R}^N) \). Lien–Tzeng–Wang [8] proved that Eq. (2) does not have a positive ground state solution in \( \Omega \) and \( \alpha(\Omega) = \alpha(\mathbb{R}^N) \). Then by Cao–Zhou [5, Proposition 3.1], Palais–Smale Decomposition Lemma 6 and Lemma 16, we have the following restricted (PS)\( _{\beta} \)-condition.

**Lemma 24.**

(i) If \( \{u_n\} \) is a (PS)\( _{\beta} \)-sequence in \( H^1_0(\Omega) \) for \( J_h \) with \( \beta < \alpha_h(\Omega) + \alpha(\Omega) \), then there exist a subsequence \( \{u_n\} \) and a nonzero \( u^0 \) in \( H^1_0(\Omega) \) such that \( u_n \to u^0 \) strongly in \( H^1_0(\Omega) \) and
\[
J_h(u^0) = \beta. \quad \text{Moreover, } u^0 \text{ is a positive solution of Eq. (1) in } \Omega;
\]
(ii) If \( \{u_n\} \subset M_h^- \) is a (PS)\( _{\beta} \)-sequence in \( H^1_0(\Omega) \) for \( J_h \) with
\[
\alpha_h(\Omega) + \alpha(\Omega) < \beta < \alpha_h^-(\Omega) + \alpha(\Omega),
\]
then there exist a subsequence \( \{u_n\} \) and a nonzero \( u^0 \in M_h^- \) such that \( u_n \to u^0 \) strongly in \( H^1_0(\Omega) \) and
\[
J_h(u^0) = \beta.
\]
Moreover, \( u^0 \) is a positive solution of Eq. (1) in \( \Omega \).
Proof. (i) Since \( \{u_n\} \) is bounded in \( H^1_0(\Omega) \), there are a subsequence \( \{u_n\} \) and a nonzero \( u^0 \in H^1_0(\Omega) \) such that \( u_n \rightharpoonup u^0 \) weakly in \( H^1_0(\Omega) \). Applying the Palais–Smale Decomposition Lemma 6, we get

\[
\alpha_h(\Omega) + \alpha(\Omega) > \beta + o(1) = J_h(u_n) = J_h(u^0) + l\alpha(\Omega) \geq \alpha_h(\Omega) + l\alpha(\Omega).
\]

Then \( l = 0 \). Hence, \( u_n \to u^0 \) strongly in \( H^1_0(\Omega) \) and \( J_h(u^0) = \beta \). Moreover, \( u^0 \) is a positive solution of Eq. (1) in \( \Omega \).

(ii) Since \( \{u_n\} \) is bounded in \( H^1_0(\Omega) \), there are a subsequence \( \{u_n\} \) and a nonzero \( u^0 \in H^1_0(\Omega) \) such that \( u_n \rightharpoonup u^0 \) weakly in \( H^1_0(\Omega) \). By Lemma 16, either \( u^0 \in M^-_h \) or \( u^0 = u_0 \). Applying the Palais–Smale Decomposition Lemma 6 to obtain

\[
\beta + o(1) = J_h(u_n) = J_h(u^0) + l\alpha(\Omega) \geq \alpha_h(\Omega) + l\alpha(\Omega).
\]

By Lemma 23(v), we have \( \alpha^-_h(\Omega) < \alpha_h(\Omega) + \alpha(\Omega) \), then \( l \leq 1 \). If \( l = 1 \) and \( u^0 = u_0 \), then \( \beta = J_h(u^0) + \alpha(\Omega) = \alpha_h(\Omega) + \alpha(\Omega) \), which is a contradiction. If \( l = 1 \) and \( u^0 \in M^-_h \), then

\[
\beta = J_h(u^0) + \alpha(\Omega) \geq \alpha^-_h(\Omega) + \alpha(\Omega),
\]

which is a contradiction. Thus, \( l = 0 \). We complete the proof. \( \Box \)

By Lemma 12(iii), there is a \((PS)_{\alpha^-_h(\Omega)}\)-sequence \( \{u_n\} \) in \( M^-_h \) for \( J_h \). Then we have the following \((PS)_{\alpha^-_h(\Omega)}\)-condition.

**Lemma 25.** Let \( \{u_n\} \subset M^-_h \) be a \((PS)_{\alpha^-_h(\Omega)}\)-sequence for \( J_h \). Then there exist a subsequence \( \{u_n\} \) and a nonzero \( u^0 \in H^1_0(\Omega) \) such that \( u_n \to u^0 \) strongly in \( H^1_0(\Omega) \). Moreover, \( u^0 \) is a positive solution of Eq. (1) such that \( J_h(u^0) = \alpha^-_h(\Omega) \).

**Proof.** By Lemma 23(v), \( \alpha^-_h(\Omega) < \alpha_h(\Omega) + \alpha(\Omega) \). Then applying Lemma 24(i), we have that there exists a positive solution \( u^0 \) of Eq. (1) such that \( J_h(u^0) = \alpha^-_h(\Omega) \). \( \Box \)

Therefore, by Lemmas 2, 13 and 25, Eq. (1) admits at least two positive solutions in \( \Omega \).

**Theorem 26.** Assume that \( h(z) \geq 0 \) and \( 0 < \|h\|_{L^2} < d(p, \alpha) \), then there are at least two positive solutions of Eq. (1) in \( \Omega \).

5. Existence of the third solution

For \( c > 0 \), we define

\[
b_c(u) = \int_{\Omega} cu^p;
\]

\[
I_c(u) = \frac{1}{2} a(u) - \frac{1}{p} b_c(u_+);
\]

\[
M_{I_c} = \{ u \in H^1_0(\Omega) \setminus \{0\} \mid \langle I'_c(u), u \rangle = 0 \}.
\]
Recall that there exist a unique $t^- = t^-(u) > 0$ and a unique $t^+ = t^+(u) > 0$ such that $t^- u \in M_h^-$ and $t^+ u \in M$. Let $\Sigma = \{u \in H^1_0(\Omega) \mid u \geq 0 \text{ and } \|u\|_{H^1} = 1\}$. Then we have the following results.

Lemma 27.

(i) For each $u \in \Sigma$, there exists a unique number $t^c(u) > 0$ such that $t^c(u)u \in M_{L_c}$ and
\[
\max_{t \geq 0} I_c(tu) = I_c(t^c(u)u) = \left(\frac{1}{2} - \frac{1}{p}\right)b_c(u)^{2/p^2};
\]
(ii) For each nonnegative $u \in H^1_0(\Omega)$ and $0 < \mu < 1$, we have
\[
(1 - \mu)I_{\frac{1}{\mu^p}}(u) - \frac{1}{2\mu}\|h\|_{L^2}^2 \leq J_h(u) \leq (1 + \mu)I_{\frac{1}{\mu^p}}(u) + \frac{1}{2\mu}\|h\|_{L^2}^2;
\]
(iii) For each $u \in \Sigma$ and $0 < \mu < 1$, we have
\[
(1 - \mu)p^{p^2}J(t^1 u) - \frac{1}{2\mu}\|h\|_{L^2}^2 \leq J_h(t^- u) \leq (1 + \mu)p^{p^2}J(t^1 u) + \frac{1}{2\mu}\|h\|_{L^2}^2;
\]
(iv) $\alpha^- > 0$ for sufficiently small $\|h\|_{L^2}$.

Proof. (i) For each $u \in \Sigma$, let $f(t) = I_c(tu) = \frac{1}{2}t^2 - \frac{1}{p}t^pb_c(u)$, then $f(t) \to -\infty$ as $t \to \infty$, $f'(t) = t - tj b_c(u)$ and $f''(t) = 1 - (p - 1)tj b_c(u)$. Let
\[
t^c(u) = \left(\frac{1}{b_c(u)}\right)^{2/p^2} > 0.
\]
Then $f'(t^c(u)) = 0$, $t^c(u)u \in M_{L_c}$ and
\[
(t^c(u))^2 f''(t^c(u)) = a(t^c(u)u) - (p - 1)b_c(t^c(u)u) = (2 - p)(t^c(u))^2 a(u) < 0.
\]
Thus, there exists a unique $t^c(u) > 0$ such that $t^c(u)u \in M_{L_c}$ and
\[
\max_{t \geq 0} I_c(tu) = I_c(t^c(u)u) = \left(\frac{1}{2} - \frac{1}{p}\right)b_c(u)^{2/p^2}.
\]
(ii) For $\mu \in (0, 1)$, we get
\[
\left|\int_\Omega hu \, dz\right| \leq \|u\|_{H^1} \|h\|_{L^2} \leq \frac{\mu}{2}\|u\|_{H^1}^2 + \frac{1}{2\mu}\|h\|_{L^2}^2.
\]
Thus, for each nonnegative $u \in H^1_0(\Omega)$, then
\[
\frac{1 - \mu}{2}\|u\|_{H^1}^2 - \frac{1}{p} \int_\Omega u^p - \frac{1}{2\mu}\|h\|_{L^2}^2 \leq J_h(u) \leq \frac{1 + \mu}{2}\|u\|_{H^1}^2 - \frac{1}{p} \int_\Omega u^p + \frac{1}{2\mu}\|h\|_{L^2}^2.
\]
(iii) Applying part (ii), we have that for each $u \in \Sigma$
\[
(1 - \mu)I_{\frac{1}{\mu^p}}(t^1 u) - \frac{1}{2\mu}\|h\|_{L^2}^2 \leq J_h(t^- u) \leq (1 + \mu)I_{\frac{1}{\mu^p}}(t^2 u) + \frac{1}{2\mu}\|h\|_{L^2}^2,
\]
where \( t^{c_1}u \in M_{I_1^{-\mu}} \) and \( t^{c_2}u \in M_{I_1^{1+\mu}} \). By part (i), then

\[
I_{I_1^{-\mu}}(t^{c_1}u) = \left( \frac{1}{2} - \frac{1}{p} \right) b_{I_1^{-\mu}}(u)^{-\frac{2}{p-2}} = (1 - \mu) \frac{2}{p-2} \left( \frac{1}{2} - \frac{1}{p} \right) b(u)^{-\frac{2}{p-2}} = (1 - \mu) \frac{2}{p-2} J(t^1u).
\]

Similarly, \( I_{I_1^{1+\mu}}(t^{c_2}u) = (1 + \mu) \frac{2}{p-2} J(t^1u) \). Hence, (iii) holds.

(iv) Applying part (iii) to obtain

\[
\max_{t \geq 0} J_h(tu) \geq (1 - \mu) \frac{p}{2} \alpha(\Omega) - \frac{1}{2\mu} \|h\|_{L^2}^2.
\]

Since \( \alpha(\Omega) > 0 \), then for each \( u \in \Sigma \) and sufficiently small \( \|h\|_{L^2} \), we have

\[
J_h(t^{-u}) = \max_{t \geq 0} J_h(tu) \geq c > 0,
\]

that is, \( \alpha_h^- > 0 \) for sufficiently small \( \|h\|_{L^2} \). \( \Box \)

Since \( \alpha_h^- > 0 \) for sufficiently small \( \|h\|_{L^2} \), we define

\[
K_h(u) = \max_{t \geq 0} J_h(tu) = J_h(t^{-u}) > 0,
\]

where \( t^{-u} \in M_{\Sigma}^c \). We observe that if \( \|h\|_{L^2} \) is sufficiently small, Bahri–Li’s minimax argument [2] also works for \( K_h \). Let

\[
\Gamma = \left\{ g \in C(B_r(0), \Sigma) \mid g|_{\partial B_r(0)} = \psi_R(z)w(z - y)/\|\psi_R(z)w(z - y)\|_{H^1} \right\}
\]

for large \( r = |y| \),

where \( y = (y_1, \ldots, y_{N-1}, 0) \) and \( \Sigma = \{ u \in H^1_0(\Omega) \mid u \geq 0 \text{ and } \|u\|_{H^1} = 1 \} \). Then we define

\[
\gamma_h(\Omega) = \inf_{g \in \Gamma} \sup_{y \in \mathbb{R}^N} K_h(g(y));
\]

\[
\gamma_0(\Omega) = \inf_{g \in \Gamma} \sup_{y \in \mathbb{R}^N} K_0(g(y)).
\]

By Lemma 27(iii), for \( 0 < \mu < 1 \), we have

\[
(1 - \mu) \frac{p}{2} \gamma_0(\Omega) - \frac{1}{2\mu} \|h\|_{L^2}^2 \leq \gamma_h(\Omega) \leq (1 + \mu) \frac{p}{2} \gamma_0(\Omega) + \frac{1}{2\mu} \|h\|_{L^2}^2. \tag{11}
\]

Let \( \Omega = (\mathbb{R}^{N-1} \setminus \overline{\Omega^{N-1}}) \times \mathbb{R} \) and \( \Omega^{N-1} \subset B^{N-1}_\rho = \{ x \in \mathbb{R}^{N-1} \mid |x| < \rho \} \). Throughout this section, assume that \( \rho \) is sufficiently small, then we have the following important lemma.

**Lemma 28.** \( \alpha(\Omega) < \gamma_0(\Omega) < 2\alpha(\Omega) \).

**Proof.** Tseng–Wang [13] proved that Eq. (2) admits at least one positive solution \( u \) in \( \Omega \) and \( J(u) = \gamma_0(\Omega) < 2\alpha(\Omega) \). Lien–Tseng–Wang [8] proved that Eq. (2) does not have a positive ground state solution in \( \Omega \) and \( \alpha(\Omega) = \alpha(\mathbb{R}^N) \). Hence, \( \alpha(\Omega) < \gamma_0(\Omega) < 2\alpha(\Omega) \). \( \Box \)
The following minimax theorem is given in Shi [11] to unify the mountain pass lemma of Ambrosetti–Rabinowitz [1] and the saddle point theorem of Rabinowitz [9].

**Theorem 29.** Let $K$ be a compact metric space, $K_0 \subset K$ a closed set, $X$ a Banach space, $\chi \in C(K_0, X)$ and let us define the complete metric space $M$ by

$$M = \left\{ g \in C(K, X) \mid g(s) = \chi(s) \text{ if } s \in K_0 \right\}$$

with the usual distance $d$. Let $\varphi \in C^1(X, \mathbb{R})$ and let us define

$$c = \inf_{g \in M} \max_{s \in K} \varphi(g(s)), \quad c_1 = \max_{\chi(K_0)} \varphi.$$

If $c > c_1$, then for each $\varepsilon > 0$ and each $f \in M$ such that

$$\max_{s \in K} \varphi(f(s)) \leq c + \varepsilon,$$

there exists $v \in X$ such that

$$c - \varepsilon \leq \varphi(v) \leq \max_{s \in K} \varphi(f(s)),$$

$$\text{dist}(v, f(K)) \leq \varepsilon^{1/2},$$

$$\|\varphi'(v)\| \leq \varepsilon^{1/2}.$$

**Lemma 30.** There exists a number $d_0 > 0$ such that if $0 < \|h\|_{L^2} < d_0$, then

$$\alpha_h(\Omega) + \alpha(\Omega) < \gamma_h(\Omega) < \alpha_h^-(\Omega) + \alpha(\Omega).$$

Moreover, there exists a positive solution $u$ of Eq. (1) in $\Omega$ such that $J_h(u) = \gamma_h(\Omega)$.

**Proof.** By Lemma 27(iii), we also have that for $0 < \mu < 1$

$$(1 - \mu) \frac{\rho}{\rho - 2} \alpha(\Omega) - \frac{1}{2\mu} \|h\|_{L^2}^2 \leq \alpha^-(\Omega) \leq (1 + \mu) \frac{\rho}{\rho - 2} \alpha(\Omega) + \frac{1}{2\mu} \|h\|_{L^2}^2.$$  

For any $\varepsilon > 0$, there exists a $d_1(\varepsilon) > 0$ such that if $\|h\|_{L^2} < d_1(\varepsilon)$, then

$$\alpha(\Omega) - \varepsilon < \alpha^-(\Omega) < \alpha(\Omega) + \varepsilon.$$

Thus,

$$2\alpha(\Omega) - \varepsilon < \alpha^-(\Omega) + \alpha(\Omega) < 2\alpha(\Omega) + \varepsilon.$$  

Using (11), for any $\delta > 0$, there exists a $d_2(\delta) > 0$ such that if $\|h\|_{L^2} < d_2(\delta)$, then

$$\gamma_0(\Omega) - \delta < \gamma_h(\Omega) < \gamma_0(\Omega) + \delta.$$  

Fix a small $0 < \varepsilon < (2\alpha(\Omega) - \gamma_0(\Omega))/2$, since $\alpha(\Omega) < \gamma_0(\Omega) < 2\alpha(\Omega)$, choosing a $\delta > 0$ such that for $\|h\|_{L^2} < d_0 = \min\{d_1, d_2\}$, we get

$$\alpha_h(\Omega) + \alpha(\Omega) < \alpha(\Omega) < \gamma_h(\Omega) < 2\alpha(\Omega) - \varepsilon < \alpha^-(\Omega) + \alpha(\Omega).$$
It is similar to Lemma 18, for \( t \geq 0 \), we have
\[
J_h(t \psi_R(z) w(z - y)) = \left( \frac{t^2}{2} - \frac{t^p}{p} \right) \frac{2p}{p - 2} \alpha(\mathbb{R}^N) + o(1)
\]
\[
= J(t w) + o(1) \leq \alpha(\mathbb{R}^N) + o(1) \quad \text{as} \ |y| \to \infty.
\]
Then
\[
K_h(\psi_R(z) w(z - y)/\|\psi_R(z) w(z - y)\|_{H^1})
\]
\[
= J_h(t^{-\psi_R(z) w(z - y)/\|\psi_R(z) w(z - y)\|_{H^1}})
\]
\[
\leq \alpha(\mathbb{R}^N) + o(1) = \alpha(\Omega) + o(1) \quad \text{as} \ |y| \to \infty,
\]
that is, \( \gamma_h(\Omega) > K_h(\psi_R(z) w(z - y)/\|\psi_R(z) w(z - y)\|_{H^1}) \) for large \( r = |y| \). Applying the Minimax Theorem 29 to obtain that \( \gamma_h(\Omega) \) is a \((PS)\)-value in \( H^1_0(\Omega) \) for \( J_h \). Therefore, by Lemmas 2 and 24(ii), we have that there exists a positive solution \( u \) of Eq. (1) in \( \Omega \) such that \( J_h(u) = \gamma_h(\Omega) \).

We can conclude the following theorem.

**Theorem 31.** Assume that \( h \in L^2(\Omega) \cap L^{(N+r)/2}(\Omega) \) (\( r > 0 \) if \( N \geq 4 \) and \( r = 0 \) if \( N = 3 \)), \( h(z) \geq 0 \) and \( 0 < \|h\|_{L^2} < \min\{d(p, \alpha), d_0\} \), where \( d_0 \) is defined as in Lemma 30. Let \( \Omega = (\mathbb{R}^{N-1} - \Omega^{N-1}) \times \mathbb{R} \) and \( \Omega^{N-1} \subset B^{N-1}_\rho = \{ x \in \mathbb{R}^{N-1} \mid |x| < \rho \} \). If \( \rho \) is sufficiently small, then there are at least three positive solutions of Eq. (1) in \( \Omega \).

**Proof.** By Lemmas 2, 13, 25 and 30, we have that Eq. (1) has at least three positive solutions in \( \Omega \).

**References**


