DISCRETE MATHEMATICS 7 (1974) 305–315. © North-Holland Publishing Company

SOME EXTREMAL PROBLEMS IN GEOMETRY

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Received 5 October 1972

Abstract. The question of how often the same distance can occur between k distinct points in n-dimensional Euclidean space E_n has been extensively studied by Paul Erdös and others. Sir Alexander Oppenheim posed the somewhat similar problem of investigating how many triangles with vertices chosen from among k points in E_n can have the same non-zero area. A subsequent article by Erdös and Purdy gave some preliminary results on this problem. Here we carry that work somewhat further and show that there cannot be more than $ck^{3-\epsilon}$ triangles with the same non-zero area chosen from among k points in E_5, where ϵ is a positive constant. Since there can be ck^3 such triangles in E_6, the result is in a certain sense best possible. The methods used are mainly combinatorial and geometrical. A main tool is a theorem on generalized graphs due to Paul Erdös.

1. Introduction

Let there be given n points $X_1, ..., X_n$ in k-dimensional Euclidean space E_k. Denote by $d(X_i, X_j)$ the distance between X_i and X_j. Let $A(X_1, ..., X_n)$ be the number of distinct values of $d(X_i, X_j)$, $1 \leq i \leq j \leq n$. Put $f_k(n) = \min A(X_1, ..., X_n)$, where the minimum is taken over all possible choices of distinct $X_1, ..., X_n$. Denote by $g_k(n)$ the maximum number of solutions of $d(X_i, X_j) = \alpha$, $1 \leq i \leq j \leq n$, where the maximum is to be taken over all possible choices of α and n distinct points $X_1, ..., X_n$. The estimation of $f_k(n)$ and $g_k(n)$ are difficult problems even for $k = 2$. It is known (see [1, 7]) that

$$(1) \qquad cn^{2/3} < f_2(n) < Cn/\sqrt{\log n},$$

$$(2) \qquad n**(1 + c/\log\log n) < g_2(n) < Cn^{3/2},$$

where c and C are positive absolute constants and $a**b$ denotes a^b.

If $k \geq 4$, the study of $g_k(n)$ becomes somewhat simpler ([4] see also [2]).

A. Oppenheim posed the problem of investigating the number of triangles chosen from n points in the plane which have the same non-zero area. This question and its generalization were first investigated in [5]. In this note I support some claims made in [5].

2. Notations

Let $n \geq 3$, $X_1, \ldots, X_n$ be n points in k-dimensional space E_k and let $\Delta > 0$.

We define $g_k^{(2)}(n; X_1, \ldots, X_n; \Delta)$ to be the number of triangles of the form $X_i X_j X_k$ having area Δ. We let

$$g_k^{(2)}(n; X_1, \ldots, X_n) = \operatorname*{Max}_{\Delta} g_k^{(2)}(n; X_1, \ldots, X_n; \Delta) ,$$

$$g_k^{(2)}(n) = \operatorname*{Max}_{X_1, \ldots, X_n} g_k^{(2)}(n; X_1, \ldots, X_n) .$$

Let P be a fixed point and define $G_k^{(2)}(n; X_1, \ldots, X_n; \Delta)$ to be the number of triangles of the form $P X_i X_j$ having area Δ. We let

$$G_k^{(2)}(n) = \operatorname*{Max}_{\substack{X_1, \ldots, X_n \\ \Delta > 0}} G_k^{(2)}(n; X_1, \ldots, X_n; \Delta) .$$

Clearly, $g_{k-1}^{(2)}(n) \leq g_k^{(2)}(n) \leq n\, G_k^{(2)}(n-1) \leq n\, G_k^{(2)}(n)$. We see that $g_k^{(2)}(n)$ is analogous to $g_k(n)$.

3. The article of Erdös and Purdy

It was shown [5] that

(3) $$c\, n^2 \log \log n \leq g_2^{(2)}(n) \leq n\, G_2^{(2)}(n) \leq 4 n^{5/2} ,$$

where c is a positive absolute constant, and

(4) $$g_2^{(2)}(n) \leq g_3^{(2)}(n) \leq n\, G_3^{(2)}(n) \leq c\, n^{3-(1/3)} .$$

A simple example, which I shall give in Section 4, shows that $G_4^{(2)}(n) \geq c\, n^2$ and $g_5^{(2)}(n) \geq c\, n^3$. It is therefore worth asking whether $g_4^{(2)}(n)$

and $g_5^{(2)}(n)$ are $o(n^3)$. The object of this note is to support the claim made in [5] that in fact $g_4^{(2)}(n) \leq g_5^{(2)}(n) \leq c\,n^{3-\epsilon}$ for some $\epsilon > 0$.

4. The example of Linz generalized

We first give the example that shows that $G_4^{(2)}(n) \geq c\,n^2$. Let $n \geq 2$ be given. Let $n = 2m + r$, where $0 \leq r < 2$. Choose a coordinate system in E_4 and put $X_i = (a_i, b_i, 0, 0)$ for $1 \leq i \leq m$, and $Y_i = (0, 0, a_i, b_i)$ for $1 \leq i \leq m + r$, where (a_i, b_i) are $m + r$ distinct real solutions of $a^2 + b^2 = 1$. Then the $m(m + r)$ triangles $O X_i Y_j$ are all congruent to the triangles with sides $1, 1, \sqrt{2}$ and therefore have the same (positive) area. Hence $G_4^{(2)}(n) \geq m(m + r) \geq \frac{1}{4}n^2 - \frac{1}{4} \geq c\,n^2$. By choosing the a_i, b_i so that some of the triangles $O\,Y_i\,Y_j$ and $O\,X_i\,X_j$ are congruent to the $O\,X_i\,Y_j$, we may improve this to $\frac{3}{8}n^2 + c\,n$, but no further.

We now show that $g_6^{(2)}(n) \geq c\,n^3$. Let $n \geq 3$ be given. Let $n = 3m + r$, where $0 \leq r < 3$. Choose a coordinate system in E_6, put $X_i = (a_i, b_i, 0, 0, 0, 0)$ for $1 \leq i \leq m$, put $Y_i = (0, 0, a_i, b_i, 0, 0)$ for $1 \leq i \leq m$, and put $Z_i = (0, 0, 0, 0, a_i, b_i)$ for $1 \leq i \leq m + r$, where (a_i, b_i) are $m + r$ distinct real solutions of $a^2 + b^2 = 1$. Then the $m^2(m + r)$ triangles $X_i\,Y_j\,Z_k$ are all equilateral triangles of side length $\sqrt{2}$. Hence $g_6^{(2)}(n) \geq m^2(m + r) \geq c\,n^3$.

5. Statement of the main theorems

Theorem 5.1. *There exist $n_1, \epsilon > 0$ such that $g_5^{(2)}(n) \leq n^{3-\epsilon}$ for $n \geq n_1$. Consequently, there exists a positive constant c such that $g_5^{(2)}(n) \leq c\,n^{3-\epsilon}$ for all n.*

Let $|S|$ denote the cardinality of the set S. We shall deduce Theorem 5.1 from the following theorem.

Theorem 5.2. *Suppose that A, B and C are finite sets in E_5 such that $|A| \geq M$, $|B| \geq N$ and $|C| \geq N$, where M and N are certain absolute constants. Then the triangles X Y Z for X in A, Y in B and Z in C cannot all have the same area, unless that area be zero.*

6. Some graph theory

By an r-graph $G^{(r)}$ we mean an object whose basic components are its elements, called vertices, and certain distinguished r-element sets of these elements, called r-sets. When $r = 2$, $G^{(r)}$ is an ordinary graph. When we say that G is a $G^{(r)}(n; m)$, we mean that G is an r-graph having n vertices and m r-sets. If G is a $G^{(r)}(n; \binom{n}{r})$, then G is the unique r-graph which has all possible r-element sets as its r-sets. We call this the complete r-graph on n vertices and denote it by $K^{(r)}(n)$. $K^{(r)}(n_1, ..., n_r)$ will denote the r-graph of $n_1 + ... + n_r$ vertices and $n_1 ... n_r$ r-sets defined as follows: The vertices are

$$X_{i_j}^{(j)}, \quad 1 \leq j \leq r, \quad 1 \leq i_j \leq n_j ,$$

and the r-sets of our r-graph are the $n_1 ... n_r$ r-sets

$$\{X_{i_1}^{(1)}, X_{i_2}^{(2)}, ..., X_{i_r}^{(r)}\}, \quad 1 \leq i_j \leq n_j, \quad 1 \leq j \leq r .$$

Denote by $f(n; K^{(r)}(l_1, ..., l_r))$ the smallest integer L so that every $G^{(r)}(n; L)$ contains a $K^{(r)}(l_1, ..., l_1)$.

In an elementary but not-trivial way, Erdös [3, Theorem 1] proves that if $n > n_0(r, l)$, then

$$(*) \qquad f(n; K^{(r)}(l, ..., l)) \leq n{**}(r - l{**}(1-r)) .$$

We shall use this result with $r = 3$, and we shall refer to the 3-sets of a 3-graph as triples in what follows.

7. The relation between the main theorems

We now prove that Theorem 5.2 implies Theorem 5.1. Let l be the maximum of M and N of Theorem 5.2, let $\epsilon = l^{-2}$ and let $X_1, ..., X_n$ be distinct points in E_5 with $n > n_0(r, l)$, where $n_0(r, l)$ is the function given in Erdös's inequality (*). It is an easy consequence of (*) that Theorem 5.2 implies

$$(5) \qquad g_5^{(2)}(n; X_1, ..., X_n) \leq n^{3-\epsilon} .$$

To see this, let $\Delta > 0$ and let $G^{(3)}$ denote the 3-graph with n vertices $X_1, \ldots, X_n$, where the triple $X_i X_j X_k$ is in $G^{(3)}$ if and only if the triangle $X_i X_j X_k$ has area Δ. Then Theorem 5.2 implies that $G^{(3)}$ does not contain a $K^{(3)}(l, l, l)$ subgraph, and (5) then follows from (*). Theorem 5.1 follows since Δ was arbitrary.

8. Some lemmas

Before proving Theorem 5.2, we must introduce some definitions and lemmas. We shall use the notation $\{x\}$ to mean least integer not less than x.

Lemma 8.1. *Let triangles $P X_i Y_j$, $1 \leq i \leq n+1$, $1 \leq j \leq H$, all have the same non-zero area Δ, where X_i, Y_j are points in real Euclidean n-dimensional space. If the $n+1$ distances $d(P, X_i)$ are all different and non-zero, then there are not more than 2^{n-1} distinct distances $d(P, Y_j)$. Hence at least $\{H/2^{n-1}\}$ of the Y_j are equidistant from P.*

Proof. Let P be the origin of coordinates. Let U_i be a unit vector parallel to $\overrightarrow{PX_i}$. The area of a triangle $O X Y$ can be written in terms of lengths and the inner product as half the square root of $|X|^2\, |Y|^2 - (X \cdot Y)^2$. For all i and j, we have $4\Delta^2 = |X_i|^2\, |Y_j|^2 - (X_i \cdot Y_j)^2$, or $|Y_j|^2 - (U_i \cdot Y_j)^2 = r_i^2$, where $r_i = 2\Delta/|X_i|$. Let C_i be the set of solutions Y of

$$(6) \qquad |Y|^2 - (U_i \cdot Y)^2 = r_i^2 \, .$$

In fact, C_i is a cylinder with axis U_i and radius r_i. Let k be the rank of the set $\{U_1, \ldots, U_{n+1}\}$. By renaming the U_i and choosing a suitable coordinate system, we may suppose that $U_i = (a_{i1}, \ldots, a_{in})$ for $1 \leq i \leq n+1$, $a_{ii} \neq 0$ for $1 \leq i \leq k$, and $a_{ij} = 0$ if $j > k$ for all i. Putting $r = |Y|$ and $Y = (y_1, \ldots, y_n)$ in (6) and solving for $Y \cdot U_i$, we obtain

$$(7) \qquad \begin{aligned} &\sum_{j=1}^{i} a_{ij}\, y_j = \pm\sqrt{r^2 - r_i^2}, \qquad 1 \leq i \leq k\,, \\ &\sum_{j=1}^{k} a_{k+1,j}\, y_j = \pm\sqrt{r^2 - r_{k+1}^2}\,. \end{aligned}$$

We shall show that r^2 is the root of a non-zero polynomial of degree at most 2^{k-1}, and the lemma will follow. Let the system of equations $\Sigma_{j=1}^{i} a_{ij} y_j = z_i$ $(1 \le i \le k)$ have the solution $y_i = \Sigma_{j=1}^{i} b_{ij} z_j$ $(1 \le i \le k)$ and suppose that $z_{k+1} = \Sigma_{j=1}^{k} a_{k+1,j} y_j$. The substituting the expression for the y_j, we get

$$z_{k+1} = \sum_{j=1}^{k} a_{k+1,j} \sum_{i=1}^{j} b_{ji} z_i = \sum_{j=1}^{k} c_j z_j \quad \text{for some } c_j.$$

There are 2^k functions $f_i(t)$ of the form $\sqrt{t-r_{k+1}^2} - \Sigma_{j=1}^{k} \pm c_j\sqrt{t-r_j^2}$ corresponding to the choices of sign. Let $P(t) = f_1(t) \ldots f_m(t)$, where $m = 2^k$. If Y is a solution of (7), then clearly $P(r^2) = 0$. It is therefore sufficient to show that P is a non-zero polynomial of degree at most 2^{k-1}. Let $W_0 = \sqrt{t-r_{k+1}^2}$ and let $W_i = c_i\sqrt{t-r_i^2}$ for $1 \le i \le k$. By induction on k, we have $\Pi(W_0 \pm W_1 \pm \ldots \pm W_k) = F_k(W_0^2, W_1^2, \ldots, W_k^2)$, where F_k is a homogeneous polynomial of degree 2^{k-1} and the product is taken over all 2^k possible combinations of signs. Hence $P(t) = F_k(t-r_{k+1}^2, c_1^2(t-r_1^2), \ldots, c_k^2(t-r_k^2))$ is a polynomial in t of degree at most 2^{k-1}.

To show that P is not the zero polynomial, we proceed as follows:

$$f_i(t) = \sqrt{t-r_{k+1}^2} - \sum_{j=1}^{k} \pm c_j\sqrt{t-r_j^2},$$

$$2f_i'(t) = 1/\sqrt{t-r_{k+1}^2} + \sum_{j=1}^{k} \pm c_j/\sqrt{t-r_j^2}.$$

Let $c_{k+1} = 1$ and let $R = r_p$ be the maximum r_j for which $c_j \neq 0$. Then $f_i'(t)$ is of constant sign for $R^2 < t < R^2 + \epsilon_i$, some positive ϵ_i, since the term $c_p/\sqrt{t-r_r^2}$ goes to infinity as $t \downarrow R^2$, and the other terms remain bounded. (If the r_i were not distinct, considerable difficulty would arise at this point.) Hence f_i has at most one zero in that interval, and P has at most m zeros in the interval $R^2 < t < R^2 + \min \epsilon_i$. The lemma is proved.

Definition 8.2. If all the points of a set B are equidistant from a point X, then we say that B is *equidistant from* X. If B is equidistant from every point X of A, then we say that B is *equidistant from* A. This relation is clearly not symmetric. If all the points of a set B are different distances from a point X, then we say that B is *separated by* X. If B is separated by every point X of A, then we say that B is *separated by* A. This relation is also not symmetric.

Lemma 8.3. *Let S and T be arbitrary sets of cardinalities M and N respectively, and suppose that the elements of $S \times T$ are divided into two classes C_1 and C_2. (Suppose that the pairs are colored two colors C_1 and C_2.) Then there is a subset T' of T of cardinality $\{N/(2^M)\}$ such that for every X in S the elements (X, Y) for Y in T' are either all in C_1 or all in C_2. (For every X, the color of the pair (X, Y) for Y in T' depends only on X.)*

Proof. Use induction on M and the pigeon-hole principle.

Lemma 8.4. *Given pairwise disjoint finite subsets A, B, C of E_k, there are subsets A' of A, B' of B, and C' of C such that B' is separated by or equidistant from A' and C' is separated by or equidistant from A'. Further if $|A| = H$, we have $|A'| = \{\frac{1}{4}H\}$, $|B'| = |B|** \, 2^{-H}$ and $|C'| = |C|** \, 2^{-\{H/2\}}$.*

Proof. Let $B_0 = B$, and $i \geq 1$ If the elements of A are $X_1, X_2, \ldots, X_H$, we define sets $B_1, B_2, \ldots, B_H$ as follows. For each X_i, we color X_i and take a subset of B_{i-1} as follows. If B_{i-1} has a subset of $\{\sqrt{|B_{i-1}|}\}$ points separated by X_i, let B_i be this subset and we color X_i red. Otherwise, by the pigeon-hole principle, there is a subset B_i of B_{i-1} of cardinality $\{\sqrt{|B_{i-1}|}\}$, equidistant from X_i, and we color X_i blue. If we do this for $1 \leq i \leq H$, we get a subset B_H of B of cardinality $|B|** \, 2^{-H}$ that is separated by all the red X_i and equidistant from all the blue X_i. Let $B' = B_H$. Then there is a subset A^* of cardinality $\{\frac{1}{2}H\}$ of A such that B' is either separated by or equidistant from A^*.

Similarly, there is a subset C' of C of cardinality $|C|** \, 2^{-K}$, $K = \{\frac{1}{2}H\}$, and a subset A' of A^* of cardinality $\{\frac{1}{4}H\}$ such that C' is either separated by or equidistant from A'. The lemma follows.

Lemma 8.5. *Let PX_iY_j, [illegible] $\Delta > 0$ for $1 \leq i \leq 3$ and $1 \leq j \leq 3$, where X_1, X_2, X_3 are distinct and Y_1, Y_2, Y_3 are distinct. Then the X_i are not collinear. By symmetry, the Y_j are not collinear.*

Proof. Suppose the X_i are collinear. Let $1 \le j \le 3$. The X_i lie on the surface of a cylinder with axis PY_j. This can happen only if the X_i lie on a line parallel to PY_j. Consequently, P and the Y_j are collinear. The distances $d(P, Y_j)$ cannot all be equal. Suppose, without loss of generality, that $d(P, Y_1) > d(P, Y_2)$. Then triangle PX_1Y_1 has a greater area than triangle PX_1Y_2, contrary to the hypothesis.

Lemma 8.6. *If in* E_4 *the cylinder* $C = \{X: |X|^2 - (U\cdot X)^2 = c^2\}$, *where* $|U| = 1$, *intersects the hyperplane* π, *then there are three possibilities.*

(i) *If* $\overrightarrow{OU}$ *is parallel to* π, *then* C *intersects* π *in a cylinder.*

(ii) *If* $\overrightarrow{OU}$ *is perpendicular to* π, *then* C *intersects* π *in a sphere.*

(iii) *If neither of the above, then* C *intersects* π *in an ellipsoid of revolution whose axis is the projection of* $\overleftrightarrow{OU}$ *onto* π.

Proof. Choose the origin O to be on π and choose the X_4 axis normal to π. Then π is the set of points (x_1, x_2, x_3, x_4) such that $x_4 = 0$. Now choose the X_1 axis lying in π, in the direction of the projection $\overrightarrow{OU^*}$ of $\overrightarrow{OU}$ onto π. Then $U = (\alpha, 0, 0, \beta)$, where $\alpha^2 + \beta^2 = 1$. The cylinder C has the equation $\sum_1^4 x_i^2 - (\alpha x_1 + \beta x_4)^2 = c^2$ and C intersects π in a surface with equation $\beta^2 x_1^2 + x_2^2 + x_3^2 = c^2$. If $\beta = 0$, we have (i); if $\beta = 1$, we have (ii); and if $0 < \beta < 1$, then we have (iii).

9. Proof of Theorem 5.2

Let A, B, C be sets of cardinality M, N, N respectively, such that the triangles XYZ for X in A, Y in B and Z in C, all have a common positive area Δ. We shall show that this leads to a contradiction if M and N are large enough, and the theorem will follow.

Let us assume that $N \ge M \ge 3$; then, by Lemma 8.5, no three points of A (or B or C) are collinear. By Lemma 8.4 and the symmetry between B and C, we may suppose without loss of generality that one of the following holds:

(1) B is separated by A, but C is equidistant from A.

(2) B is separated by A and C is separated by A.

(3) B is equidistant from A and C is equidistant from A.

This application of Lemma 8.4 reduces M and N. From now on, M and N are for the new sets.

Let $A = \{X_1, \ldots, X_M\}, B = \{Y_1, \ldots, Y_N\}$. First of all, (2) leads immediately to a contradiction. Take one point X of A and six points $Y_1, \ldots, Y_6$ of B. Then, by Lemma 8.1, there are at most 16 points $Z_1, \ldots, Z_{16}$ such that $\{Z_1, \ldots, Z_{16}\}$ is separated by X. We get a contradiction if $N \geq 17$.

Secondly, (3) leads to a contradiction; (3) implies that the affine hull of A is orthogonal to the affine hull of B and the affine hull of C. We shall find a subsets B' of B and C' of C whose affine hulls are orthogonal. Let X be a fixed point of A, let B' be a subset of order three of B, let d be the common distance of B from X and let e be the common distance of C from X. Then $4\Delta^2 = e^2 d^2 - \{(Y-X)\cdot(Z-X)\}^2$ or $|(Y-X)\cdot(Z-X)| = \sqrt{e^2 d^2 - 4\Delta^2}$ for all (Y, Z) in $B' \times C$. By Lemma 8.3, there is a subset C' of C of order $\{\frac{1}{8}N\}$ such that for each Y in B', $(Y-X)\cdot(Z-X)$ has a constant sign as Z ranges over C'. Hence for Z, Z' in C' and Y, Y' in B', $(Y-X)\cdot(Z-X) = (Y-X)\cdot(Z'-X)$, $(Y-X)\cdot(Z-Z') = 0$, $(Y-Y')\cdot(Z-Z') = 0$, and the affine covers of C' and B' are orthogonal. Since no three points of A (or B or C) can be collinear and three pairwise orthogonal planes cannot exist in $\mathbf{R}^5$, we obtain a contradiction for $M \geq 3$ and $N \geq 17$.

We next show that (1) leads to a contradiction. This is the last and hardest case. We start by reducing B to be $\{Y_1, \ldots, Y_M\}$ by throwing away $N-M$ points. Now $4\Delta^2 = |X-Z|^2 |Y-X|^2 - \{(Y-X)\cdot(Z-X)\}^2$ for all (X, Y, Z) in $A \times B \times C$. Hence $|(Z-X)\cdot(Y-X)| = \sqrt{|X-Z|^2 |Y-X|^2 - 4\Delta^2}$, and the right-hand side is independent of Z since $|X-Z|$ is independent of Z. Let $\gamma_1, \ldots, \gamma_r$, where $r = M^2$, be an enumeration of $A \times B$. Let us 2-color $G^{(2)} = (A \times B) \times C$ as follows: If $(Z-X)\cdot(Y-X) \geq 0$, then color $((X, Y), Z)$ red. Otherwise, color $((X, Y), Z)$ blue. By Lemma 8.3, there is a subset C' of C of cardinality $\{N/(2^r)\}$ such that $(Z-X)\cdot(Y-X)$ is of constant sign as Z ranges over C' with (X, Y) fixed. Hence for (X, Y) in $A \times B$ and Z, Z' in C', $(Z-Z')\cdot(Y-X) = 0$; for Y, Y' in B, Z, Z' in C' and X, X' in A, we have $(Z-Z')\cdot(Y-Y') = (Z-Z')\cdot(X-X') = 0$. Hence the affine hull of C' is orthogonal to the affine hull of $A \cup B$. Let us assume that $N \geq 2{*}{*}(M^2+1)+1$, so that the order $\{N/(2^r)\}$ of C' is at least 3, so that C' contains three non-collinear points.

Hence the dimension of the affine hull of C' is at least two, and this forces $A \cup B$ to lie in a three dimensional subspace π.

If C' is also contained in π, then the whole configuration is in $\mathbf{R}^3$, and if M and N are large enough, we have a contradiction by (4). We may therefore suppose the existence of a point Z of C' that is not in π. Let Z^* be the orthogonal projection of Z onto π. The points of B lie on cylinders with axes $\overleftrightarrow{X_i Z}$ $(1 \le i \le M)$, which, by Lemma 8.6, intersect π in surfaces ξ_i which are either cylinders, spheres, or ellispoids of revolution with axes $\overleftrightarrow{X_i Z^*}$. Call these surfaces ξ_i. By the same lemma, ξ_i cannot be a cylinder since $\overleftrightarrow{X_i Z}$ is not parallel to π.

Also by Lemma 8.6, ξ_i is a sphere only if $X_i = Z^*$. Since no three points of the set A are collinear, there exist two points, say X_1 and X_2, of A such that X_1, X_2, Z^* are not collinear. This implies that neither X_1 nor X_2 coincide with Z^*, so that by Lemma 8.6, ξ_1 and ξ_2 are ellipsoids of revolution with axes of revolution $\overleftrightarrow{X_1 Z^*}$ and $\overleftrightarrow{X_2 Z^*}$.

Suppose that B has a nine-point subset B^* that is equidistant from Z. Then B^* lies on a sphere S^* having center Z^* and lying in π, and for $i = 1, 2$, each ξ_i intersects the sphere S^* in a pair of circles C_i and C_i' whose centers lie on the line $\overleftrightarrow{X_i Z^*}$. For $i, j = 2$ and $j \neq i$, C_i is distinct from C_j and C_j', since the normals $\overleftrightarrow{X_i Z^*}$ and $\overleftrightarrow{X_j Z^*}$ are not parallel, due to the fact that X_1, X_2, Z^* are non collinear. Two distinct circles on the surface of a sphere in $\mathbf{R}^3$ intersect in at most two points. Hence $(C_1 \cup C_1') \cap (C_2 \cup C_2') = C_1 \cap C_2 \cup C_1' \cap C_2 \cup C_1 \cap C_2' \cup C_1' \cap C_2'$ is a set of order less than nine containing a set B^* of order nine, which is absurd.

Hence there exists a set B' of cardinality $\{\frac{1}{8}M\}$, which is separated by Z. Let us suppose that $M \ge 41$ and take B' to have cardinality at least 6. By Lemma 8.1, there exists a subset A' of A of cardinality $R = \{M/16\}$ that is equidistant from Z; let d be the common distance of the points of A' from Z. Let B'' be a subset of three elements of B'. Then $4\Delta^2 = d^2|Y-Z|^2 - \{(Y-Z)\cdot(X-Z)\}^2$ or $|(Y-Z)\cdot(X-Z)| = \sqrt{d^2|Y-Z|^2 - 4\Delta^2}$ for all $(X, Y) \in A' \times B''$. The right-hand side is independent of X. By Lemma 8.3, there is a subset A'' of A of order $S = \{\frac{1}{8}R\}$ such that $(Y-Z)\cdot(X-Z)$ is of constant sign for fixed Y in B'' as X ranges over A''. Hence $(Y-Z)\cdot(X-X') = 0$ for all X, X' in A'' and Y in B''. Hence $(Y-Y')\cdot(X-X') = 0$ for all X, X' in A'' and Y, Y' in B'', and the affine hull of A'' is orthogonal to the affine hull of B''. Combining this with our earlier result, we see that the affine hulls of A'', B'' and C' are pairwise orthogonal. To get a contradiction, it is sufficient to ensure that each of these sets has at least three elements. If $M = 257$, then $R = 17$ and $S = 3$. If $N \geq 2{*}{*}(M^2 + 1) + 1$, we obtain the desired contradiction.

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