Helge Tverberg
A celebration of a life in mathematics

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Helge Arnulf Tverberg was born in Bergen, Norway on 6th March, 1935, and has been associated with the University of Bergen throughout his career, beginning as an undergraduate in 1954, as he recounts in [25]. He was elected a Fellow of the Norwegian Academy of Science and Letters (*Det Norske Videnskaps-Akademi*), in 1988. The editors of Axel Thue's *Selected Mathematical Papers* [12], described themselves as sharing with Thue "the dual nationality of being mathematicians and Norwegians"; and without doubt Helge Tverberg belongs to this distinguished tradition of an international scientific outlook rooted firmly in a strong local identity. The University of Bergen, perhaps reflecting the historical tradition of Bergen itself as a member of the Hanseatic League, has the reputation of being Norway's most internationally minded university. However, Helge Tverberg stands out even in this setting, since his colleagues at the University like to say of him that if anyone in Norway has heard of some foreign mathematician it will be Tverberg.

Many family names in Norway refer to geographical features, rather than, say, occupations, as might be more common in other countries—not unnaturally perhaps in view of Norway's dramatic scenery. As a case in point, *Tverberg* connotes a mountain that runs cross-wise to a series of parallel ridges and valleys, with a similar meaning for the street name, *Tverrgaten*. Without endorsing nominative determinism, there does seem to be an aspect here also of Helge Tverberg's own distinctive outlook: a happy observation of Jonathan Miller on first hearing the late Peter Cook perform in 1959 comes to mind in this regard.

> One knew one was in the presence of comedy at rightangles to all the comedy we'd heard.

It is an endearing, and immediately recongnizable, characteristic of Helge Tverberg too that he inhabits the *orthogonal complement* alike of daily discourse and mundane mathematics. Tverberg's researches have truly cut across many areas of mathematics, always providing an incisive complement to the work of specialists in those areas—for example, Tverberg's eclectic interests in algebra, analysis, and number theory, were early at work in his investigations of the irreducibility of polynomials, on which he completed a thesis in 1968, in addition to several publications. More technically,

Tverberg has been much fascinated by *transversals*, and *intersections*, and the suitably whimsical sounding *ham sandwich theorem*. His reading, mathematical and general, has been prodigious, and his wry sense of humour, like his mathematics, depends much on the startling, but ever playful, juxtaposition of recondite information. This has made him much valued as an advisor or referee, alas perhaps detracting from his own output, even beyond what his own high standards and natural diffidence would censor. Likewise, it would be entirely typical that he would be equally familiar with, for example, the early sketches from *Beyond the Fringe* and be able to place Jonathan Miller's comment on Peter Cook.

However, by the same token, it would be a tough challenge to survey the mathematical work of Helge Tverberg, for the brevity of his notes belies their breadth and quality. Fortunately, he allowed himself to be persuaded to contribute an essay [25] reflecting on Norwegian contributions to combinatorial mathematics and on his own working life. Overcoming his reservations about whether anyone would be interested in reading this essay, he has achieved a vignette of rare charm and distinction, perfectly in character, and an illuminating contribution to what Hadamard called the psychology of invention in the mathematical field [8]. Bryan Birch, who himself figures in [25], mused that it was a pity that Helge Tverberg had never met Thoralf Skolem and so could not bring out what an "extraordinarily nice, unassuming, person" Skolem was, "apart from being a fine mathematician". What does come through from reading [25] is that Helge Tverberg has that same modest disposition, coupled with integrity and intellect, as Skolem, living for his mathematics and for his mathematical friends—as, of course, they are already well aware. And, in discussing Norwegian figures from the past like Skolem, he has an unerring eye for points that will make the reader want to go back and learn more about them. (An earlier draft of [25] did indeed prompt the authors of [1] to return to the work of Axel Thue and make explicit reference to it in their second edition.)

Still, some remarks on a few selected items from Helge Tverberg's publications may not be out of place, as helping to give a fuller picture of Tverberg's life in mathematics. To begin at the beginning, his first paper [19], in 1958, announces the impress of his mind. Claude Shannon and Warren Weaver had presented a characterization of the *information function* in their book *The Mathematical Theory of Communication* [18], published in 1949, reflecting Shannon's war-time research. In the Russian school, Khinchin had published, in [10], a characterization under additional assumptions in 1953, and Faddeev had been able to pare back these assumptions to those of Shannon and Weaver, in a paper [5] in 1956, with both articles cited in the literature in English by 1958. Tverberg is familiar with this background, and moves deftly to prove a stronger characterization by weakening the clearly technical assumption that the information function is continuous, defending this move with judicious dispatch: "If my weakening of the conditions is insignificant from an information-theoretic point of view, I do not think it is so from a purely mathematical one." The whole exercise only takes a couple of pages; and Tverberg's pioneering judgement has been amply confirmed—a survey [16] in 1987 of generalizations of the functional equation satisfied

by the information function ran to 108 references; and more recently a whole book [3] has been devoted to the topic.

Playfulness is a more fugitive quality, more associated with conversation than the cold print of a mathematical journal. However, [22] preserves one instance of the highly resonant timbre to Helge Tverberg's cast of mind. The number 23 sticks in the memory of many mathematicians since it is the smallest for which, given a random group of people of that size, the probability that two share a birthday exceeds the probability that all the birthdays fall on different days of the year—it seems that most people suppose that a larger group would be needed. But Tverberg's first thought on learning, during a visit to Australia in 1988, that until that year there had been 23 Prime Ministers since Federation, was to wonder about their birthdays. By the time he got around to looking them up, on another visit in 1995, Australia had a new Prime Minister. It was not a surprise that no two of the first 23 shared a birthday, but what piqued Tverberg's amusement was that the new Prime Minister's birthday was the same as that of the first Prime Minister, born 95 years earlier. We owe it to Joe Gani, who was present as Tverberg casually remarked on this diverting discovery, that it was reported locally at the time. But it deserves wider retelling.

The working life of most scientists is scattered with loose ends never tied together in publications. Helge Tverberg is no exception, but his *obiter dicta* have a memorable charm. Among his many sidelines is combinatorial probability theory: this is reflected in [21]; and he has also tried his hand with the celebrated *Monty Hall* problem of the Cadillac and the goats, in [23] (which includes the Australian Prime Ministers for good measure). So, when asked for a Christmas brain teaser for the local student newspaper in 1998, he aired anew this famous dilemma of whether to switch or not, when Monty opens another box to reveal a goat. Even Erdős, despite pioneering contributions in probabilistic graph theory and probabilistic number theory, was flummoxed by this old chestnut, as recounted in both recent biographies [9,17]. It was only to be expected then that it would once again provoke discussion, and, indeed, two of the professors of linguistics in Bergen got into an argument about it. There are, of course, several ways to see that switching is the way to go: you might imagine two players, one always staying with their first choice, the other always switching, as in [13]; or, again, as in one of the biographies [17], you might consider, say, one million boxes, with Monty opening up all but one other box than that first chosen to reveal 999,998 goats. One of the professors of linguistics had yet another twist: what would happen if there were a large number of boxes opened *sequentially* with the offer of a new choice each time? Tverberg was delighted to find, on working through the recurrence relation for the probability $p(k)$ that the player picks the box with the Cadillac at their kth choice, that $p(n-1)$ was a rather familiar quantity, namely the probability in the even older derangement, or *jeu de rencontre*, problem of, say, the hats, of at least one of n people ending up with the right hat. Or, as Tverberg summed it up, "everyone getting the wrong hat and getting a Cadillac are complementary events".

As is well known, one approach to the derangement problem is *via* the *inclusion–exclusion* principle, which goes back at least to the beginning of the 18th century, in

work of Montmort and of Bernoulli (see [11]). Tverberg was recently teaching a course which covered this principle, but where some of the students, although not all, might well have met it before. In order to interest everyone, he thought to try an experiment, inviting the students to consider what happens as you transfer elements from one set to another, until one set contains all the others. It turns out that, if you do this, the two sides of the inclusion–exclusion *identity* are conserved separately at each step, until finally you reach a trivial equality between the two sides. So, the identity is proved. While this was novel for Tverberg, he later found, in his characteristically thorough way, that Zeilberger [26] had earlier drawn attention to a somewhat similar, if perhaps not quite so simple, argument, and that, moreover, a whole thesis [2] had only lately been devoted to *improved* inclusion–exclusion identities.

Several of Tverberg's colleagues and former students have pressed the case to say a bit more at this point of Tverberg as a teacher. As this example shows, he has a gift for extemporaneous innovation in the classroom. However, it is notoriously difficult to catch such fleeting moments on paper. Tverberg recasts, in [25], the choice Lomas presented to Hardy more amusingly in terms of local statues in Bergen, and he often observes that certainly the sea gulls seem to prefer that of Christian Michelsen. Continuing in this vein, he asks what would happen if a statue of a gull were placed on top of the figure of Michelsen. Clearly, this is the start of a monumental induction that is guaranteed to hold the attention of students in teetering expectation. There is a similar lively awareness of just what students are most likely to remember in a story Tverberg tells against himself. Once, on an autumnal afternoon, when attention might have been flagging, he broke off his exposition to describe his method of catching the flies that tend to seek sanctuary in the warmth of Norwegian homes at that season—you come up behind them with the nozzle of the vacuum cleaner, and just suck them in, which neatly avoids breaking things in the vain attempt to swat them. And he added that, whatever the class might remember of the course, they would be sure to recall this lesson in fly catching. Years later, someone vaguely familiar greeted Tverberg while out skiing, and it turned out that they had been in his class. This erstwhile student confessed regretfully that he could not claim to remember much about the course, but he still recalled that afternoon Tverberg had told them how to catch flies.... But a student attracted to mathematics might well reflect afterwards that theorems too can be caught by a mind alive to the potential of such stratagems.

Probably the best known, certainly one of the most frequently cited, of all Helge Tverberg's *publications* is another two-page gem [20], from 1982. Ron Graham and Henry Pollak had proved, in [6] in 1971, that, if the complete graph is decomposed into edge-disjoint complete bipartitite subgraphs, then the number of these subgraphs is at least the number of vertices less one; and this is best possible, with the obvious exception of when there is only a single vertex—in a subsequent paper [7], Graham and Pollak pointed out that this theorem had been established earlier in a somewhat different form by a colleague at Bell Labs, Hans Witsenhausen, who suggested the approach using eigenvalues which they implemented. This is clearly an attractive combinatorial result, and it has generated great interest—but it is also a worrisome one, in that no

strictly combinatorial proof has been given, and all proofs to date draw instead on ideas from linear algebra. Of these, Tverberg's proof, in [20], has long been about the most popular, so it was entirely fitting that Martin Aigner and Günter Ziegler based their exposition in [1] on it, commenting that "[it] may be the most transparent".

Helge Tverberg recalls in [25] how he came to work on generalizations of Radon's theorem. Johann Radon had shown, in his now classic paper [14] of 1921, that any set S of $d + 2$ points in R^d can be partioned into two disjoint subsets the intersection of whose closed convex hulls is, however, non-empty, a result at the heart of convexity theory, as Jürgen Eckhoff sets out in his magisterial survey [4]. This topic has been a central theme too in Tverberg's own research, to which he has returned in some half dozen papers. The most recent of these [24] represents perhaps the apogee and consummation of his explorations in this area. Possibly in self-recognition of this, Tverberg lingers attentively over the history of the subject and the contributions of others, and is at special pains to make his own line of reasoning almost as though anyone might have thought of it for themselves. But a perceptive, anonymous referee shrewdly recognized the merit of Tverberg's work through his transparency.

> The author describes a beautiful, simple but clever, proof of a conjecture of Reay [in [15]] that asserts that any set of $2(k - 1)d + 2$ points in R^d can be partitioned into k parts so that the intersection of the convex hulls of all of them is at least one-dimensional, that is, contains a segment of positive length. The proof follows from the author's well known generalization of Radon's theorem, and despite the attempts of the author to describe it here in an extremely modest way, explaining how the proof is suggested naturally by the form of the number $2(k - 1)d + 2 = 2((k - 1)d + 1)$, I think this is an extremely clever, short proof, which certainly deserves publication.

How many of us are ever likely to have such a delightful report, from a referee who both knows our work and appreciates our character so well? That seems to be the ultimate accolade for any author.

Acknowledgements

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