A multiplicity theorem for problems with the $p$-Laplacian

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Received 13 July 2005; accepted 11 November 2006

Communicated by L. Gross

Abstract

We consider a nonlinear elliptic problem driven by the $p$-Laplacian, with a parameter $\lambda \in \mathbb{R}$ and a nonlinearity exhibiting a superlinear behavior both at zero and at infinity. We show that if the parameter $\lambda$ is bigger than $\lambda_2 = \lambda_{2,p}(Z)$, then the problem has at least three nontrivial solutions. Our approach combines the method of upper–lower solutions with variational techniques involving the Second Deformation Theorem. The multiplicity result that we prove extends an earlier semilinear (i.e. $p = 2$) result due to Struwe [M. Struwe, Variational Methods, Springer-Verlag, Berlin, 1990].

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Keywords: Multiple nontrivial solutions; Superlinear nonlinearity; Upper and lower solutions; Eigenvalues of the $p$-Laplacian; Second deformation theorem

1. Introduction

In this paper we prove a multiplicity theorem for nonlinear elliptic problems driven by the $p$-Laplacian. So suppose $Z \subseteq \mathbb{R}^N$ is a bounded domain with a $C^2$-boundary $\partial Z$. The problem under consideration is the following:

\[
\begin{aligned}
- \text{div}(\|Dx(z)\|^{p-2}Dx(z)) &= \lambda \|x(z)\|^{p-2}x(z) - f(z, x(z)) & \text{a.e. on } Z \\
x|_{\partial Z} &= 0, & \lambda \in \mathbb{R}, \ 1 < p < \infty.
\end{aligned}
\]

(1.1)
We are interested in multiplicity results when the nonlinearity \( f(z, x) \) exhibits a “super-linear” behavior both at zero and at \( \pm \infty \). In the past this problem was investigated in the case \( p = 2 \) (semilinear case). First, Ambrosetti, Mancini [2] proved that if \( \lambda > \lambda_1 \) (\( \lambda_1 \) being the principal eigenvalue of \( (-\Delta_p, W_0^{1,p}(Z)) \)), then the problem has two nontrivial solutions of constant sign (one positive and the other negative). Soon thereafter Struwe [11] improved the result and proved that if \( \lambda > \lambda_2 \) the problem (1.1) has three nontrivial solutions. Subsequently Ambrosetti, Lupo [1] slightly improved the work of Struwe [11] and also presented an approach based on Morse theory. This, of course, required that the nonlinearity \( f(z, \cdot) \) is \( C^1 \). The most general result for the semilinear case can be found in Struwe [12, p. 132], who succeeded in eliminating the differentiability condition on the nonlinearity \( f \) and simplified the argument of Ambrosetti, Lupo [1]. We remark, however, that still Struwe [12] requires that the nonlinearity \( f \) (which he assumes it to be independent of \( z \)), is Lipschitz continuous. When \( p \neq 2 \) (nonlinear problem), we are not aware of any such multiplicity results for problem (1.1). Here we present such a generalization of the result of Struwe [12].

2. Preliminaries

First let us briefly recall some basic facts about the spectrum of \( (-\Delta_p, W_0^{1,p}(Z)) \). So we consider the following nonlinear eigenvalue problem:

\[
\begin{cases}
-\text{div}\left(\|Dx(z)\|^{p-2}Dx(z)\right) = \lambda |x(z)|^{p-2}x(z) & \text{a.e. on } Z \\
x|_{\partial Z} = 0, \quad \lambda \in \mathbb{R}, \quad 1 < p < \infty.
\end{cases}
\]  

The least real number \( \lambda \) for which problem (2.1) has a nontrivial solution is called the first eigenvalue of \( (-\Delta_p, W_0^{1,p}(Z)) \) and it is denoted by \( \lambda_1 \). The first eigenvalue \( \lambda_1 \) is positive, isolated and simple (i.e. the corresponding eigenspace is one-dimensional). There is a variational characterization of \( \lambda_1 > 0 \), via the Rayleigh quotient, i.e.

\[
\lambda_1 = \min \left\{ \frac{\|Dx\|_p^p}{\|x\|_p^p} : x \in W_0^{1,p}(Z), \, x \neq 0 \right\}.
\]  

This minimum is realized at the normalized principal eigenfunction \( u_1 \). Note that if \( u_1 \) minimizes the Rayleigh quotient, then so does \( |u_1| \) and so it follows that \( u_1 \) does not change sign on \( Z \). Thus we may assume that \( u_1 \geq 0 \). Moreover, from the nonlinear regularity theory (see Lieberman [10]), we know that \( u_1 \in C^1_0(\bar{Z}) \). In addition, via the nonlinear strict maximum principle of Vazquez [13], we have that \( u_1(z) > 0 \) for all \( z \in Z \) and \( \frac{\partial u_1}{\partial n}(z) < 0 \) for all \( z \in \partial Z \). If we consider the ordered Banach space \( C^1_0(\bar{Z}) = \{ x \in C^1(\bar{Z}) : x(z) = 0 \text{ for all } z \in \partial Z \} \) with positive cone

\[
C^1_0(\bar{Z})_+ = \{ x \in C^1_0(\bar{Z}) : x(z) \geq 0 \text{ for all } z \in \bar{Z} \},
\]

we know that \( \text{int } C^1_0(\bar{Z})_+ \neq \emptyset \) and is given by

\[
\text{int } C^1_0(\bar{Z})_+ = \left\{ x \in C^1_0(\bar{Z})_+ : x(z) > 0 \text{ for all } z \in Z \text{ and } \frac{\partial x}{\partial n}(z) < 0 \text{ for all } z \in \partial Z \right\}.
\]

Therefore \( u_1 \in \text{int } C^1_0(\bar{Z})_+ \).
The Lusternik–Schnirelmann theory, in addition to \( \lambda_1 > 0 \), gives us a whole strictly increasing sequence \( \{\lambda_n\}_{n \geq 1} \) of eigenvalues such that \( \lambda_n \to +\infty \). These are the so-called “Lusternik–Schnirelmann or variational eigenvalues” of \((-\Delta_p, W^{1,p}_0(Z))\). If \( p = 2 \) (linear eigenvalue problem), then these are all the eigenvalues. If \( p \neq 2 \) (nonlinear eigenvalue problem), we do not know if this is the case. However, we can say the following. Since \( \lambda_1 > 0 \) is isolated we can define

\[
\lambda_2^* = \inf\{\lambda: \lambda \text{ is an eigenvalue of } (-\Delta_p, W^{1,p}_0(Z)), \lambda > \lambda_1\} > \lambda_1.
\]

Anane, Tsouli [3] proved that \( \lambda_2^* = \lambda_2 \), i.e. the second eigenvalue and the second variational eigenvalue coincide. Therefore the Lusternik–Schnirelmann theory provides a variational characterization of the second eigenvalue \( \lambda_2 \). An alternative variational characterization of \( \lambda_2 \) was produced by Cuesta, de Figueiredo, Gossez [5] as a byproduct of their study of the first nontrivial curve of the Fučik spectrum of \((-\Delta_p, W^{1,p}_0(Z))\). They proved that

\[
\lambda_2 = \inf_{\gamma_0 \in \Gamma_0} \max_{u \in \gamma_0([-1,1])} \|Du\|_p^p,
\]

where \( \Gamma_0 = \{\gamma_0 \in C([-1,1], S): \gamma_0(-1) = -u_1, \gamma_0(1) = u_1\}, S = W^{1,p}_0(Z) \cap \partial B^{1,p}_1(Z) \) and \( \partial B^{1,p}_1(Z) = \{u \in L^p(Z): \|u\|_p = 1\} \).

Our analysis of problem (1.1) will use the so-called Second Deformation Theorem. For easy reference we recall the result here. Details can be found in Chang [4, p. 23] and Papageorgiou, Papageorgiou [7, p. 366] and [8, p. 617]. First let us introduce some notation. Suppose \( X \) is a Banach space and \( \varphi \in C^1(X) \). For every \( c \in \mathbb{R} \), we set

\[
\varphi^c = \{x \in X: \varphi(x) \leq c\} = \varphi^{-1}((-\infty, c]),
\]

\[
K = \{x \in X: \varphi'(x) = \nabla \varphi(x) = 0\} \quad \text{(the set of critical points of } \varphi) \quad \text{and}
\]

\[
K_c = \{x \in K_c: \varphi(x) = c\} \quad \text{(the set of critical points of } \varphi \text{ with energy level } c).\]

In what follows by \( X^* \) we denote the topological dual of \( X \) and by \( \langle \cdot, \cdot \rangle \) the duality brackets for the pair \((X, X^*)\). Recall that \( \varphi \) is said to satisfy the “Palais–Smale condition at level \( c \in \mathbb{R} \)” (PS\(_d\)-condition for short), if any sequence \( \{x_n\}_{n \geq 1} \subseteq X \) such that

\[
\varphi(x_n) \to c \quad \text{and} \quad \varphi'(x_n) \to 0 \quad \text{in } X^* \quad \text{as } n \to \infty,
\]

has a strongly convergent subsequence.

The theorem that follows is known in the literature as the Second Deformation Theorem. In it we allow \( b = +\infty \), in which case \( \varphi^b \setminus K_b = X \).

**Theorem 2.1.** If \( \varphi \in C^1(X), a \in \mathbb{R}, a < b \leq +\infty \), \( \varphi \) satisfies the PS\(_d\)-condition for every \( c \in [a, b] \), \( \varphi \) has no critical values in \((a, b)\) and \( \varphi^{-1}(a) \) contains at most a finite number of critical points of \( \varphi \), then there exists a \( \varphi \)-decreasing homotopy \( h: [0, 1] \times (\varphi^b \setminus K_b) \to \varphi^b \) such that

\[
h(1, \varphi^b \setminus K_b) \subseteq \varphi^a, \quad h(0, \cdot) = id|_{\varphi^b \setminus K_b}
\]

and \( h(t, x) = x \) for all \((t, x) \in [0, 1] \times \varphi^a\).
Remark 2.2. The $\varphi$-decreasing property of the homotopy $h(t, x)$ means that

$$\varphi(h(t, x)) \leq \varphi(h(s, x))$$

for all $t, s \in [0, 1]$, $s \leq t$ and all $x \in X$.

The conclusion of Theorem 2.1 says that $\varphi^a$ is a strong deformation retract of $\varphi^b \setminus K_b$.

3. Multiplicity theorem

The hypotheses on the nonlinearity $f(z, x)$ are the following.

$H(f)$: $f : Z \times \mathbb{R} \to \mathbb{R}$ is a function such that $f(z, 0) = 0$ a.e. on $Z$ and

(i) for all $x \in \mathbb{R}$, $z \to f(z, x)$ is measurable;

(ii) for almost all $z \in Z$, $x \to f(z, x)$ is continuous and $f(z, x)x \geq 0$;

(iii) for almost all $z \in Z$ and all $x \in \mathbb{R}$, we have

$$|f(z, x)| \leq a(z) + c|x|^{r-1}$$

with $a \in L^\infty(Z)_+, \ c > 0, \ p < r < p^*$;

(iv) $\lim_{x \to 0} \frac{f(z, x)}{|x|^{p-2}x} = 0$ uniformly for almost all $z \in Z$;

(v) $\lim_{|x| \to +\infty} \frac{f(z, x)}{|x|^{p-2}x} = +\infty$ uniformly for almost all $z \in Z$.

Remark 3.1. When $p = 2$ hypotheses $H(f)$(iv) and (v) imply that the nonlinearity is superlinear both at zero and at infinity.

Now by virtue of hypotheses $H(f)$(iii) and (v), given $\mu > 0$, we can find $\beta_\mu \in L^\infty(Z)_+, \ \beta_\mu \neq 0$, such that

$$f(z, x) > \mu|x|^{p-2}x - \beta_\mu(z) \quad \text{for a.a.} \ z \in Z \ \text{and all} \ x \geq 0 \quad \text{and} \quad (3.1)$$

$$f(z, x) < \mu|x|^{p-2}x + \beta_\mu(z) \quad \text{for a.a.} \ z \in Z \ \text{and all} \ x \leq 0. \quad (3.2)$$

We consider the following auxiliary problem:

$$\begin{cases}
-\text{div}(\|Dx(z)\|^{p-2}Dx(z)) = (\lambda - \mu)|x(z)|^{p-2}x(z) + \beta_\mu(z) \quad \text{a.e. on} \ Z \\
x|_{\partial Z} = 0.
\end{cases} \quad (3.3)$$

Proposition 3.2. If $\lambda > \lambda_1$ and we choose $\mu > \lambda - \lambda_1$, then problem (3.3) has a solution $\bar{x} \in \text{int} \ C^1_0(Z)_+$.

Proof. Consider the operator $A : W^{1,p}_0(Z) \to W^{-1,p'}(Z)$ ($1/p + 1/p' = 1$) defined by

$$\langle A(x), y \rangle = \int_Z \|Dx(z)\|^{p-2}(Dx(z), Dy(z))_{\mathbb{R}^N} \, dz$$

for all $x, y \in W^{1,p}_0(Z)$. 

Hereafter by $\langle \cdot, \cdot \rangle$ we denote the duality brackets for all the pair $(W_0^{1,p}(Z), W^{-1,p'}(Z))$. It is easy to check that $A$ is bounded, demicontinuous, monotone, hence it is maximal monotone. Also let $K : W_0^{1,p}(Z) \to L^{p'}(Z)$ be defined by

$$K(x)(\cdot) = (\lambda - \mu)|x(\cdot)|^{p-2}x(\cdot).$$

Exploiting the compact embedding of $W_0^{1,p}(Z)$ into $L^{p}(Z)$, we see that $K$ is completely continuous. Therefore $A - K : W_0^{1,p}(Z) \to W^{-1,p'}(Z)$ is pseudomonotone. Also

$$\langle A(x) - K(x), x \rangle = \|Dx\|_p^p - (\lambda - \mu)\|x\|_p^p. \quad (3.4)$$

If $\lambda \leq \mu$, then $A - K$ is coercive.

If $\lambda > \mu$, then by hypothesis $\lambda - \mu = \lambda_1 - \varepsilon > 0$ for some $\varepsilon > 0$ and so

$$\langle A(x) - K(x), x \rangle \geq \|Dx\|_p^p - \frac{\lambda_1 - \varepsilon}{\lambda_1}\|Dx\|_p^p$$

$$= \frac{\varepsilon}{\lambda_1}\|Dx\|_p^p \quad \text{(see (2.2) and (3.4))},$$

$$\Rightarrow A - K \text{ is coercive.}$$

Therefore $A - K$ is coercive. But a pseudomonotone, coercive operator is surjective. So we can find $\bar{x} \in W_0^{1,p}(Z)$ such that

$$A(\bar{x}) - K(\bar{x}) = \beta\mu. \quad (3.5)$$

This implies that $\bar{x} \in W_0^{1,p}(Z)$ solves problem (3.3). Since $\beta\mu \neq 0$, from (3.5) it follows that $\bar{x} \neq 0$. From the nonlinear regularity theory we have that $\bar{x} \in C^1_0(\bar{Z})$. In (3.5) above we act with the test function $-\bar{x}^- \in W_0^{1,p}(Z)$ and obtain

$$\|D\bar{x}^-\|_p^p \leq (\lambda - \mu)\|\bar{x}^-\|_p^p \quad \text{(recall $\beta\mu \geq 0$).}$$

Again, if $\lambda - \mu \leq 0$, then by Poincare’s inequality, we have $\bar{x}^- = 0$, i.e. $\bar{x} \geq 0$.

If $\lambda > \mu$, then $\lambda - \mu = \lambda_1 - \varepsilon$ for some $\varepsilon > 0$ and so

$$\|D\bar{x}^-\|_p^p \leq (\lambda_1 - \varepsilon)\|\bar{x}^-\|_p^p,$$

a contradiction to (2.2), unless $\bar{x}^- = 0$, i.e. $\bar{x} \geq 0$. So we have proved that $\bar{x} \geq 0$, $\bar{x} \neq 0$. Then from (3.3) it follows that

$$\text{div}(\|D\bar{x}(z)\|^{p-2}D\bar{x}(z)) \leq |\lambda - \mu|\|\bar{x}(z)\|^{p-2}\bar{x}(z) \quad \text{a.e. on } Z. \quad (3.6)$$

Invoking the nonlinear strict maximum principle of Vazquez [13], from (3.6) we infer that $\bar{x} \in \text{int } C^1_0(\bar{Z})_+$. □
**Definition 3.3.**

(a) A function \( \bar{x} \in W^{1,p}(Z) \) such that \( \bar{x}|_{\partial Z} \geq 0 \) and
\[
\int_Z \|D\bar{x}\|^{p-2}(D\bar{x}, Du)_{\mathbb{R}^N} \, dz \geq \lambda \int_Z |\bar{x}|^{p-2}\bar{x}u \, dz - \int_Z f(z, \bar{x})u \, dz
\]
for all \( u \in W^{1,p}_0(Z) \), \( u \geq 0 \), is an “upper solution” for problem (1.1).

(b) A function \( x \in W^{1,p}(Z) \) such that \( x|_{\partial Z} \leq 0 \) and
\[
\int_Z \|Dx\|^{p-2}(Dx, Du)_{\mathbb{R}^N} \, dz \leq \lambda \int_Z |x|^{p-2}x \, dz - \int_Z f(z, x)u \, dz
\]
for all \( u \in W^{1,p}_0(Z) \), \( u \geq 0 \), is a “lower solution” for problem (1.1).

**Corollary 3.4.** If \( \lambda > \lambda_1 \) and we choose \( \mu > \lambda - \lambda_1 > 0 \), then the solution \( \bar{x} \in \text{int} C^1_0(\bar{Z})_+ \) of problem (3.3) obtained in Proposition 3.2 is an upper solution for the problem (1.1).

**Proof.** This follows at once from (3.1).

Because of hypothesis \( H(f)(iv) \), given \( \varepsilon > 0 \), we can find \( \delta = \delta(\varepsilon) > 0 \) such that
\[
f(z, x) < \varepsilon |x|^{p-2}x \quad \text{for a.a. } z \in Z \text{ and all } x \in [0, \delta] \quad \text{and} \quad (3.7)
f(z, x) > \varepsilon |x|^{p-2}x \quad \text{for a.a. } z \in Z \text{ and all } x \in [-\delta, 0]. \quad (3.8)
\]

Recall that \( u_1 \in \text{int} C^1_0(\bar{Z})_+ \). We can find \( \xi > 0 \) such that \( \xi u_1(z) \in [0, \delta] \) for all \( z \in \bar{Z} \). To produce a lower solution \( x \) of problem (1.1), we will need the following simple fact about ordered Banach spaces.

**Lemma 3.5.** If \( X \) is an ordered Banach space with order cone \( K \) such that \( \text{int} K \neq \emptyset \) and \( x_0 \in \text{int} K \), then for every \( y \in X \), we can find \( \theta = \theta(y) > 0 \) such that \( \theta x_0 - y \in \text{int} K \).

**Proof.** Since \( x_0 \in \text{int} K \), we can find \( \delta > 0 \) such that
\[
\bar{B}_\delta(x_0) = \{ x \in X : \|x - x_0\| \leq \delta \} \subseteq \text{int} K.
\]
Consider \( y \in X \), \( y \neq 0 \) (if \( y = 0 \), then the result is trivially true for all \( \theta > 0 \)). Then
\[
x_0 \pm \delta \frac{y}{\|y\|} \in \bar{B}_\delta(x_0) \subseteq \text{int} K, \quad \Rightarrow \quad \frac{\|y\|}{\delta} x_0 - y \in \text{int} K.
\]
So if \( \theta = \theta(y) = \frac{\|y\|}{\delta} \), then \( \theta x_0 - y \in \text{int} K \). \( \square \)

Using this lemma, we can choose \( \xi > 0 \) small enough such that \( \xi u_1 \leq \bar{x} \). So we have
\[
\xi u_1(z) \in [0, \delta] \quad \text{for all } z \in \bar{Z} \text{ and } \xi u_1 \leq \bar{x}. \quad (3.9)
\]
We set \( \bar{x} = \xi u_1 \). Evidently \( \bar{x} \in \text{int} C^1_0(\bar{Z})_+ \) and \( \bar{x} \) (hence \( \bar{x} \)) too depends on \( \varepsilon > 0 \).
Proposition 3.6. If $\lambda > \lambda_1$, then $\bar{x} \in \text{int} C^1_0(\bar{Z})_+$ defined above is a lower solution for problem (1.1).

Proof. Fix $\varepsilon > 0$ such that $\lambda = \lambda_1 + \varepsilon$. Then we have

$$-\text{div}\left(\|D\varphi(z)\|^{p-2} D\varphi(z)\right) = \lambda_1 \|\varphi(z)\|^{p-2} \varphi(z) = (\lambda - \varepsilon) \|\varphi(z)\|^{p-2} \varphi(z)$$

$$< \lambda \|\varphi(z)\|^{p-2} \varphi(z) - f(z, \varphi(z)) \quad \text{a.e. on } Z \text{ (see (3.7) and (3.9))},$$

$$\Rightarrow \varphi \in \text{int} C^1_0(\bar{Z})_+ \text{ is a lower solution for problem (1.1)}. \qed$$

For the ordered pair $\{x, \bar{x}\}$, we consider the truncation map $\tau_+: Z \times \mathbb{R} \to \mathbb{R}$ defined by

$$\tau_+(z, x) = \begin{cases} \varphi(z) & \text{if } x \leq \varphi(z), \\ x & \text{if } \varphi(z) < x < \bar{x}(z), \\ \bar{x}(z) & \text{if } x \geq \bar{x}(z). \end{cases}$$

Clearly this is a Caratheodory function, i.e. it is measurable in $z \in Z$ and continuous in $x \in \mathbb{R}$, hence jointly measurable. We set

$$f_+(z, x) = f(z, \tau_+(z, x)), \quad F_+(z, x) = \int_0^x f_+(z, r) dr, \quad F(z, x) = \int_0^x f(z, r) dr,$$

$$\phi^\lambda_+(x) = \frac{1}{p} \|Dx\|_p^p - \frac{\lambda}{p} \|x\|_p^p + \int Z F_+(z, x(z)) dz \quad \text{for all } x \in W^{1,p}_0(Z) \quad \text{and}$$

$$\phi^\lambda(x) = \frac{1}{p} \|Dx\|_p^p - \frac{\lambda}{p} \|x\|_p^p + \int Z F(z, x(z)) dz \quad \text{for all } x \in W^{1,p}_0(Z).$$

We know that $\phi^\lambda_+, \phi^\lambda \in C^1(W^{1,p}_0(Z))$ and the critical points of $\phi^\lambda$ are solutions of problem (1.1). Also we introduce the order interval

$$E_+ = [\underline{x}, \bar{x}] = \{x \in W^{1,p}_0(Z): \underline{x}(z) \leq x(z) \leq \bar{x}(z) \text{ a.e. on } Z\}.$$
\begin{align*}
\varphi^\lambda_+(x) &= \frac{1}{p} \|Dx\|^p_p - \frac{\lambda}{p} \|x\|^p_p + \int_Z F_+(z, x(z)) \, dz \\
& \geq \frac{1}{p} \|Dx\|^p_p - \frac{\lambda - \mu}{p} \|x\|^p_p - c_1 \|Dx\|_p \quad \text{for some } c_1 > 0 \text{ (see (3.10))} \\
& = \frac{1}{p} \left(1 - \frac{\lambda - \mu}{\lambda_1}\right) \|Dx\|^p_p - c_1 \|Dx\|_p \quad \text{(see (2.2))},
\end{align*}

\Rightarrow \varphi^\lambda_+ \text{ is coercive (recall } \lambda - \mu < \lambda_1).$

Also it is easy to see that \(\varphi^\lambda_+\) is weakly lower semicontinuous on \(W^{1,p}_0(Z)\). Therefore by the Weierstrass theorem, we can find \(x_0 \in E_+\) such that

\[ \varphi^\lambda_+(x_0) = \inf_{E^+_+} \varphi^\lambda_+. \]

For any \(y \in E_+\), we set \(\theta(t) = \varphi^\lambda_+(ty + (1-t)x_0)), t \in [0, 1].\) Then

\[ \theta(0) \leq \theta(t) \quad \text{for all } t \in [0, 1], \]

\[ \Rightarrow 0 \leq \theta'(0^+), \]

\[ \Rightarrow 0 \leq \int_Z |x_0|^{p^2 - 2} x_0(y - x_0) \, dz + \int_Z f_+(z, x_0(z))(y - x_0) \, dz \]

for all \(y \in E_+.\)  \hfill (3.11)

For any \(v \in W^{1,p}_0(Z)\) and any \(\varepsilon > 0,\) we set

\[ y(x) = \begin{cases} 
\bar{x}(z) & \text{if } z \in \{x_0 + \varepsilon v < \bar{x}\}, \\
x_0(z) + \varepsilon v(z) & \text{if } z \in \{\bar{x} < x_0 + \varepsilon v < \bar{x}\}, \\
x_0(z) & \text{if } z \in \{\bar{x} < x_0 + \varepsilon v\}. 
\end{cases} \]

Evidently \(y \in E_+.\) We use it as a test function in (3.11) and obtain

\[ 0 \leq \varepsilon \int_{\{\bar{x} < x_0 + \varepsilon v < \bar{x}\}} \|Dx_0\|^{p^2 - 2}(Dx_0, Dv)_{\mathbb{R}^N} \, dz - \lambda \varepsilon \int_{\{\bar{x} < x_0 + \varepsilon v < \bar{x}\}} |x_0|^{p^2 - 2} x_0 v \, dz \]

\[ + \varepsilon \int_{\{\bar{x} < x_0 + \varepsilon v < \bar{x}\}} f_+(z, x_0) v \, dz + \int_{\{x_0 + \varepsilon v \leq \bar{x}\}} \|Dx_0\|^{p^2 - 2}(Dx_0, D\bar{x} - Dx_0)_{\mathbb{R}^N} \, dz \]

\[ - \lambda \int_{\{x_0 + \varepsilon v \leq \bar{x}\}} |x_0|^{p^2 - 2} x_0(\bar{x} - x_0) \, dz + \int_{\{x_0 + \varepsilon v \leq \bar{x}\}} f_+(z, x_0)(\bar{x} - x_0) \, dz \]

\[ + \int_{\{x_0 + \varepsilon v \geq \bar{x}\}} \|Dx_0\|^{p^2 - 2}(Dx_0, D\bar{x} - Dx_0)_{\mathbb{R}^N} \, dz - \lambda \int_{\{x_0 + \varepsilon v \geq \bar{x}\}} |x_0|^{p^2 - 2} x_0(\bar{x} - x_0) \, dz \]

\[ + \int_{\{x_0 + \varepsilon v \geq \bar{x}\}} f_+(z, x_0)(\bar{x} - x_0) \, dz. \]
\[= \varepsilon \int_{Z} \|Dx_0\|^{p-2}(Dx_0, Dv)_{\mathbb{R}^N} \, dz - \lambda \varepsilon \int_{Z} |x_0|^{p-2}x_0 v \, dz\]

\[+ \varepsilon \int_{Z} f_+(z, x_0) v \, dz - \int_{\{x_0 + \varepsilon v \geq \bar{x}\}} \|D\bar{x}\|^{p-2}(D\bar{x}, D(x_0 + \varepsilon (v - \bar{x}))_{\mathbb{R}^N} \, dz\]

\[+ \lambda \int_{\{x_0 + \varepsilon v \geq \bar{x}\}} |\bar{x}|^{p-2}\bar{x}(x_0 + \varepsilon v - \bar{x}) \, dz - \int_{\{x_0 + \varepsilon v \geq \bar{x}\}} f_+(z, \bar{x})(x_0 + \varepsilon v - \bar{x}) \, dz\]

\[+ \varepsilon \int_{\{x_0 + \varepsilon v \leq \bar{x}\}} f_+(z, x_0) v \, dz - \int_{\{x_0 + \varepsilon v \leq \bar{x}\}} \|D\bar{x}\|^{p-2}(D\bar{x}, D(x_0 - \varepsilon v))_{\mathbb{R}^N} \, dz\]

\[+ \lambda \int_{\{x_0 + \varepsilon v \leq \bar{x}\}} |x_0|^{p-2}x_0(x_0 - \varepsilon v) \, dz + \int_{\{x_0 + \varepsilon v \leq \bar{x}\}} f_+(z, \bar{x})(x_0 - \varepsilon v) \, dz\]

\[- \lambda \int_{\{x_0 + \varepsilon v \leq \bar{x}\}} (f_+(z, x_0) - f_+(z, \bar{x}))(x_0 + \varepsilon v - \bar{x}) \, dz\]

\[- \lambda \int_{\{x_0 + \varepsilon v \geq \bar{x}\}} (f_+(z, \bar{x}) - f_+(z, x_0))(\bar{x} - x_0 - \varepsilon v) \, dz\]

\[+ \int_{\{x_0 + \varepsilon v \geq \bar{x}\}} (\|D\bar{x}\|^{p-2}D\bar{x} - \|Dx_0\|^{p-2}Dx_0, D(x_0 - \bar{x}))_{\mathbb{R}^N} \, dz\]

\[- \lambda \int_{\{x_0 + \varepsilon v \geq \bar{x}\}} (|\bar{x}|^{p-2}\bar{x} - |x_0|^{p-2}x_0)(x_0 - \bar{x}) \, dz\]

\[- \int_{\{x_0 + \varepsilon v \leq \bar{x}\}} (\|D\bar{x}\|^{p-2}D\bar{x} - \|Dx_0\|^{p-2}Dx_0, D(\bar{x} - x_0))_{\mathbb{R}^N} \, dz\]

\[+ \lambda \int_{\{x_0 + \varepsilon v \leq \bar{x}\}} (|\bar{x}|^{p-2}\bar{x} - |x_0|^{p-2}x_0)(\bar{x} - x_0) \, dz\]

\[+ \varepsilon \int_{\{x_0 + \varepsilon v \geq \bar{x}\}} (\|D\bar{x}\|^{p-2}D\bar{x} - \|Dx_0\|^{p-2}Dx_0, Dv)_{\mathbb{R}^N} \, dz\]

\[+ \varepsilon \int_{\{x_0 + \varepsilon v \leq \bar{x}\}} (\|D\bar{x}\|^{p-2}D\bar{x} - \|Dx_0\|^{p-2}Dx_0, Dv)_{\mathbb{R}^N} \, dz\]

\[- \lambda \varepsilon \int_{\{x_0 + \varepsilon v \geq \bar{x}\}} (|\bar{x}|^{p-2}\bar{x} - |x_0|^{p-2}x_0) v \, dz\]

\[- \lambda \varepsilon \int_{\{x_0 + \varepsilon v \leq \bar{x}\}} (|\bar{x}|^{p-2}\bar{x} - |x_0|^{p-2}x_0) v \, dz.\]  

(3.12)
If \( u = (x_0 + \varepsilon v - \bar{x})^+ \in W^{1,p}_0(Z)_+ \), then because \( \bar{x} \) is an upper solution of (1.1), we have

\[
- \int_{\{x_0 + \varepsilon v \geq \bar{x}\}} \| D\bar{x} \|^p \int_{\{x_0 + \varepsilon v \geq \bar{x}\}} (D\bar{x}, D(x_0 + \varepsilon v - \bar{x}))_{\mathbb{R}^N} \, dz + \lambda \int_{\{x_0 + \varepsilon v \geq \bar{x}\}} |\bar{x}|^{p-2}\bar{x}(x_0 + \varepsilon v - \bar{x}) \, dz \\
- \int_{\{x_0 + \varepsilon v \geq \bar{x}\}} f_+(z, \bar{x})(x_0 + \varepsilon v - \bar{x}) \, dz \leq 0. \tag{3.13}
\]

Similarly if we let \( u = (\bar{x} - x_0 - \varepsilon v)^+ \in W^{1,p}_0(Z)_+ \), then because \( \bar{x} \) is a lower solution of (1.1), we have

\[
\int_{\{x_0 + \varepsilon v \leq \bar{x}\}} \| D\bar{x} \|^p \int_{\{x_0 + \varepsilon v \leq \bar{x}\}} (D\bar{x}, D(x_0 - \varepsilon v))_{\mathbb{R}^N} \, dz - \lambda \int_{\{x_0 + \varepsilon v \leq \bar{x}\}} |\bar{x}|^{p-2}\bar{x}(x_0 - \varepsilon v) \, dz \\
+ \int_{\{x_0 + \varepsilon v \leq \bar{x}\}} f_+(z, \bar{x})(x_0 - \varepsilon v) \, dz \leq 0. \tag{3.14}
\]

The map \( \psi_p : \mathbb{R}^N \rightarrow \mathbb{R}^N \) defined by

\[
\psi_p(y) = \begin{cases} 
\| y \|^{p-2} y & \text{if } y \neq 0, \\
0 & \text{if } y = 0,
\end{cases}
\]

is a strictly monotone homeomorphism. So

\[
\int_{\{x_0 + \varepsilon v \geq \bar{x}\}} (\| D\bar{x} \|^p \int_{\{x_0 + \varepsilon v \geq \bar{x}\}} D\bar{x} - \| Dx_0 \|^p \int_{\{x_0 + \varepsilon v \geq \bar{x}\}} Dx_0 - D\bar{x})_{\mathbb{R}^N} \, dz \leq 0 \tag{3.15}
\]

and

\[
- \int_{\{x_0 + \varepsilon v \leq \bar{x}\}} (\| D\bar{x} \|^p \int_{\{x_0 + \varepsilon v \leq \bar{x}\}} D\bar{x} - \| Dx_0 \|^p \int_{\{x_0 + \varepsilon v \leq \bar{x}\}} Dx_0 - D\bar{x})_{\mathbb{R}^N} \, dz \leq 0. \tag{3.16}
\]

In addition since \( x_0 \leq \bar{x} \) and \( \bar{x} - x_0 \leq \varepsilon v \) on \( \{x_0 + \varepsilon v \geq \bar{x}\} \), we have

\[
-\lambda \int_{\{x_0 + \varepsilon v \geq \bar{x}\}} (|\bar{x}|^{p-2}\bar{x} - |x_0|^{p-2}x_0)(x_0 - \bar{x}) \, dz \leq \lambda \varepsilon \int_{\{x_0 + \varepsilon v \geq \bar{x}\}} (|\bar{x}|^{p-2}\bar{x} - |x_0|^{p-2}x_0) \, dz. \tag{3.17}
\]

Similarly because \( \bar{x} \leq x_0 \) and \( \bar{x} - x_0 \geq \varepsilon v \) on \( \{x_0 + \varepsilon v \leq \bar{x}\} \), we have

\[
\lambda \int_{\{x_0 + \varepsilon v \leq \bar{x}\}} (|\bar{x}|^{p-2}\bar{x} - |x_0|^{p-2}x_0)(x_0 - \bar{x}) \, dz \leq \lambda \varepsilon \int_{\{x_0 + \varepsilon v \leq \bar{x}\}} (|\bar{x}|^{p-2}\bar{x} - |x_0|^{p-2}x_0) \, dz. \tag{3.18}
\]
Moreover, because $\bar{x}, \tilde{x} \in C^1_0(\bar{Z})_+$, we have

\[
- \int_{\{x_0 + \varepsilon v \leq \bar{x}\}} \left( f_+(z, x_0) - f_+(z, \bar{x}) \right) (\bar{x} - x_0 - \varepsilon v) \, dz = \int_{\{x_0 + \varepsilon v \leq \bar{x}\}} \left( f_+(z, x_0) - f_+(z, \bar{x}) \right) (\bar{x} - x_0 - \varepsilon v) \, dz
\]

\[
\leq c_2 \varepsilon \int_{\{x_0 + \varepsilon v \leq \bar{x} < x_0\}} (-v) \, dz \text{ for some } c_2 > 0 \tag{3.19}
\]

(see hypothesis $H(f)$ (iii) and recall $\bar{x} \leq x_0$). Similarly, we obtain

\[
- \int_{\{x_0 + \varepsilon v \geq \tilde{x}\}} \left( f_+(z, \tilde{x}) - f_+(z, x_0) \right) (\tilde{x} - x_0 - \varepsilon v) \, dz
\]

\[
\leq c_3 \varepsilon \int_{\{x_0 + \varepsilon v \geq \tilde{x} > x_0\}} v \, dz \text{ for some } c_3 > 0 \tag{3.20}
\]

(see hypothesis $H(f)$ (iii) and recall $x_0 \leq \tilde{x}$).

We return to (3.12) and use (3.13) $\Rightarrow$ (3.20). So we get

\[
0 \leq \varepsilon \int_{Z} \|Dx_0\|^{p-2}(Dx_0, Dv) \rho_N \, dz - \lambda \varepsilon \int_{Z} |x_0|^{p-2}x_0v \, dz + \varepsilon \int_{Z} f_+(z, x_0)v \, dz
\]

\[
+ c_2 \varepsilon \int_{\{x_0 + \varepsilon v \leq \bar{x} < x_0\}} (-v) \, dz + c_3 \varepsilon \int_{\{x_0 + \varepsilon v \geq \tilde{x} > x_0\}} v \, dz.
\]

We divide with $\varepsilon > 0$ and let $\varepsilon \downarrow 0$. Note that if by $| \cdot |_N$ we denote the Lebesgue measure on $\mathbb{R}^N$, then

\[
|\{x_0 + \varepsilon v \leq \bar{x} < x_0\}|_N \downarrow 0 \text{ and } |\{x_0 + \varepsilon v \geq \tilde{x} > x_0\}|_N \downarrow 0 \text{ as } \varepsilon \downarrow 0.
\]

So in the limit as $\varepsilon \downarrow 0$, we have

\[
0 \leq \int_{Z} \|Dx_0\|^{p-2}(Dx_0, Dv) \rho_N \, dz - \lambda \int_{Z} |x_0|^{p-2}x_0v \, dz + \int_{Z} f_+(z, x_0)v \, dz. \tag{3.21}
\]

Since $v \in W_0^{1,p}(Z)$ was arbitrary, from (3.21) it follows that

\[
\left\{ \begin{array}{l}
- \text{div}(\|Dx_0(z)\|^{p-2}Dx_0(z)) - \lambda |x_0(z)|^{p-2}x_0(z) = f(z, x_0(z)) \text{ a.e. on } Z \\
x|_{\partial Z} = 0.
\end{array} \right. \tag{3.22}
\]

(note that $f_+(z, x_0(z)) = f(z, x_0(z))$).
Because of the nonlinear regularity theory, we have that \( x_0 \in C^1_0(\bar{Z}) \). From (3.1) and the comparison principles of Guedda, Veron [9], we have

\[
\bar{x} - x_0 \in \text{int } C^1_0(\bar{Z})_+.
\]

Similarly because of (3.7) and the results of Guedda, Veron [9], we have

\[
x_0 - x \in \text{int } C^1_0(\bar{Z})_+.
\]

From the definition of \( f_+ \) it follows that \( x_0 \) is a \( C^1_0(\bar{Z}) \)-local minimizer of \( \varphi_\lambda \). Invoking Theorem 1.1 of Garcia Azorero, Manfredi, Peral Alonso [6] we infer \( x_0 \) is a \( W^{1,p}_0(Z) \)-local minimizer of \( \varphi_\lambda \). □

From (3.22) above we see that \( x_0 \in \text{int } C^1_0(\bar{Z})_+ \) is a solution of problem (1.1).

We repeat a similar argument on the negative semiaxis of \( \mathbb{R} \). Because of (3.2), this time we consider the following auxiliary problem:

\[
\begin{align*}
-\text{div}(\|Dx(z)\|^{p-2}Dx(z)) &= (\lambda - \mu)|x(z)|^{p-2}x(z) - \beta \mu(z) \quad \text{a.e. on } Z \\
x|_{\partial Z} &= 0.
\end{align*}
\]

(3.23)

Solving (3.23) (with an argument as in the proof of Proposition 3.2), we obtain \( v \in -\text{int } C^1_0(\bar{Z})_+ \), which can be shown to be a lower solution for problem (1.1) (see (3.2)). Similarly as before by virtue of (3.8), with \( \xi > 0 \) as in the definition of \( \bar{x} \), we will have that \( \bar{v} = \xi(-u_1) \) is an upper solution for problem (1.1).

We introduce the Caratheodory truncation map \( \tau_-(z,x) \) with respect to the order pair \( \{v, \bar{v}\} \) and then using it we define \( f_-(z,x) = f(z, \tau_-(z,x)) \), \( F_-(z,x) \int_0^x f_-(z,r)dr \) and

\[
\varphi^{\lambda}_-(x) = \frac{1}{p}\|Dx\|^p_\lambda - \frac{\lambda}{p}\|x\|^p_\lambda + \int_Z F_-(z,x(z))dz \quad \text{for all } x \in W^{1,p}_0(Z).
\]

Working on the order interval \( E_- = [v, \bar{v}] = \{x \in W^{1,p}_0(Z) : v(z) \leq x(z) \leq \bar{v}(z) \text{ a.e. on } Z\} \) as in the proof of Proposition 3.7, we obtain the following result:

**Proposition 3.8.** If hypotheses \( H(f) \) hold and \( \lambda > \lambda_1 \), then there exists \( v_0 \in E_- \) which is a local minimizer of \( \varphi^{\lambda}_- \).

Evidently \( v_0 \in -\text{int } C^1_0(\bar{Z})_+ \) is a solution of problem (1.1). Therefore we have produced a second nontrivial solution for problem (1.1). Note that both solutions \( x_0, v_0 \in C^1_0(\bar{Z}) \) have constant sign. Next we will show that if \( \lambda > \lambda_2 \), then we can deduce the existence of a third nontrivial solution distinct from the other two.

**Theorem 3.9.** If hypotheses \( H(f) \) hold and \( \lambda > \lambda_2 \), then problem (1.1) has at least three distinct solutions \( x_0, v_0, y_0 \in C^1_0(\bar{Z}) \).

**Proof.** From Propositions 3.7 and 3.8, we have two nontrivial solution \( x_0, v_0 \in C^1_0(\bar{Z}) \) which are of constant sign and both are local minimizers of the Euler functional \( \varphi \). We assume that these are the only nontrivial critical points of \( \varphi^{\lambda}_- \) or otherwise we are done. We choose \( \delta > 0 \)
small such that
\[
\varphi^\lambda(v_0) < \inf \{ \varphi(x): x \in \partial B_\delta(v_0) \} \quad \text{and} \quad \varphi^\lambda(x_0) < \inf \{ \varphi(x): x \in \partial B_\delta(v_0) \}.
\]

Here for \( u \in W^{1,p}_0(Z) \), \( \partial B_\delta(u) = \{ x \in W^{1,p}_0(Z): \| x - u \| = \delta \} \). Let \( \varphi^\lambda(v_0) \leq \varphi^\lambda(x_0) \), \( V = \partial B_\delta(x_0) \), \( W_0 = \{ v_0, x_0 \} \) and \( W = \{ v_0, x_0 \} = \{ x \in W^{1,p}_0(Z): v_0(z) \leq x(z) \leq x_0(z) \text{ a.e. on } Z \} \). Then the pair \((W_0, W)\) links with \( V \) in \( W^{1,p}_0(Z) \) (see Gasinski, Papageorgiou [8, p. 631] and Struwe [12, p. 115]). Due to hypothesis \( H(f)(v) \), we can easily check that \( \varphi^\lambda \) is coercive from which it follows in a straightforward manner that \( \varphi^\lambda \) satisfies the PS\(_c\)-condition for all \( c \in \mathbb{R} \). So from the minimax theorem for linking sets (see Gasinski, Papageorgiou [8, p. 633]) we can find \( y_0 \in W^{1,p}_0(Z) \) such that \( \varphi'(y_0) = 0 \) (i.e. \( y_0 \) is a critical point of \( \varphi \)) and
\[
\varphi^\lambda(v_0), \varphi^\lambda(x_0) < \varphi^\lambda(y_0) = \inf_\gamma \max_{\gamma \in \Gamma} \varphi(\gamma(t)),
\]
where \( \Gamma = \{ \gamma \in C([-1, 1], W^{1,p}_0(Z)): \gamma(-1) = v_0, \gamma(1) = x_0 \} \).

We will generate an admissible path \( \gamma \in \Gamma \), such that \( \varphi|_{\gamma} < 0 \). So from (3.24) it follows that \( \varphi^\lambda(y_0) < 0 = \varphi^\lambda(0) \), i.e. \( y_0 \neq 0 \). Let \( S = W^{1,p}_0(Z) \cap \partial B^L_p(Z) \) and \( S_c = W^{1,p}_0(Z) \cap C^1_0(\bar{Z}) \cap \partial B^L_1(Z) \). Then \( S_c \) is dense in \( S \). Given \( \delta_0 > 0 \), because of (2.3), we can find \( y_0 \in \Gamma_0 = \{ y_0 \in C([-1, 1], S): y_0(-1) = -u_1 \text{ and } y_0(1) = u_1 \} \) such that \( y_0([-1, 1]) \subseteq S_c \) and
\[
\max \{ \| Du \|_p: u \in y_0([-1, 1]) \} \leq \lambda_2 + \delta_0.
\]
We choose \( \delta_0 > 0 \) so that
\[
\lambda_2 + \delta_0 < \lambda.
\]

Also because of hypothesis \( H(f)(iv) \) (see also (3.7) and (3.8)), given \( \varepsilon > 0 \) we can find \( \delta_0 = \delta(\varepsilon) > 0 \) such that
\[
F(z, x) \leq \frac{\varepsilon}{p} |x|^p \quad \text{for a.a. } z \in Z, \text{ all } |x| \leq \delta.
\]
(3.26)

Since \( y_0([-1, 1]) \subseteq S_c \) and \( -v_0, x_0 \in \text{int } C^1_0(\bar{Z})_+ \), we can choose \( \varepsilon > 0 \) small such that
\[
|\varepsilon u(z)| \leq \delta \quad \text{for all } z \in \bar{Z}, \text{ all } u \in y_0([-1, 1]), \quad \lambda_2 + \delta_0 + \varepsilon < \lambda
\]
and \( \varepsilon u \in W = \{ v_0, x_0 \} \) for all \( u \in y_0([-1, 1]) \).

Therefore, if \( u \in y_0([-1, 1]) \), we have
\[
\varphi^\lambda(\varepsilon u) = \frac{\varepsilon^p}{p} \| Du \|_p^p - \frac{\lambda \varepsilon^p}{p} \| u \|_p^p + \int_Z F(z, \varepsilon u(z)) \, dz
\leq \frac{\varepsilon^p}{p} (\lambda_2 + \delta_0) - \frac{\lambda \varepsilon^p}{p} + \frac{\varepsilon^{p+1}}{p} \quad \text{(see (3.25), (3.26) and recall that } \| u \|_p = 1)\]
\[
= \frac{\varepsilon^p}{p} (\lambda_2 + \delta_0 + \varepsilon - \lambda) < 0 \quad \text{(recall the choice of } \varepsilon > 0).\]
So, if we consider the path $\varepsilon \gamma_0$, this path joins $-\varepsilon u_1$ and $\varepsilon u_1$ and we have shown that

$$\varphi^\lambda|_{\varepsilon \gamma_0} < 0. \quad (3.27)$$

Next we will create another continuous path $\gamma_+: [0, 1] \to W^{1,p}_0(Z)$ which joins $\varepsilon u_1$ and $x_0$ and such that $\varphi^\lambda|_{\gamma_+} < 0$. To this end, we consider the truncation map

$$\hat{\tau}_+(x) = \begin{cases} x & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}$$

We set $\hat{f}_+(z, x) = f(z, \hat{\tau}_+(x))$, $\hat{F}_+(z, x) = \int_0^x \hat{f}_+(z, r) \, dz$ and

$$\hat{\varphi}_+(x) = \frac{1}{p} \|Dx\|_p^p - \frac{\lambda}{p} \|x^+\|_p^p + \int Z \hat{F}_+(z, x(z)) \, dz \quad \text{for all } x \in W^{1,p}_0(Z).$$

By virtue of hypothesis $H(f)(v)$, we see that $\hat{\varphi}_+$ is coercive, hence it satisfies the PS$_c$-condition for all $c \in \mathbb{R}$. Also from our hypothesis about the critical points of $\varphi^\lambda$, it follows that $\{0, x_0\}$ are the only critical points of $\hat{\varphi}_+$ and $\hat{\varphi}_+(x_0) = \inf_{W^{1,p}_0(Z)} \hat{\varphi}_+$. We set

$$\hat{b}_+ = \hat{\varphi}_+(\varepsilon u_1) \quad \text{and} \quad \hat{a}_+ = \hat{\varphi}_+(x_0) = \inf_{W^{1,p}_0(Z)} \hat{\varphi}_+.$$

Applying Theorem 2.1, we can find $h \in C([0, 1] \times [\hat{a}_+, \hat{b}_+])$ such that

$$h(t, x) = x \quad \text{for all } t \in [0, 1] \text{ and all } x \in [\hat{a}_+, \hat{b}_+],$$

$$h(0, x) = x \quad \text{for all } x \in \hat{a}_+, \quad \text{and}$$

$$h(1, x) = x_0 \quad \text{for all } x \in \hat{b}_+. $$

Then we consider the continuous path $\gamma_+(t) = h(t, \varepsilon u_1)$ for all $t \in [0, 1]$ which joins $\varepsilon u_1$ and $x_0$. We have

$$\varphi^\lambda(\gamma_+(t)) \leq \hat{\varphi}_+(\gamma_+(t)) \leq \hat{b}_+ = \hat{\varphi}_+(\varepsilon u_1) < 0 \quad \text{(see } (3.27)),$$

$$\Rightarrow \varphi^\lambda|_{\gamma_+} < 0. \quad (3.28)$$

In a similar fashion we produce a continuous path $\gamma_-: [0, 1] \to W^{1,p}_0(Z)$ joining $-\varepsilon u_1$ and $v_0$ such that

$$\varphi^\lambda|_{\gamma_-} < 0. \quad (3.29)$$

Concatenating the paths $\varepsilon \gamma_0$, $\gamma_+$, $\gamma_-$, we produce a path $\gamma \in \Gamma$ such that

$$\varphi^\lambda|_{\gamma} < 0 \quad \Rightarrow \varphi^\lambda(y_0) < 0 = \varphi^\lambda(0), \quad \text{i.e. } y_0 \neq 0.$$
Moreover, from the nonlinear regularity theory (see Lieberman [10]) it follows that $y_0 \in C^1_0(\bar{Z})$. □

**Remark 3.10.** Even in the semilinear case ($p = 2$) our theorem is more general than that of Struwe [12, p. 132], who assumes that the nonlinearity is Lipschitz continuous.

**Acknowledgment**

The authors wish to thank the referee for his constructive criticism and remarks.

**References**


