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Quasi-isometries and ends of groups

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Abstract

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We study quasi-isometries of groups. We show that the number of ends, the semistability of an end, and being simply-connected at ∞ is preserved by quasi-isometries.

0. Introduction

Two metric spaces (X, d_X) and (Y, d_Y) are called *quasi-isometric* if there are functions $f: X \to Y$ and $g: Y \to X$ and constants K and L so that for all $x, x_1, x_2 \in X$ and $y, y_1, y_2 \in Y$ the following holds:

$$d_Y(f(x_1), f(x_2)) \leq K \cdot d_X(x_1, x_2) + L \,, \tag{0.1}$$

$$d_X(g(y_1), g(y_2)) \leq K \cdot d_Y(y_1, y_2) + L \,, \tag{0.2}$$

$$d_X(x, g \circ f(x)) \leq K \,, \tag{0.3}$$

$$d_Y(y, f \circ g(y)) \leq K \,, \tag{0.4}$$

(see [1]). We will refer to the pair (f, g) as the quasi-isometry. Note that if L is non-zero then neither f nor g need be continuous.

Two finitely generated groups G and H are said to be *quasi-isometric* if there are finite generating sets S for G and T for H so that the associated Cayley graphs $\Gamma(G, S)$ and $\Gamma(H, T)$, endowed with the path metrics, are quasi-isometric. It turns out that this does not depend on the generating sets chosen.

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Quasi-isometries of groups have been studied in [1] and [3]. Group-theoretic properties that are invariant under a quasi-isometry are called *geometric*. Some geometric properties of groups are

(1) the property of being virtually nilpotent,

(2) the property of being virtually free,

(3) the property of being virtually abelian.

In this paper we show that the number of ends and the type of ends—whether they are semistable or simply-connected—are invariant under quasi-isometries (the definitions of a semistable or simply connected end are given below). The invariance of the number of ends is mentioned in [1] but is only proven for hyperbolic groups.

The layout of the paper is as follows. In Section 1 we interpret the definition of quasi-isometry for graphs. In Section 2 we turn to the number of ends. In Section 3 we consider semistability. In Section 4 we look at simple-connectivity at infinity.

1. Quasi-isometries of graphs

From this point on, assume, unless stated otherwise, that each graph is connected and is endowed with the path metric (where each edge has length 1). We will show that quasi-isometries of graphs are particularly nice.

Suppose Γ and Δ are graphs. We will say that they are *combinatorially quasi-isometric* if there is a constant N, PL maps $\alpha : \Gamma \to \Delta$ and $\beta : \Delta \to \Gamma$ with $\alpha(\Gamma^{(0)}) \subset \Delta^{(0)}$ and $\beta(\Delta^{(0)}) \subset \Gamma^{(0)}$ so that for all $a, a' \in \Gamma$ and $b, b' \in \Delta$ the following holds:

$$d_\Delta(\alpha(a), \alpha(a')) \leq N \cdot d_\Gamma(a, a') , \tag{1.1}$$

$$d_\Gamma(\beta(b), \beta(b')) \leq N \cdot d_\Delta(b, b') , \tag{1.2}$$

$$d_\Gamma(\alpha, \beta \circ \alpha(a)) \leq N , \tag{1.3}$$

$$d_\Delta(b, \alpha \circ \beta(b)) \leq N . \tag{1.4}$$

We will refer to the pair (α, β) as a combinatorial quasi-isometry between Γ and Δ.

The way we will obtain combinatorial quasi-isometries is by defining the maps on the zero-skeleta and extending. For this reason we will refer to a pair of maps (α_0, β_0) as a *vertex combinatorial quasi-isometry* between Γ and Δ if $\alpha_0 : \Gamma^{(0)} \to \Delta^{(0)}$, $\beta_0 : \Delta^{(0)} \to \Gamma^{(0)}$, and there is a constant M so that for all $v, v' \in \Gamma^{(0)}$ with v, v' adjacent and $u, u' \in \Delta^{(0)}$ with u, u' adjacent the following holds:

$$d_\Delta(\alpha_0(v), \alpha_0(v')) \le M , \tag{1.1'}$$

$$d_\Gamma(\beta_0(u), \beta_0(\mu')) \le M , \tag{1.2'}$$

$$d_\Gamma(v, \beta_0 \circ \alpha_0(v)) \le M , \tag{1.3'}$$

$$d_\Delta(u, \alpha_0 \circ \beta_0(u)) \le M . \tag{1.4'}$$

Note that if Γ and Δ are the Cayley graphs of a group G with respect to finite generating sets S and T, then $(\mathrm{id}_G, \mathrm{id}_G)$ is a vertex combinatorial quasi-isometry with constant

$$M = \max\{|t_i|_S, |s_i|_T \colon s_i \in S,\ t_i \in T\} .$$

Here $|*|_U$ denotes the length function with respect to the generating set U. Equivalently $|w|_U = d_{\Gamma(G,U)}(w,1)$.

Observe that any combinatorial quasi-isometry yields a vertex combinatorial quasi-isometry by restriction. The converse is also true.

Lemma 1.1. *If (α_0, β_0) is a vertex combinatorial quasi-isometry between Γ and Δ, then there are extensions α and β of α_0 and β_0, respectively, so that the pair (α, β) is a combinatorial quasi-isometry between Γ and Δ.*

Proof. Let M be a constant for which formulas (1.1')–(1.4') hold. If e is an (oriented) edge in Γ with initial vertex v_0 and terminal vertex v_1 define α on e by mapping it to some geodesic path from $\alpha_0(v_0)$ to $\alpha_0(v_1)$. Map e in a piecewise linear fashion. Note that the path need not be unique, but it is of length $\le M$.

Doing this for all edges defines α. Similarly define β. We need to verify formulas (1.1)–(1.4).

The triangle inequality clearly implies formulas (1.1) and (1.2), with constant M.

If $a \in \Gamma$, let $a_0 \in \Gamma$ be a vertex with $d_\Gamma(a, a_0) \le \frac{1}{2}$. Then by formula (1.1'), $d_\Delta(\alpha(a), \alpha(a_0)) \le \frac{1}{2}M$. Applying β and using formula (1.2') gives

$$d_\Gamma(\beta \circ \alpha(a), \beta \circ \alpha(a_0)) \le \tfrac{1}{2}M^2 .$$

Applying formula (1.3') and the triangle inequality yields

$$d_\Gamma(\beta \circ \alpha(a), a_0) \le \tfrac{1}{2}M^2 + M .$$

But $d_\Gamma(a_0, a) \le \frac{1}{2}$, so we get $d_\Gamma(\beta \circ \alpha(a),) \le \frac{1}{2}M^2 + M + \frac{1}{2}$. Similarly for $\alpha \circ \beta$. Thus formulas (1.1)–(1.4) hold with constant $N = \frac{1}{2}M^2 + M + \frac{1}{2}$. $\square$

As a consequence, we see that two different finite generating sets of a group yield Cayley graphs that are combinatorially quasi-isometric.

Note that the concept of a combinatorial quasi-isometry is, a priori, more stringent than that of a quasi-isometry. However, using the preceding result, we have the following:

Proposition 1.2. *Quasi-isometric graphs are combinatorially quasi-isometric.*

Proof. Suppose (f, g) is a quasi-isometry of graphs Γ and Δ. By Lemma 1.1, it suffices to construct a vertex combinatorial quasi-isometry (α_0, β_0).

Define $\alpha_0 : \Gamma^{(0)} \to \Delta^{(0)}$ by letting $\alpha_0(a_0)$ be a vertex of Δ of minimal distance to $f(a_0)$, i.e. with $d_\Delta(\alpha_0(a_0), f(a_0)) \le \frac{1}{2}$. Similarly define β_0. We need to verify formulas (1.1')–(1.4').

The triangle inequality and formula (0.1) imply that, for adjacent vertices a and a' of Γ,

$$d_\Delta(\alpha_0(a), \alpha_0(a')) \le L + K + 1 .$$

Similarly, formula (1.2') holds for β_0 with constant $L + K + 1$.

Suppose $a \in \Gamma^{(0)}$. Then $d_\Delta(\alpha_0(a), f(a)) \le \frac{1}{2}$. Applying formula (0.2) yields $d_\Gamma(g(\alpha_0(a)), g(f(a))) \le \frac{1}{2}K + L$. But $d_\Gamma(\beta_0(\alpha_0(a)), g(\alpha_0(a))) \le \frac{1}{2}$ by the definition of β_0 and $d_\Gamma(g(f(a)), a) \le K$ by formula (0.3). Hence the triangle inequality gives

$$d_\Gamma(\beta_0 \circ \alpha_0(a), a) \le \tfrac{3}{2}K + L + \tfrac{1}{2} .$$

Thus formula (1.3') holds with constant $\frac{3}{2}K + L + \frac{1}{2}$. Similarly for formula (1.4'). Take $M = \max(L + K + 1, \frac{3}{2}K + L + \frac{1}{2})$. $\square$

Before proving our next result, we need a definition. If A is a subcomplex of a graph Ω and $n \ge 0$, let $\mathrm{Nbhd}_n(A)$ denote the closed n-neighborhood of A in Ω,

$$\mathrm{Nbhd}_n(A) = \{b \mid \exists a \in A \text{ with } d_\Omega(a, b) \le n\} .$$

We use this terminology in the proof of the following:

Proposition 1.3. *Suppose (α, β) is a combinatorial quasi-isometry of locally finite graphs. Then the maps α and β are proper maps.*

Proof. Let $\alpha : \Gamma \to \Delta$. In order to prove that α is proper, it suffices to show that the preimage of a finite subcomplex is contained in a finite subcomplex (such a preimage will be a subcomplex of a subdivision).

Assume N is the constant for which formulas (1.1)–(1.4) hold. Let $C \subset \Delta$ be a finite subcomplex. Suppose $x \in \alpha^{-1}(C)$. Then $\alpha(x) \in C$ which implies $\beta(\alpha(x)) \in$

$\beta(C)$. Formula (1.3) yields $d_{\Gamma}(x, \beta(\alpha(x))) \leq N$. Hence

$$\alpha^{-1}(C) \subset \mathrm{Nbhd}_N(\beta(C)) .$$

As Γ is locally finite we can conclude that $\mathrm{NbHd}_N(\beta(C))$ is a finite subcomplex. Hence α is proper. Similarly β is proper. □

2. Ends

We take here for our approach to ends that found in [4]. Let us recall a few of the definitions.

If K is a locally finite connected CW-complex, then two proper maps $r,s : [0, \infty) \to K$ are said to *converge to the same end of* K if given any compact set $C \subset K$ there is an integer $n(c)$ so that $r([n(C), \infty))$ and $s([n(C), \infty))$ lie in the same component of $K \backslash C$. We write $E(K)$ for the set of all proper maps $[0, \infty) \to K$. The relation of converging to the same end is an equivalence relation on $E(K)$. Let $\mathscr{E}(K)$, *the set of ends of* K, be the set of equivalence classes. Denote the equivalence class containing r by $[r]$. Note that each equivalence class $[r]$ contains PL maps and thus it suffices to look at one-skeleta. The *number of ends of* K is the cardinality of $\mathscr{E}(K)$.

If $f : K \to L$ is a proper map then composing proper rays in K with f yields a function $E(K) \to E(L)$ that clearly respects the equivalence relation of converging to the same end. Thus f induces a function $f_{\mathscr{E}} : \mathscr{E}(K) \to \mathscr{E}(L)$. Further this association is functorial. One can identify $f_{\mathscr{E}}$ with $(f \restriction K^{(1)})_{\mathscr{E}}$.

If G is a finitely generated group with finite generating set S then *the number of ends of* G is defined to be the number of ends of the Cayley graph $\Gamma(G, S)$. As is well known, this is independent of the finite generating set chosen (this will also follow from Proposition 2.2).

We turn now to showing that the number of ends is a geometric property. We start with a lemma.

Lemma 2.1. *Let $\gamma : \Gamma \to \Gamma$ be a proper map of a locally finite graph. Assume there is a constant N so that $d_{\Gamma}(x, \gamma(x)) \leq N$ for all $x \in \Gamma$. Then $\gamma_{\mathscr{E}} : \mathscr{E}(\Gamma) \to \mathscr{E}(\Gamma)$ is the identity.*

Proof. Suppose $r : [0, \infty) \to \Gamma$ is a proper ray. Let $C \subset \Gamma$ be compact. As Γ is locally finite, $\mathrm{Nbhd}_n(C)$ is compact for any integer n. Using the fact that r is proper choose $n(C)$ large enough so that

$$r([n(C), \infty)) \subset \Gamma \backslash \mathrm{Nbhd}_N(C) .$$

Let A be the component of $\Gamma \backslash C$ that contains $r([n(C), \infty))$. Let $n \geq n(C)$. We have $d_{\Gamma'}(\gamma(r(n)), r(n)) \leq N$. There is a path between $\gamma(r(n))$ and $r(n)$ that misses C by our choice of $n(C)$. Hence

$$\gamma \circ r([n(C), \infty)) \subset A .$$

It follows that r and $\gamma \circ r$ converge to the same end of Γ. $\square$

Applying this lemma and the results of Section 1 yields the following proposition:

Proposition 2.2. *Quasi-isometric graphs have the same number of ends.*

Proposition. Apply Propositions 1.2, 1.3, formulas (1.3), (1.4), Lemma 2.1 and functorality. $\square$

Note that since different finite generating sets yield quasi-isometric Cayley graphs we could use the preceding to obtain yet another proof that the number of ends does not depend on the generating set.

As a corollary we get the following:

Corollary 2.3. *Finitely generated groups that are quasi-isometric have the same number of ends.* $\square$

Applying Stallings' classification theorem (see [5]) yields the following corollary:

Corollary 2.4. *If G and H are quasi-isometric finitely generated groups then G splits over some finite subgroup iff H splits over some finite subgroup.* $\square$

Note that free products do not necessarily preserve the property of being quasi-isometric. For example $\mathbb{Z}_2$ is quasi-isometric to $\mathbb{Z}_3$ (any two finite groups are quasi-isometric) but $\mathbb{Z}_2 * \mathbb{Z}_2$ is not quasi-isometric to $\mathbb{Z}_3 * \mathbb{Z}_3$ as they have a different number of ends.

3. Semistability

We turn to the definition of semistability (see [4]). If K is a locally finite connected CW-complex and $r : [0, \infty) \to K$ is a proper map, then we say that the end $[r]$ is *semistable* if whenever $s \in [r]$ then s and r are properly homotopic. The cellular approximation theorem lets us replace the map r by a cellular map r'. Since r and r' are easily seen to be properly homotopic, we may assume that both r and s are cellular maps. We say that K is *semistable* iff each end of K is semistable.

Suppose G is a finitely presented group. Let X be some finite two-complex with fundamental group being G, and denote by $\tilde{X}$ the universal cover of X. We say that G is *semistable* iff $\tilde{X}$ is semistable. This is independent of the finite complex chosen.

We also need to recall the notion of a Dehn function (see [2]). Let K be a two-complex. If w is a null-homotopic edge loop in K then there is a Van Kampen diagram that w bounds, i.e. a combinatorial map $j : D \to K$ of a finite simply connected planar complex with boundary cycle mapping to w (note that the diagrams are not unique). We denote by $a(D)$ the number of two-cells of D and call it the area of the diagram. A diagram D is called *minimal* for w if it has minimal area. The Dehn function of K, denoted δ_K, is the least function $f : \mathbb{N} \to \mathbb{N} \cup \{\infty\}$ so that any null-homotopic edge loop w in K of length $\leq n$ bounds a Van Kampen diagram of area $\leq f(n)$. The key fact about Dehn functions that we will need is that the Dehn function of a cover of a finite complex is finite-valued [2, Proposition 1.4].

If A is a subcomplex of $K^{(1)}$ define full(A) to be the union of A together with all two-cells with boundary contained in A (so full(A) may be thought of as the full subcomplex of K determined by A). Extend to subcomplexes of two-complexes the definition of $\mathrm{Nbhd}_n(C)$ by setting

$$\mathrm{Nbhd}_n(C) = \mathrm{full}(\mathrm{Nbhd}_n(C \cap K^{(1)}))\,.$$

We turn to a technical lemma.

Lemma 3.1. *Let $\tilde{X}$ be the universal cover of a finite two-complex and m be a constant. Then there is a constant m' so that if w is a edge loop in $\tilde{X}$ of length $\leq m$ then w bounds a Van Kampen diagram $j : D \to \tilde{X}$ with $j(D) \subset \mathrm{Nbhd}_{m'}(w)$.*

Proof. Let $M = \delta_{\tilde{X}}(m) < \infty$ and

$$n = \max(\text{the length of } \partial F \mid F \text{ is a face of } X) < \infty\,.$$

Define $m' = M \cdot n$.

Suppose w is as above. Then taking $j : D \to \tilde{X}$ to be a Van Kampen diagram of area $\leq M$, we see that any vertex in $j(D)$ is of distance $\leq M \cdot n$ from some vertex of w. The result follows. $\square$

Note that the preceding can easily be seen to be false for complexes that are not covering spaces. For instance, consider a wedge of infinitely many disks with finer and finer triangulations, but all having boundary of length three. Wedge them together at their centers and let w vary among the boundaries.

If K is a two-complex then we say that K *has bounded two-cells* if there is an upper bound to the lengths of the boundaries of the two-cells (counted as possibly unreduced words). With this in mind, the previous lemma yields the following:

Proposition 3.2. Let $\tilde{X}$ *be the universal cover of a finite two-complex and* N *be a constant. If* K *is a locally finite two-complex having bounded two-cells and* $\gamma : K^{(1)} \to \tilde{X}^{(1)}$ *is a proper map that sends vertices to vertices and edges to edge paths of length* $\leq N$, *then there is a PL extension* $\bar{\gamma} : K \to \tilde{X}$ *which is proper.*

Proof. Suppose f is a two-cell of K. Let $w = \gamma(\partial f)$. This is a edge loop in $\tilde{X}$. Take a Van Kampen diagram $j : D_f \to \tilde{X}$ of minimal area and define $\bar{\gamma}$ on f by mapping f to D_f and then to $\tilde{X}$ using the map j (note that the diagram is not necessarily unique). This may be done in a cellular fashion. Doing this for each two-cell of K defines an extension $\bar{\gamma}$ of γ. We need to see that $\bar{\gamma}$ is proper.

It suffices to prove that the preimage under $\bar{\gamma}$ of a finite subcomplex B of $\tilde{X}$ is contained in a finite subcomplex of K (it will then be a closed subset of a compact set).

Let c be an upper bound to the lengths of the boundaries of the two-cells of K. Then $N \cdot c$ is an upper bound for the lengths of the edge loops in $\tilde{X}$ of the form $w = \gamma(\partial f)$ for some two-cell f of K. Take m' to be the constant from Lemma 3.1 corresponding to the constant $m = N \cdot c$. The subcomplex $A = \gamma^{-1}(\mathrm{Nbhd}_{m'+c}(B))$ is contained in a finite subcomplex of $K^{(1)}$ since γ is proper and $\tilde{X}$ is locally finite. Then by Lemma 3.1 and the definition of $\bar{\gamma}$, we have

$$\bar{\gamma}^{-1}(B) \subset \mathrm{full}(A) ,$$

which is a finite subcomplex, by local finiteness of K. It follows that $\bar{\gamma}$ is proper. $\square$

We apply the preceding to obtain the following corollary:

Corollary 3.3. *Suppose* $\tilde{X}$ *and* $\tilde{Y}$ *are universal covers of finite two-complexes and* (α, β) *is a combinatorial quasi-isometry between* $\tilde{X}^{(1)}$ *and* $\tilde{Y}^{(1)}$. *There are cellular extensions* $\bar{\alpha} : \tilde{X} \to \tilde{Y}$ *and* $\bar{\beta} : \tilde{Y} \to \tilde{X}$ *which are proper.*

Proof. This follows immediately from Proposition 3.2 when one observes that both $\tilde{X}$ and $\tilde{Y}$ have bounded two-cells. $\square$

Furthermore, we have the following:

Proposition 3.4. *Suppose* $\tilde{X}$ *and* $\tilde{Y}$ *are universal covers of finite two-complexes and* (α, β) *is a combinatorial quasi-isometry between* $\tilde{X}^{(1)}$ *and* $\tilde{Y}^{(1)}$. *If* $r : [0, \infty) \to \tilde{X}^{(1)}$ *is a proper cellular map then* $\beta \circ \alpha \circ r$ *is proper. Moreover,* r *and* $\beta \circ \alpha \circ r$ *are properly homotopic.*

Proof. That $\beta \circ \alpha \circ r$ is proper is a consequence of Proposition 1.3.

We will construct a two-complex K and a proper map of K into $\tilde{X}$ that will show that r and $\beta \circ \alpha \circ r$ are properly homotopic.

We start by defining the one-skeleton of K. Take two copies l_1 and l_2 of $[0, \infty)$. Denote points in each copy by $l_1(t)$ and $l_2(t)$, where $t \geq 0$. For $i = 0, 1, 2, \ldots$, let

$$l_i = d_{\tilde{X}^{(1)}}(r(i), \beta(\alpha(r(i)))) .$$

Observe that by formula (1.3) there is a constant N with $l_i \leq N$ for each i. For $i = 0, 1, 2, \ldots$, attach a path P_i from $l_1(i)$ to $l_2(i)$ of length l_i. If $l_i = 0$ then we identify the vertices $l_1(i)$ and $l_2(i)$. One can view the resulting one-complex as an infinite ladder, but one where the rungs are of possibly varying lengths (possibly even zero).

Now attach two-cells to each path of the form

$$([l_1(i), l_1(i+1)])(P_{i+1})([l_2(i), l_2(i+1)])^{-1}(P_i)^{-1} .$$

Thus we are filling in the steps. Call the resulting two-complex K.

We define a cellular map $h : K^{(1)} \to \tilde{X}$ by taking h to be r on l_1, $\beta \circ \alpha \circ r$ on l_2, and mapping each path P_i to some geodesic path in $\tilde{X}$ between $r(i)$ and $\beta(\alpha(r(i)))$.

Note that the maps r and $\beta \circ \alpha \circ r$ are proper and the paths P_i are of bounded length. These facts can be used to see that h is proper; we omit the details.

Now Proposition 3.2 applies. We get a proper extension $\bar{h} : K \to \tilde{X}$ which is a proper homotopy between the rays r and $\beta \circ \alpha \circ r$. □

This yields the following proposition:

Proposition 3.5. *Suppose $\tilde{X}$ and $\tilde{Y}$ are universal covers of finite two-complexes and (α, β) is a combinatorial quasi-isometry between $\tilde{X}^{(1)}$ and $\tilde{Y}^{(1)}$. Then $\alpha_{\mathscr{E}}$ and $\beta_{\mathscr{E}}$ map semistable ends to semistable ends.*

Proof. As mentioned above, it suffices to consider cellular rays. Suppose a cellular proper ray r represents an end of $\tilde{Y}$ that is semistable. We will show that $\beta \circ r$ represents an end of $\tilde{X}$ that is also semistable. By symmetry, this suffices.

Let $s : [0, \infty) \to \tilde{X}$ be a proper cellular map converging to the same end as $r' = \beta \circ r$. We need to see that s and r' are properly homotopic.

By Corollary 3.3, α and β extend to proper maps $\bar{\alpha} : \tilde{X} \to \tilde{Y}$ and $\bar{\beta} : \tilde{Y} \to \tilde{X}$.

By Lemma 2.1, formulas (1.3), (1.4), and functorality $\alpha \circ r'$ and $\alpha \circ s$ represent the same end as r. Hence they are properly homotopic. Let H be a proper homotopy between them. Then $\bar{\beta} \circ H$ is a proper homotopy between $\beta \circ \alpha \circ r'$ and $\beta \circ \alpha \circ s$. By Proposition 3.4, $\beta \circ \alpha \circ r'$ and r' are properly homotopic and $\beta \circ \alpha \circ s$ and s are properly homotopic.

Combining the proper homotopies we see that s and r' are properly homotopic as desired. □

The above can also be used to show that the definition of semistability for finitely presented groups does not depend on the finite two-complex chosen (as any two such complexes have combinatorially quasi-isometric one-skeletons).

We also obtain the following:

Corollary 3.6. *Suppose G and H are finitely presented groups. If G and H are quasi-isometric then G is semistable iff H is semistable.* $\square$

4. Simple connectivity at infinity

We start by recalling a definition. A space A is *simply connected at infinity* if given a compact set $C \subset A$ there is a compact set $D_A(C)$ in A containing C such that any loop in $A \backslash D_A(C)$ is null-homotopic in $A \backslash C$. Clearly, we can restrict ourselves to subcomplexes and edge loops.

We have the following:

Proposition 4.1. *Suppose $\tilde{X}$ and $\tilde{Y}$ are universal covers of finite two-complexes and (α, β) is a combinatorial quasi-isometry between $\tilde{X}^{(1)}$ and $\tilde{Y}^{(1)}$. Then $\tilde{X}$ is simply-connected at infinity iff $\tilde{Y}$ is simply connected at infinity.*

Proof. Let N be the constant of the combinatorial quasi-isometry as in formula (1.3). Extend α and β to proper maps, also denoted α and β. Take m' to be the constant of Lemma 3.1 associated to $m = N^2 + 2N + 1$.

Suppose $C \subset \tilde{X}$ is a finite subcomplex. Define

$$D_{\tilde{X}}(C) = \mathrm{Nbhd}_{N+m'}(C) \cup \alpha^{-1}(D_{\tilde{Y}}(\beta^{-1}(C))) ,$$

which is a compact subset of $\tilde{X}$ since α and β are proper and $\tilde{X}$ is locally finite.

Let w be an edge loop in $\tilde{X} \backslash D_{\tilde{X}}(C)$. We will show that w is null-homotopic in $\tilde{X} \backslash C$. This will finish the proof.

By the choice of N, we have image$(\beta \circ \alpha \circ w) \subset \tilde{X} \backslash \mathrm{Nbhd}_{m'}(C)$. If e is an edge of w, then $\beta\alpha e$ is an edge path in $\tilde{X}$ of length $\leq N^2$. Choose edge paths P_0 and P_1 of lengths $\leq N$ from $e(0)$ to $\beta\alpha e(0)$ and from $e(1)$ to $\beta\alpha e(1)$ respectively. The edge loop

$$U = e^{-1} \cdot P_0 \cdot \beta\alpha e \cdot P_1^{-1}$$

is of length $\leq N^2 + 2N + 1 = m$ and has image contained in $\tilde{X} \backslash \mathrm{Nbhd}_{m'}(C)$. By our choice of m', it follows that U is null-homotopic in $\tilde{X} \backslash C$. This is true for each edge of w. We can conclude that w is homotopic to $\beta \circ \alpha \circ w$ in $\tilde{X} \backslash C$. Hence it suffices to show that $\beta \circ \alpha \circ w$ is null-homotopic $\tilde{X} \backslash C$.

Now image$(\alpha \circ w) \subset \tilde{Y} \backslash D_{\tilde{Y}}(\beta^{-1}(C))$ by our construction of $D_{\tilde{X}}(C)$ and the fact that w is a loop missing $D_{\tilde{X}}(C)$. By the definition of $D_{\tilde{Y}}(\beta^{-1}(C))$, it follows that $\alpha \circ w$ is null-homotopic in $\tilde{Y} \backslash \beta^{-1}(C)$. Let H be a homotopy showing this. Then $\beta \circ H$ is a homotopy in $\tilde{X} \backslash C$ from $\beta \circ w$ to a constant, as desired. □

A finitely presented group G is *simply connected at infinity* if some finite two-complex X with $\pi_1(X) = G$ is simply connected at infinity. The preceding proposition shows that this definition does not depend on the choice of X. Also, the above yields the following corollary:

Corollary 4.2. *Suppose G and H are finitely presented groups. If G and H are quasi-isometric then G is simply connected at infinity iff H is simply connected at infinity.* □

The proof of Proposition 4.1 actually shows a bit more. We first need to recall another approach to ends. Start by viewing an end e of a space A as a choice, for each compact $C \subset A$, of a non-compact connected component $e(C)$ of $A \backslash C$, where $C_1 \subset C_2$ implies $e(C_2) \subset e(C_1)$. A proper map $f : A \to B$ induces a map on the ends as follows: take $C \subset B$ compact; since f is proper, $f^{-1}(C) \subset A$ is compact; let e be an end of A which chooses the component U of $A \backslash f^{-1}(C)$; finally, let f map the end e to the end of B that chooses the component of $B \backslash C$ that contains $f(U)$ (which is connected). An end e of A is *simply connected* if given a compact $C \subset A$ there is a compact D in A containing C such that any loop in $e(D)$ is null-homotopic in $e(C)$. Then the argument in the proof of Proposition 4.1 shows the following:

Corollary 4.3. *Suppose $\tilde{X}$ and $\tilde{Y}$ are universal covers of finite two-complexes and (α, β) is a combinatorial quasi-isometry between $\tilde{X}^{(1)}$ and $\tilde{Y}^{(1)}$. Then $\alpha_{\mathscr{E}}$ and $\beta_{\mathscr{E}}$ map simply connected ends to simply connected ends.* □

Needless to say, the above can be applied to show that quasi-isometries of groups send simply connected ends to simply connected ends.

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