Deformations of Plane Graphs

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Communicated by the Editors

Received May 3, 1982

It is proven that, if \( \Gamma_0 \) and \( \Gamma_1 \) are isomorphic strictly convex graphs such that their outer polygons correspond to each other and have the same orientations, then \( \Gamma_0 \) can be continuously deformed into \( \Gamma_1 \) such that, at each stage, the graph under consideration is convex. This extends a result of Cairns (Ann of Math. 45 (2) (1944), 207–217; Amer. Math. Monthly 51 (1944), 247–252) and proves a conjecture of Grünbaum and Shepard ("Proceedings, 8th British Combinatorial Conf.", 1981). This result is applied to prove an analogous conjecture by Grünbaum and Shepard on deformations of straight graphs in general and it is shown how the proof method also can be used to verify a conjecture of Robinson ("Proceedings, 8th British Combinatorial Conf.", 1981) on deformations of rectanguloid curves.

1. INTRODUCTION

A convex graph is a finite 2-connected plane graph such that each face (region) is bounded by a convex polygon. If we move a vertex of a convex graph along a bounded straight line segment (and move its incident edges accordingly and keep all other vertices and edges fixed) such that, at each stage, we have a convex graph, then we perform a simple convex deformation of the graph. By a convex deformation we mean a finite sequence of simple convex deformations. For straight graphs (i.e., plane graphs such that all edges are straight line segments) we define a simple straight deformation and a straight deformation analogously.

Consider two convex graphs \( \Gamma_0 \) and \( \Gamma_1 \) with outer polygons (cycles) \( \Sigma_0, \Sigma_1 \) and suppose that there exists an isomorphism of \( \Gamma_0 \) onto \( \Gamma_1 \) (when these are viewed as abstract graphs) such that \( \Sigma_0 \) is taken to \( \Sigma_1 \) and such that the clockwise orientation of \( \Sigma_0 \) and \( \Sigma_1 \) is preserved. Then we prove in Section 3 as a main result that \( \Gamma_1 \) can be obtained from \( \Gamma_0 \) by a convex deformation. The restriction of this result to the 3-connected case was conjectured by Grünbaum and Shepard [6] and it implies the result of Cairns [4, 5] on straight deformations of triangulations (other extensions of Cairns' result...
were obtained by Bing and Starbird [2, 3] and Ho [7, 8]). In addition, it has other applications as described below.

In Section 4 we prove that, if $f_0$ and $f_1$ are straight graphs such that there exists an orientation preserving homeomorphism of the Euclidean plane taking $f_0$ onto $f_1$, then $f_0$ and $f_1$ can be extended to triangulations $A_0$ and $A_1$, respectively, such that $A_0$ and $A_1$ satisfy the condition of the result in Section 3. Thus, in particular, $f_1$ can be obtained from $f_0$ by a straight deformation. This was also conjectured by Grünbaum and Shepard [6].

In Section 5 we show how the method in Section 3 can be modified to prove the conjecture of Robinson [9] that, if two closed curves each made up by vertical and horizontal straight line segments have the same turns sequence, then one can be obtained from the other by a so-called rectanguloid deformation.

Our terminology is the same as in [10]. In particular, abstract graphs are denoted by capital italic letters and plane graphs by capital greek letters. In addition, a plane graph isomorphic to the abstract graph $G$ will be denoted $\tilde{G}$ and similarly for subgraphs and vertices and edges of $G$ and $\tilde{G}$. The main result of this paper depends on the characterization of convex graphs given in [10, Theorem 5.1] and the reader is assumed to be familiar with that result and its proof. We first apply [10, Theorem 5.1] to establish some reduction results on abstract graphs having convex representations.

2. Abstract Graphs with Convex Representations

Consider a planar 2-connected graph $G$ and a facial cycle $S$ (i.e., $S$ is the boundary of a face in some plane representation of $G$) and let $S$ be partitioned into paths $P_1, P_2, \ldots, P_k$ such that $P_i$ and $P_{i+1}$ have precisely an endvertex in common for $i = 1, 2, \ldots, k$ (where $P_{k+1} = P_1$). Let $\Sigma$ be a convex $k$-gon in the plane. Then Theorem 5.1 in [10] asserts that $\Sigma$ can be extended to a convex graph $\Gamma$ isomorphic to $G$ such that $\Sigma$ is the outer polygon whose segments represent the paths $P_i$ ($1 \leq i \leq k$) if and only if

(i) each vertex $x$ of $V(G) \setminus V(S)$ of degree at least 3 is joined to $S$ by three paths which are pairwise disjoint except for $x$;

(ii) no $S$-component of $G$ has all its vertices of attachment on the same $P_i$;

(iii) each cycle which has no edge in common with $S$ has at least three vertices of degree at least 3.

In the proof of [10, Theorem 5.1] the angle $\pi$ may occur at many vertices. Of course it must occur at the interior vertices of the paths $P_1, P_2, \ldots, P_k$ and at all vertices of degree 2. However, a close inspection of the proof of [10,
Theorem 5.1] shows that the angle $\pi$ can be avoided at all other vertices (after having performed the induction step in the proof of [10, Theorem 5.1] we make, if necessary, small displacements of the paths $P'_1, P'_2, \ldots, P'_r$ that occur in the proof). Such a convex graph will be called strictly convex. In the proof of Theorem 3.3 of the present paper it is essential that we can make small displacements of vertices and split up vertices that correspond to contracted edges and for that it is convenient to avoid the angle $\pi$. (In the type of graph illustrated in Fig. 1 $\pi$ occurs at all but three vertices. If we perform a convex deformation on this graphs such that the outer cycle is kept fixed, then we can only move the six "inner" vertices.) If we avoid $\pi$ (except at $\Sigma$) in a convex deformation of $G$ we speak of a strictly convex deformation.

Condition (iii) means that $G$ is a subdivision of a graph $G_0$ (without multiple edges) such that no vertex in $V(G_0) \setminus V(S)$ has degree 2 and condition (i) implies that this graph is almost 3-connected in the sense that each separating set of two vertices is contained in $S$.

We first prove some results on abstract graphs with convex representations which are analogous to results on 3-connected graphs discussed in [12]. We refer throughout Sections 2 and 3 to the above graph $G$. The first result is analogous to a result of Barnette and Grünbaum [1] and its proof is similar to the proof of [11, Lemma 4.1].

**Proposition 2.1.** If $G$ has order at least 5 and $V(G) \setminus V(S) \neq \emptyset$ and each vertex of $V(G) \setminus V(S)$ has degree at least 3, then $G - E(S)$ has an edge $e$ such that $G - e$ has a convex representation with $\Sigma$ representing $S$.

**Proof.** We consider a proper subgraph $H$ of $G$ such that $H$ contains $S$ and satisfies (i)–(iii) and such that $|E(H)|$ is maximum under these restrictions. We shall prove that $E(G) \setminus E(H)$ consists of a single edge.

If $H$ has a path $P$ of length at least 2 such that the ends $x$ and $y$ of $P$ have degree at least 3 in $H$ and all intermediate vertices have degree 2 in $H$ and $H$
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is edge-disjoint from $S$, then $G - \{x, y\}$ has a path $P'$ from $P$ to $H - V(P)$ and now $H \cup P'$ satisfies (i) - (iii). Hence $H \cup P' = G$ and $P'$ contains one edge $e$.

So assume all vertices of $H \setminus V(S)$ have degree at least 3 in $H$. If $V(H) = V(G)$, then the proof is easily completed. So assume there is a vertex $x$ in $V(G) \setminus V(H)$. We shall obtain a contradiction from this. Let $P'_1, P'_2, P'_3$ be three paths from $x$ to $H$ which are pairwise disjoint except for $x$. Let $x_1, x_2, x_3$ be the other endvertices of $P'_1, P'_2, P'_3$, respectively. Then $H' = H \cup P'_1 \cup P'_2 \cup P'_3$ satisfies (i)-(iii) unless $x_1, x_2$ and $x_3$ are at the same $P_i$. But in that case there exists a path $P''$ from $P_i$ to $H - V(P_i)$ such that $H \cup P''$ is a proper subgraph of $G$ satisfying (i)-(iii), a contradiction to the maximality of $H$. So we can assume that $H' = G$ and that each of $P'_1, P'_2, P'_3$ has length 1. By the maximality of $H$, $H \cup P'_1 \cup P'_2$ does not satisfy (i)-(iii) so it must fail to satisfy (iii), i.e., $x_1$ and $x_2$ are adjacent. More generally, we can assume that $x_1, x_2, x_3$ span a $K_3$. This $K_3$ does not equal $S$ (because $G$ has order $>4$) so assume the edge $e' = x_1, x_2$ is not in $S$. Then $(H \cup P'_1 \cup P'_2) - e'$ satisfies (i)-(iii) and contradicts the maximality of $H$. This completes the proof.

We mention a result which is related to Proposition 2.1. Since we shall not need the result in this paper we omit the proof (a proof can be found in [12]). The first part of Proposition 2.2 is "dual" to Proposition 2.1 and the second is analogous to the observation of H. Whitney that, for each edge $e$ of a 2-connected graph $H$ of order at least 3, either $H - e$ or $H/e$ (which denotes the graph obtained from $H$ by contracting $e$ and replacing all multiple edges in the resulting graph by single edges) is 2-connected.

**Proposition 2.2.** If all paths $P_1, P_2, ..., P_k$ in the aforementioned graph $G$ have length one and $G$ has at least one vertex not in $S$, then $G - E(S)$ contains an edge $e$ such that $G/e$ has a convex representation with $\Sigma$ representing $S$. If, in addition, all vertices in $V(G) \setminus V(S)$ have degree at least 3, then for each edge $e \in E(G) \setminus E(S)$, either $G - e$ or $G/e$ has a convex representation with $\Sigma$ representing $S$.

### 3. Convex Deformations of Graphs

Propositions 2.1 and 2.2 suggest that the main result of this paper be proved by contracting or deleting a suitable edge and then using an inductive argument. The difficulty of the proof lies in finding such an edge. Suppose $\Gamma$ is a convex graph isomorphic to $G$ such that $\Sigma$ represents $S$. Then an edge $e = xy$ of $E(G) \setminus E(S)$ is removable in $\Gamma$ if $\Gamma - e$ is convex and $e$ is contractible into $y$ in $\Gamma$ if we can move $\bar{x}$ into $\bar{y}$ (along $\bar{e}$) such that, at each stage, we
have a convex graph isomorphic to $G$. We say that $e$ is contractible if it is contractible into either $x$ or $y$. With this notation we have

**Lemma 3.1.** If $G \not= S$, then there exists a convex deformation $\Gamma'$ of $\Gamma$ such that $\Sigma$ is fixed throughout the deformation and such that, for some edge $e = xy \in E(G) \setminus E(S)$, either

(a) $\tilde{e}$ is removable in $\Gamma'$ and each of $x, y$ belongs to $S$ or has degree at least 4, or

(b) $x$ has degree 3 and $\tilde{e}$ is contractible into $\tilde{y}$ in $\Gamma'$.

Moreover, if $\Gamma$ is strictly convex, then the deformation can be chosen to be strictly convex.

**Proof.** We can assume that all vertices of $G - V(S)$ have degree at least 3 in $G$. If $G - V(S)$ has a vertex $x$ of degree 3 (in $G$) such that two of the neighbours of $x$ are adjacent, then $x$ can be contracted into its third neighbour so assume that no such vertex $x$ exists.

Suppose that $\Gamma$ has no removable edge. This means that, for every edge $e = xy$ in $E(G) \setminus E(S)$, the deletion of $\tilde{e}$ from $\Gamma$ results in an angle exceeding $\pi$ either at $\tilde{x}$ or at $\tilde{y}$. If no angle exceeds $\pi$ at $x$ in $\Gamma - \tilde{e}$, then we assign an orientation to $e$ from $x$ towards $y$. (Since $\Gamma$ has no removable edge, no edge is directed in both directions). If $e$ is not directed from $x$ to $y$, then all edges of $\Gamma$ incident with $\tilde{x}$, except possibly the two edges consecutive to $\tilde{e}$, are directed away from $\tilde{x}$. So, if $x$ has degree at least 4, then at most two edges are not directed away from $x$ and these two edges are consecutive. Also, if $x$ is on $S$, then all edges of $E(G) \setminus E(S)$ incident with $x$ are directed away from $x$.

Let $A$ denote the set of vertices of $V(G) \setminus V(S)$ which have degree 3 in $G$. We claim that there is some connected component of the subgraph $G(A)$ of $G$ induced by $A$ such that at least two edges are directed from $V(G) \setminus A$ towards this component. For suppose this were false. Then we contract each component of $G(A)$ into a single vertex and direct all undirected edges incident with that vertex towards $V(G) \setminus A$. In this way we obtain a graph where no vertex has larger indegree than outdegree (because $G$ satisfies (i)) and some vertex of $S$ has larger outdegree than indegree. This contradiction proves our claim.

So $G$ has a path or cycle $x x_1 x_2 \cdots x_m y$, where $m \geq 1$ and possibly $x = y$ such that each $x_i$, $i = 1, 2, \ldots, m$, has degree 3 and the edges $xx_1$ and $yx_m$ are directed away from $x$ and $y$, respectively. Among all convex deformations of $\Gamma$ we choose one, say $\Gamma'$, such that $m$ is least possible (we allow $x$ to have degree 3 but $y$ should have degree at least 4 or be in $S$). If $m = 0$, then $\tilde{x}\tilde{y}$ is removable so assume $m \geq 1$. Let $x_1'$ be the neighbour of $x_1$ distinct from $x$ and $x_2$. We now move $\tilde{x}_1$ towards $\tilde{x}_1'$. This will result in a contraction of
\(\bar{x}_1\bar{x}'_1\) unless the angle \(\pi\) occurs at \(\bar{x}_2\) at some stage of the contraction process. If that happens, then the edges \(\bar{x}_2\bar{x}_1\) and \(\bar{x}_2\bar{x}_3\) must form the angle \(\pi\) because of the minimality of \(m\) (see Fig. 2).

Now we move successively \(\bar{x}_2\) towards its neighbour \(\bar{x}'_2\) (distinct from \(\bar{x}_1\) and \(\bar{x}_3\)) and \(\bar{x}_1\) towards \(\bar{x}'_1\) and this will result in a contraction of either \(x_1x'_1\) or \(x_2x'_2\) unless the angle \(\pi\) occurs at \(\bar{x}_3\). We proceed like this and will eventually contract one of the edges incident with the path \(x_1x_2\cdots x_m\). Figure 3 illustrates this part of the proof. We may think of moving \(\bar{x}_1\) towards \(\bar{x}'_1\) such that the path \(\bar{x}_1\bar{x}_2\cdots\bar{x}_t\) remains a straight line segment. (The broken line of Fig. 3 shows the path \(\bar{x}_1\bar{x}_2\cdots\bar{x}_t\) at some stage of the contraction process.) Then we show that the same can be accomplished by a sequence of simple convex deformations (beginning by moving \(\bar{x}_1\) in the "wrong" direction). It is important here that none of the edges of the path \(x_1x_2\cdots x_t\) are in triangles.

This completes the first part of the proof. In order to prove the second part we note that, in the proof of the first part, we only create new angles \(\pi\) at vertices of degree 3. By performing appropriate small displacements of these vertices we can avoid the angle \(\pi\).

Fig. 2. Creation of a path or cycle with smaller \(m\).

Fig. 3. Contracting an edge in the proof of Lemma 3.1.
Using Lemma 3.1 we can go a step further.

**Lemma 3.2.** If $G - V(S)$ has at least one edge and $\Gamma$ is strictly convex, then there exists a strictly convex deformation $\Gamma'$ of $\Gamma$ such that $\Sigma$ is fixed and some edge $\bar{e}$ of $\Gamma'$ is contractible.

*Proof* (by induction on $|E(G - V(S))|$). If $G - V(S)$ has exactly one edge, then this is contractible in $\Gamma$ so we proceed to the induction step. By Lemma 3.1, we can assume that there exists a strictly convex deformation $\Gamma''$ such that an edge $\bar{e}$ is removable and both ends of $e$ have degree at least 4 or belong to $S$. The lemma now follows by applying the induction hypothesis to $\Gamma'' - \bar{e}$.

We can now prove the main result.

**Theorem 3.3.** Let $\Gamma_0$ and $\Gamma_1$ be two strictly convex graphs representing the graph $G$ such that $\Gamma_0$ and $\Gamma_1$ have the same outer polygon $\Sigma$ (representing the cycle $S$ in $G$). Then $\Gamma_1$ can be obtained from $\Gamma_0$ by a strictly convex deformation.

*Proof* (by induction on $|E(G)| - |E(S)|$). If $V(G) = V(S)$, there is nothing to prove so we proceed to the induction step and assume that $V(G) \neq V(S)$. If $G$ has a separating set $\{u, v\}$ of two vertices, then $u$ and $v$ are in $S$ and, by adding the line segment between $\bar{u}$ and $\bar{v}$ (if this is not already present), we split $\Gamma_0$ and $\Gamma_1$ up into two smaller convex graphs and we apply the induction hypothesis.

So assume $G$ is connected. We can also assume that $|V(G) \setminus V(S)| \geq 2$, for otherwise $G$ is a wheel and the proof is easy to complete. By Lemma 3.2, $G$ contains edges $e_0$ and $e_1$ (not in $S$) such that $\Gamma_0$ and $\Gamma_1$ have strictly convex deformations $\Gamma'_0$ and $\Gamma'_1$ in which $\bar{e}_0$ (resp. $\bar{e}_1$) is contractible.

Consider first the case where $e_0 = e_1$. We then contract $\bar{e}_0$ in $\Gamma'_0$ and $\bar{e}_1$ in $\Gamma'_1$ and obtain thereby $\Gamma''_0$ and $\Gamma''_1$, respectively. By the induction hypothesis, $\Gamma''_1$ may be regarded as a strictly convex deformation of $\Gamma''_0$ and it is easy to modify this so as to obtain a strictly convex deformation of $\Gamma''_0$ into $\Gamma'_1$ (keeping the ends of $\bar{e}_0$ and $\bar{e}_1$ sufficiently close together).

So we may assume that $e_0 \neq e_1$. Then each vertex of $G - V(S)$ is joined to at least two vertices of $G - V(S)$. For if $x \in V(G) \setminus V(S)$, then $x$ is joined to only one vertex $y$ of $V(G) \setminus V(S)$, then $\bar{x}$ can be contracted into $\bar{y}$ in both $\Gamma_0$ and $\Gamma_1$. Also, we can assume that no triangle has a vertex (in $V(G) \setminus V(S)$) of degree 3 for otherwise, we can contract an edge in both $\Gamma_0$ and $\Gamma_1$ as in the beginning of the proof of Lemma 3.1. Now contract $\bar{e}_0$ in $\Gamma'_0$. By Lemma 3.2, the resulting graph has a strictly convex deformation in which some edge $e'_0$ is contractible. From this we conclude that $\Gamma'_0$ (and hence also $\Gamma''_0$) has a strictly
convex deformation such that $e'_a$ is contractible. (Instead of contracting $\bar{e}_0$ in $\Gamma_0$ we keep the ends of $\bar{e}_0$ close together in the deformation process above. The only problem that may occur here is that an end of $e_0$ has degree 3 and is joined to both ends of $e'_a$ in which case the angle $\pi$ will occur when we contract $e'_a$ in $\bar{G}$. But this possibility is excluded by the assumption above that no triangle has a vertex of degree 3). So there are at least two possibilities for choosing each of $e_0$ and $e_1$ (a fact we shall use later).

Now let $x$ be a vertex which is not incident with any of $e_0$, $e_1$ such that $x$ is a corner of $\Sigma$. Since $G$ is 3-connected, $G - x$ is 2-connected and in $\Gamma_0$ and $\Gamma_1$ the outer cycle consists of a path $P''$ of $S$ together with a path $P'$ that is partitioned into paths $P'_1, P'_2, \ldots, P'_r$ (compare the proof of [10, Theorem 5.1]) by the neighbours of $x$. If we draw $P'_1, \ldots, P'_r$ as straight line segments such that $P'' \cup P'_1 \cup \ldots \cup P'_r$ form a convex polygon $\Sigma'$ and each $P'_i$, $1 \leq i \leq r$, is inside $\Sigma$, then $\Sigma'$ can be extended to a strictly convex graph $\Gamma_2$ isomorphic to $G - x$ since $G - x$ satisfies (i)-(iii). If $G - x = P' \cup P''$, the proof is easy to complete, so assume this is not so. Then, by Proposition 2.1, $G - x$ has an edge $e'$ not in $P' \cup P''$ such that $\Sigma'$ can be extended to a convex graph $\Gamma_2$ isomorphic to $G - x - e'$. Since there are at least two possible choices for each of $e_0$ and $e_1$ and $\Sigma$ has at least three corners we can choose $x, e_0, e'$ such that there are still two possible choices for $e_1$.

Now the idea of the last part of the proof is to show that there are strictly convex deformations (except for the ends of $e'$ if these (or one of these) have degree 3) of $\Gamma_0$ and $\Gamma_1$ such that $e'$ is removable in both deformations. From now on we concentrate on $\Gamma_0$. We are going to contract $e_0$ in $\Gamma_0$ and then use induction in order to move $P' \cup P''$ (viewed as a subgraph of the resulting graph) "close to" $\Sigma'$ (which is a subgraph of $\Gamma_2$) and then deforming the interior of $\Sigma'$ (again using induction) so that $e'$ becomes removable. More precisely, we proceed as follows:

Assume first $e_0 \in E(P')$. Then $(G/e_0) - x$ is 2-connected and can be represented as a strictly convex graph $\Gamma'_3$ such that $P''$ is represented by part of $\Sigma$ and $(P'/e_0) \cup P''$ is represented by a convex polygon $\Sigma''$ such that the corners are either corners of $\Sigma$ or correspond to neighbours of $x$ in $G/e_0$. By splitting up $\bar{e}_0$ we obtain a strictly convex representation of $G - x$. We denote this by $\Gamma'_3$ and its outer polygon by $\Sigma'$. Note that $\bar{e}_0$ is contractible in $\Gamma'_3$ and that $\Sigma'$ has one corner more than $\Sigma''$ if $x$ is joined to both ends of $e_0$. By the induction hypothesis of the proof of Theorem 3.3, there exists a strictly convex deformation $\Gamma_4$ of $\Gamma_3$ such that $\bar{e}'$ is removable in $\Gamma_4$. Note that $\Gamma'_3$ can be extended to a convex representation of $G$. This representation is not in general strictly convex but we can make it into a strictly convex representation $\Gamma''_3$ by moving the intermediate vertices of $P'_1, P'_2, \ldots, P'_r$ a little and, provided the displacements are sufficiently small, we may regard the above deformation of $\Gamma'_3$ into $\Gamma_4$ as a strictly convex deformation of $\Gamma''_3$ into a strictly convex representation of $G$ in which $e'$ is removable. We also make
the above displacements so small that $\tilde{e}_0$ is still contractible in $\Gamma''_3$ (note that these displacements cannot, in general, be achieved by a convex deformation and also note that in the resulting convex graph (where $e'$ is removable) it may happen that $e'_0$ is not contractible; however, this will not be needed). Now in each of $\Gamma''_0$ and $\Gamma''_3$ we can contract $\tilde{e}_0$ and obtain, by the induction hypothesis, $\Gamma''_0/\tilde{e}_0$ from $\Gamma''_0/\tilde{e}_0$ by a strictly convex deformation and so $\Gamma''_0$ can be obtained from $\Gamma'_0$ by a strictly convex deformation. This proves that there exists a strictly convex deformation of $\Gamma_0$ such that $e'$ becomes removable.

If $e_0$ has at most one end in $P' \cup P''$, the proof is similar (note that we may now have $e' = e_0$). Now $(G - x)/e_0$ satisfies (i)–(iii) (because $G/e_0$ does) and so, as above, we can represent $G - x$ as a strictly convex graph $\Gamma_3$ with outer polygon $\Sigma'$ such that $e_0$ is contractible in $\Gamma_3$ and, by the induction hypotheses, there exists a strictly convex deformation $\Gamma_4$ of $\Gamma_3$ such that $\tilde{e}'$ is removable in $\Gamma_4$ (also when $e' = e_0$). As above we define $\Gamma''_3$ such that $\Gamma''_3$ is a strictly convex representation of $G$ in which $\tilde{e}_0$ is contractible and such that $\Gamma''_0$ has a strictly convex deformation into a graph in which $\tilde{e}'$ is removable.

It remains to consider the case where $e_0$ joins a vertex of $P'$ with a vertex of $P''$ (see Fig. 4).

Now $e_0$ partitions $P' \cup P''$ into two paths which, together with $e_0$, form outer polygons of graphs $H_1, H_2$ satisfying (i)–(iii) (where the corners of the outer polygons are thought of as the corners of $\Sigma$, the ends of $e_0$ and the neighbours of $x$). Since each vertex of $G - V(S)$ has at least two neighbours in $G - V(S)$, we can assume that one of the graphs $H_1, H_2$ (say $H_1$) has an edge not in $P' \cup P'' \cup \{e_0\}$ and hence, by Proposition 2.1, $H_1$ has an edge $e''$

![Fig. 4. The case where $e_0$ joins $P'$ and $P''$.](image-url)
such that $H_1 - e''$ satisfies (i)-(iii). Then also $(G - x) - e''$ satisfies (i)-(iii) and so we can let $e''$ play the role of $e'$ in case $e' = e_0$. (We can still have $e_1 \neq e'$ because there are two possible choices for $e_2$.)

So we can assume $e_0 \neq e'$. The proof in this case is a slight modification of the proof in the two previous cases. Now, we represent $G$ as a convex graph $\Gamma_S$ with outer polygon $\Sigma$ such that $\Gamma_s - x$ is the union of two strictly convex graphs $\Gamma'_s$ and $\Gamma''_s$ having $\tilde{e}_0$ in common and such that $\tilde{e}_0$ is contractible in $\Gamma_s$ (this is indicated in Fig. 4, where $\Gamma'_s = \tilde{H}_1$ and $\Gamma''_s = \tilde{H}_2$). We can do this by [10, Theorem 5.1]. Now there exists a strictly convex deformation of $\Gamma'_s$ such that $\tilde{e}'$ becomes removable. By making small displacements of the vertices of $P'$ we obtain a strictly convex graph $\Gamma_0$ representing $G$ such that $\tilde{e}_0$ is contractible in $\Gamma_0$ and such that $\Gamma_0$ has a strictly convex deformation in which $\tilde{e}'$ is removable. By contracting $e_0$ we also get a strictly convex deformation of $\Gamma_0/\tilde{e}_0$ into $\Gamma'_0/\tilde{e}_0$ (and hence of $\Gamma_0$ into $\Gamma'_0$). In each case we have shown that there exists a strictly convex deformation (except for the ends of $e'$) of $\Gamma_0$ such that $\tilde{e}'$ is removable. Since the same can be proved for $\Gamma_1$ we can complete the proof by applying the induction hypothesis to $G - e'$.

We can now prove the conjecture of Grünbaum and Shepard.

**Theorem 3.4.** Let $G$ be a 2-connected planar graph and $S$ a facial cycle of $G$. Suppose $\Gamma_0$ and $\Gamma_1$ are strictly convex graphs isomorphic to $G$ such that, in both $\Gamma_0$ and $\Gamma_1$, $S$ is the outer polygon and that $S$ is strictly convex and has the same orientation in $\Gamma_0$ and $\Gamma_1$. Then $\Gamma_1$ may be regarded as a convex deformation of $\Gamma_0$.

**Proof** (by induction on $|E(G)| - |E(S)|$). If $G = S$ the theorem is easily proved. On the other hand, if $G \neq S$, then, by Proposition 2.1, there is an edge $e$ in $E(G) \setminus E(S)$ such that $G - e$ has a convex representation with outer polygon $S$. By Lemma 3.3 there is a strictly convex deformation (except possibly for the ends of $e$) $\Gamma'_0$ (resp. $\Gamma'_1$) of $\Gamma_0$ (resp. $\Gamma_1$) such that $\tilde{e}$ is removable in $\Gamma'_0$ and $\Gamma'_1$. Now the theorem follows by applying the induction hypothesis to $\Gamma'_0 - \tilde{e}$ and $\Gamma'_1 - \tilde{e}$.

Note that the condition of Theorem 3.4 is weaker than the condition in Grünbaum and Shepard's conjecture that there exists an orientation preserving homeomorphism taking $\Gamma_0$ onto $\Gamma_1$. On the other hand, Theorem 3.4 implies that the two conditions are equivalent.

### 4. Straight Deformations of Plane Graphs

The following result, when combined with Theorem 3.4, implies an analog of Theorem 3.4 for arbitrary straight graphs.
THEOREM 4.1. Let $\Gamma_0$ and $\Gamma_1$ be isomorphic straight graphs such that there exists an orientation preserving homeomorphism of the Euclidean plane taking $\Gamma_0$ onto $\Gamma_1$. Then $\Gamma_0$ and $\Gamma_1$ can be extended (by adding edges and vertices) to isomorphic triangulations $\Delta_0$ and $\Delta_1$ such that the outer cycle of $\Delta_0$ corresponds to the outer cycle of $\Delta_1$ and has the same orientation.

Proof. We first draw two triangles $\Sigma_0$ and $\Sigma_1$ containing $\Gamma_0$ (respectively $\Gamma_1$) in the interior. It is well known that we can extend $\Gamma_0 \cup \Sigma_0$ and $\Gamma_1 \cup \Sigma_1$ to isomorphic maximal planar graphs $\Gamma_0'$ (respectively $\Gamma_1'$) such that all edges are polygonal arcs. We may regard $\Gamma_0'$ and $\Gamma_1'$ as isomorphic straight graphs containing $\Gamma_0 \cup \Sigma_0$ and $\Gamma_1 \cup \Sigma_1$ (by regarding each polygonal arc as a path rather than an edge) so in order to complete the proof it is sufficient to show that the interior of two cycles, say $\Theta_0$ and $\Theta_1$, of the same length can be triangulated such that the resulting straight graphs are isomorphic.

It is easy to see that the interior of $\Theta_0$ can be triangulated by adding edges only. Whenever we add an edge $e$ to $\Theta_0$ we add the corresponding edge to $\Theta_1$ but as a polygonal arc with as few (say $p$) straight line segments as possible. We regard both $e$ and the corresponding polygonal arc inside $\Theta_1$ as paths of length $p$. We call the resulting graphs $\Theta_0'$ and $\Theta_1'$, respectively. We now consider any face of $\Theta_1'$ and triangulate it like we triangulated the interior of $\Theta_0$. Whenever we add an edge to $\Theta_1$ we add the corresponding edge to $\Theta_0'$ and we can do this by a straight line segment because all faces of $\Theta_0'$ are convex and, by the minimality of $p$, we never have to add an edge to $\Theta_0'$ joining two vertices on the same straight line segment. This completes the proof.

THEOREM 4.2. If $\Gamma_0$ and $\Gamma_1$ are isomorphic straight graphs such that there exists an orientation preserving homeomorphism of the plane taking $\Gamma_0$ onto $\Gamma_1$, then $\Gamma_1$ can be obtained from $\Gamma_0$ by a straight deformation.

Proof. By Theorem 4.1, we extend $\Gamma_0$ and $\Gamma_1$ to triangulations $\Delta_0$ and $\Delta_1$ and Theorem 4.2 now follows by applying Theorem 3.4 (or Cairn's result [4]) to these triangulations.

5. DEFORMATIONS OF RECTANGULOID CURVES

A rectanguloid curve is a simple closed plane curve made up of vertical and horizontal straight line segments. We start with any of the vertical straight line segments moving upwards and record the sequence of left and right turns. As pointed out by Robinson [9], a sequence of $r-s$ and $l-s$ (corresponding to "right" and "left", respectively) describe a rectanguloid curve anticlockwise if and only if it has four more $l-s$ than $r-s$ (see Fig. 5).
A rectanguloid deformation of a rectanguloid curve is defined in the obvious way. In Fig. 6 we indicate how part of a rectanguloid curve can be replaced by the dotted lines provided that the shaded rectangles do not intersect the curve. Clearly such deformations preserve the turns sequence. Conversely, Robinson [9] conjectured

**Theorem 5.1.** If two rectanguloid curves $C_1, C_2$ have the same turns sequence, then one can be obtained from the other by a rectanguloid deformation.

**Proof** (by induction on the length of the turns sequence). If the turns sequence is \( llll \) there is nothing to prove so assume this is not so. Consider a subsequence \( rll \) in the turns sequence and the corresponding part of $C_1$ (see Fig. 7a).

We first claim that there is a rectanguloid deformation of $C_1$ fixing the part of $C_1$ in Fig. 7a such that we obtain the configuration described in Fig. 7b such that the shaded rectangle does not intersect the curve. Having proved this claim the result follows easily: We deform each of $C_1$ and $C_2$ into $C'_1$ and $C'_2$ such that the situation of Fig. 7b arises. Then we replace part of the curves by the dotted line so that the subsequence \( rll \) is replaced by \( l \) and, by the induction hypothesis, one of the resulting curves can be obtained from the other by a rectanguloid deformation and it is easy to modify this deformation to a rectanguloid deformation of $C'_1$ into $C'_2$. 

![Fig. 6. Simple rectanguloid deformations.](image)
So it remains to prove our claim and this will also be done by induction on the length of the turns sequence.

The part of $C_1$ in Fig. 7a has the extension in Fig. 8a, b, or c. If the situation in Fig. 8a occurs, then $e$ can perform the desired deformation or else we can transform some subsequence $rll$ or $lrr$ to $l$ or $r$, respectively, by modifying a part of $C_1$ that intersects the shaded area and if Fig. 8b occurs (which means that $rll$ is succeeded by $r$), then there is another subsequence $rll$ in the turns sequence which also corresponds to the situation in Fig. 8a or b. We concentrate on that part instead and conclude that somewhere we can transform $rll$ to $l$ or $lrr$ to $r$ or $rll$ to $r$ or $lrl$ to $l$. (The two latter possibilities occur if we obtain the situation in Fig. 8b such that the shaded area does not intersect the curve.) The case of Fig. 8c is treated similarly. We leave the tedious details for the reader. In each case we obtain a curve with a shorter turns sequence and we prove our claim by induction.

**REFERENCES**