Available online at www.sciencedirect.com

SCIENCE DIRECT®

Topology and its Applications 153 (2006) 2386–2412

Topology and its Applications

www.elsevier.com/locate/topol

Compatible relations on filters and stability of local topological properties under supremum and product

Francis Jordan, Frédéric Mynard *

Department of Mathematical Sciences, Georgia Southern University, 0203 Georgia Ave., Room 3008, Statesboro, GA 30460-8093, USA

Received 7 December 2004; received in revised form 8 August 2005; accepted 22 August 2005

Abstract

An abstract scheme using particular types of relations on filters leads to general unifying results on stability under supremum and product of local topological properties. We present applications for Fréchetness, strong Fréchetness, countable tightness and countable fan-tightness, some of which recover or refine classical results, some of which are new. The reader may find other applications as well.

MSC: 54B10; 54A10; 54A20

Keywords: Product spaces; Fréchet; Strongly Fréchet and productively Fréchet spaces; Tightness; Fan-tightness; Absolute tightness; Tight points

1. Introduction

A large number of topological properties fail to be stable under finite products or even by supremum of topologies. Among such properties are a lot of fundamental local topological features such as Fréchetness, strong Fréchetness, countable tightness and countable fan-tightness, to cite a few (see the next section for definitions). Consequently, the quest

* Corresponding author.
E-mail addresses: fjordan@georgiasouthern.edu (F. Jordan), fmynard@georgiasouthern.edu (F. Mynard).

0166-8641/$ – see front matter
doi:10.1016/j.topol.2005.08.008

for conditions on the factor spaces to ensure that the product space has the desired local property has attracted a lot of attention, e.g., [1,8,12–15,18,21,24,23,25–31,16,17] for Fréchetness and strong Fréchetness alone; [1,4,2,5,6,21,20] for tightness and fan-tightness.

In this paper, we propose a unified approach to this class of problems, obtaining as byproducts of our theory both refinements and generalizations of known results and entirely new theorems. More specifically, we are interested in the following type of problem: Let $\mathcal{P}$ and $\mathcal{Q}$ be local topological properties.

Question 1. Characterize topological spaces X such that $X \times Y$ has property $\mathcal{Q}$ for every space Y with property $\mathcal{P}$.

In most cases, investigations have been restricted to $\mathcal{P} = \mathcal{Q}$.

The first crucial observation is that if $\mathcal{P}$ and $\mathcal{Q}$ are local topological properties, they are characterized by a property of the neighborhood filters. Of course, the property is not stable under product because the corresponding class of filters is not. The second crucial observation is that, even if these classes of filters behave badly with respect to the product operation, they are almost always defined via other better behaved classes of filters. For instance, a topological space is *Fréchet* if whenever a point x is in the closure of a subset A, there exists a sequence (equivalently a countably based filter) on A that converges to x.

If $A \subseteq X$, then the *principal filter of* A is $A^{\uparrow} = \{B \subseteq X\colon\ A \subseteq B\}$. Similarly, if $\mathcal{A} \subset 2^X$, we denote $\mathcal{A}^{\uparrow}$ the family of subsets of X that contains an element of $\mathcal{A}$. However, we will often identify subsets of X with their principal filters, that is, $A^{\uparrow}$ is often simply denoted A. Two collections of sets $\mathcal{F}$ and $\mathcal{G}$ *mesh*, in symbol $\mathcal{F} \# \mathcal{G}$, if $F \cap G \neq \emptyset$ for every $F \in \mathcal{F}$ and every $G \in \mathcal{G}$. The supremum $\mathcal{F} \vee \mathcal{G}$ of two collections of sets $\mathcal{F}$ and $\mathcal{G}$ exists when $\mathcal{F} \# \mathcal{G}$ and stands for the collection of intersections $\{F \cap G\colon\ F \in \mathcal{F} \text{ and } G \in \mathcal{G}\}$. However, when $\mathcal{F}$ and $\mathcal{G}$ are filters we will understand $\mathcal{F} \vee \mathcal{G}$ to be the filter $(\mathcal{F} \vee \mathcal{G})^{\uparrow}$ generated by the collection of intersections.

With these notations, the definition of Fréchetness in terms of closure rephrases in terms of neighborhood filter as follows: $\mathcal{F}$ is a Fréchet filter (a filter of neighborhood in a Fréchet space) if whenever $A \# \mathcal{F}$, there exists a countably based filter (equivalently, a sequence) $\mathcal{L}$ which is finer than $A \vee \mathcal{F}$. This definition depends on the class $\mathbb{F}_1$ of principal filters and on the class $\mathbb{F}_\omega$ of countably based filters, which are both productive.

We will take an approach based on relations between filters. For example, we consider the relation $\triangle$ on the set $\mathbb{F}(X)$ of filters on X defined by $\mathcal{F} \triangle \mathcal{H}$ if

$$\mathcal{H} \# \mathcal{F} \implies \exists \mathcal{L} \in \mathbb{F}_\omega\colon\ \mathcal{L} \geqslant \mathcal{F} \vee \mathcal{H}.$$

A filter is Fréchet if and only if it is in the $\triangle$-relation with every principal filter. In other words the class of Fréchet filters is $\mathbb{F}_1^{\triangle}$ [20,9] where we denote by $\mathbb{J}^{\star}$ the filters that are in the $\star$-relation with every filter of the class $\mathbb{J}$.

Under mild conditions on $\mathbb{J}$ and on relations $\star$ and $\square$, we characterize filters whose supremum (in Section 3) and whose product (Section 4) with a filter of the class $\mathbb{J}$ or $\mathbb{J}^{\star}$ is a filter of $\mathbb{J}^{\star}$ or $\mathbb{J}^{\square}$. The quests for stability under supremum and under product turn out to be intimately related (Section 4). These abstract results lead to a large collection of significant concrete corollaries, because most classical local topological properties, like

Fréchetness, can be characterized in terms of neighborhood filters of the type $\mathbb{J}^\star$ for classes $\mathbb{J}$ and relations $\star$ that fulfill the needed conditions.

In Section 4, we show how solutions for the problem of stability under product are obtained as an abstractly defined subclass, called kernel, of the class of solutions for the problem of stability under supremum. From the technical viewpoint, the difficulties (that attracted attention to this type of problems for so long) lie in the internal characterization of kernels. In Section 5, we characterize a variety of such kernels, obtaining as byproducts improvements of classical results as well as entirely new results. Finally, our abstract approach allows us to clarify (Section 6) the relationships between all these properties, improving again upon the known results.

We present applications related to three relations only, but the theory is designed for further applications with other examples of relations and of classes of filters.

2. Sup-compatible relations on $\mathbb{F}(X)$

Let R denote a relation on a set X ($R \subseteq X \times X$). As usual, $Rx = \{y \in X\colon (x, y) \in R\}$ and if $F \subseteq X$, $RF = \bigcup_{x\in F} Rx$. The *polar of* a subset F of X (with respect to R) is

$$F^R = \bigcap_{x\in F} Rx,$$

with the convention that $\emptyset^R = X$. An immediate consequence of the definitions is that

Lemma 2. *If R is a symmetric relation on X and $F \subseteq X$, then*

(1) $F \subseteq F^{RR}$;
(2) $F^{RRR} = F^R$.

Moreover, if $F \subseteq F^R$ then $F \subseteq F^{RR} \subseteq F^R$.

The symbol $\neg$ denotes negation. For instance, $x(\neg R)y$ means that $(x, y) \notin R$.

The set of filters on a given set X is denoted by $\mathbb{F}(X)$. A symmetric relation $\star$ on $\mathbb{F}(X)$ for which $\mathcal{F} \star \mathcal{G}$ whenever $\mathcal{F}$ and $\mathcal{G}$ do not mesh and which verifies

$$\mathcal{F} \star (\mathcal{G} \vee \mathcal{H}) \implies (\mathcal{F} \vee \mathcal{H}) \star \mathcal{G}, \tag{$\bullet$}$$

whenever $\mathcal{G} \# \mathcal{F}$ and $\mathcal{F} \# \mathcal{H}$ is called *sup-compatible* or $\vee$*-compatible*.

We are going to consider several particular classes of filters. In general $\mathbb{J}$ and $\mathbb{D}$ will denote generic classes of filters. A filter of the class $\mathbb{J}$ is called a $\mathbb{J}$*-filter*. The collection of $\mathbb{J}$-filters on X is denoted $\mathbb{J}(X)$. However, we will often omit X when the underlying set considered is clear. In particular, we will frequently consider the classes $\mathbb{F}_1$ and $\mathbb{F}_\omega$ of principal and countably based filters, respectively.

Example 3. The relation $\triangle$ on $\mathbb{F}(X)$ defined by $\mathcal{F} \triangle \mathcal{H}$ if

$$\mathcal{H} \# \mathcal{F} \implies \exists \mathcal{L} \in \mathbb{F}_\omega\colon \mathcal{L} \geqslant \mathcal{F} \vee \mathcal{H}, \tag{1}$$

is a $\vee$-compatible relation. Then $\mathbb{F}_1^{\triangle}$ is the class of *Fréchet filters* [20,9]. As noticed in the introduction, a topological space is Fréchet if and only if all its neighborhood filters are Fréchet.

Analogously, $\mathbb{F}_\omega^{\triangle}$ is the class of *strongly Fréchet filters* [20,9], that is, of filters $\mathcal{F}$ satisfying (1) for every countably based filter $\mathcal{H}$. Recall that a topological space X is *strongly Fréchet* if for every $x \in X$ and every decreasing countable collection $A_n \subseteq X$ such that $x \in \mathrm{cl}(A_n)$ for every n, there exists a sequence $x_n \in A_n$ that converges to x. It is easy to see that a topological space is strongly Fréchet if and only if all its neighborhood filters are strongly Fréchet.

Example 4. Consider the relation $\diamondsuit$ on $\mathbb{F}(X)$ defined by $\mathcal{F} \diamondsuit \mathcal{H}$ if

$$\mathcal{H} \# \mathcal{F} \implies \exists A \in \mathbb{F}_1,\ |A| \leqslant \omega\colon\ A \# \mathcal{F} \vee \mathcal{H}.$$

This is a $\vee$-compatible relation. Then $\mathbb{F}_1^{\diamondsuit}$ is the class of *countably tight filters*. Recall that a topological space X is countably tight [1] if for every $x \in X$ and $A \subseteq X$ such that $x \in \mathrm{cl}(A)$, there exists a countable subset $B \subseteq A$ such that $x \in \mathrm{cl}(B)$. It is easy to see that a space X is countably tight if and only if all its neighborhood filters are countably tight.

Example 5. A topological space is *countably fan-tight* [2] if for every countable family $(A_n)_\omega$ of subsets such that $x \in \bigcap_{n\in\omega} \mathrm{cl}(A_n)$, there exists finite subsets B_n of A_n such that $x \in \mathrm{cl}(\bigcup_{n\in\omega} B_n)$.

We call a filter $\mathcal{F}$ *countably fan-tight* if whenever $A_n \# \mathcal{F}$, there exists finite subsets B_n of A_n such that $\bigcup_{n\in\omega} B_n \# \mathcal{F}$. It was observed in [2, Remark 1] that the definition of fan-tightness is unchanged if we only consider *decreasing* countable collections $(A_n)_\omega$. Clearly, a space is countably fan-tight if and only if all its neighborhood filters are countably fan-tight. Consider the relation $\dagger$ on $\mathbb{F}(X)$ defined by $\mathcal{F} \dagger \mathcal{H}$ if

$$(A_n)_\omega \# (\mathcal{F} \vee \mathcal{H}) \implies \exists B_n \subseteq A_n,\ |B_n| < \omega,\ \left(\bigcup_{n\in\omega} B_n\right) \# \mathcal{F} \vee \mathcal{H}.$$

This is a $\vee$-compatible relation. By [2, Remark 1], $\mathcal{F} \dagger \mathcal{H}$ if for any *decreasing* countable filter base $(A_n)_\omega$ meshing with $\mathcal{F} \vee \mathcal{H}$, there exists finite sets $B_n \subseteq A_n$ such that $(\bigcup_{n\in\omega} B_n) \# \mathcal{F} \vee \mathcal{H}$.

Lemma 6. *The following are equivalent*:

(1) *$\mathcal{F}$ is countably fan tight*;
(2) *$\{E_k\}_{k\in\omega} \# \mathcal{F}$, then there exist finite sets $B_k \subseteq E_k$ such that $\{\bigcup_{n\leqslant k} B_k\}_{n\in\omega} \# \mathcal{F}$*;
(3) *If $\{E_k\}_{k\in\omega}$ is a decreasing countable filter base and $\{E_k\}_{k\in\omega} \# \mathcal{F}$, then there exist finite sets $B_k \subseteq E_k$ such that $\{\bigcup_{n\leqslant k} B_k\}_{n\in\omega} \# \mathcal{F}$*;
(4) *$\mathcal{F} \in \mathbb{F}_\omega^{\dagger}$*;
(5) *$\mathcal{F} \in \mathbb{F}_1^{\dagger}$*.

Proof. (1) $\Rightarrow$ (2). Fix $n \in \omega$. Since $E_k \# \mathcal{F}$ for all $k \geqslant n$, there exist finite sets $\{B_k^n\}_{k\geqslant n}$ such that $B_k^n \subseteq E_k$ and $(\bigcup_{k\geqslant n} B_k^n) \# \mathcal{F}$. For every $k \in \omega$ let $B_k = \bigcup_{k\geqslant n} B_k^n$. Notice $B_k \subseteq E_k$ and

B_k is finite for every $k \in \omega$. Let $n \in \omega$ and $F \in \mathcal{F}$. There is a $k \geqslant n$ such that $F \cap B_k^n \neq \emptyset$. Since $k \geqslant n$, we have $B_k^n \subseteq B_k$. Thus, $k \geqslant n$ and $B_k \cap F \neq \emptyset$. Therefore, $(\bigcup_{n \leqslant k} B_k) \# \mathcal{F}$ for every $n \in \omega$.

(2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (5) are straightforward. (5) $\Rightarrow$ (1) was observed in Example 5.

(3) $\Rightarrow$ (4) follows from [2, Remark 1]. Indeed, if $\mathcal{F}$ is as in (3), $\mathcal{H}$ and $\mathcal{A}$ are countably based filters with decreasing filter bases $(H_n)_{n\in\omega}$ and $(A_n)_{n\in\omega}$ respectively, such that $(A_n)_n \# (\mathcal{H} \vee \mathcal{F})$, then $(A_n \cap H_n)_n \# \mathcal{F}$. Therefore, there exists finite sets $B_n \subset A_n \cap H_n$ such that $\{\bigcup_{n\leqslant k} B_k\}_{n\in\omega} \# \mathcal{F}$. Clearly, $\{\bigcup_{n\leqslant k} B_k\}_{n\in\omega} \# (\mathcal{F} \vee \mathcal{H})$. In particular, $(\bigcup_{n\in\omega} B_n) \# (\mathcal{F} \vee \mathcal{H})$ and $\mathcal{F} \in \mathbb{F}_\omega^\dagger$. □

3. Stability of local properties under supremum

We call a class $\mathbb{J}$ of filters $(\mathbb{D}, \mathbb{M})$-*steady* if $\mathcal{F} \vee \mathcal{H} \in \mathbb{M}(X)$ whenever $\mathcal{F} \in \mathbb{J}(X)$ and $\mathcal{H} \in \mathbb{D}(X)$. If $\mathbb{J}$ is $(\mathbb{D}, \mathbb{J})$-steady we say that $\mathbb{J}$ is $\mathbb{D}$-*steady*. If $\mathbb{J}$ is $\mathbb{J}$-steady, we simply say that $\mathbb{J}$ is *steady*. By Lemma 2, we only generate the two classes $\mathbb{J}^\star$ and $\mathbb{J}^{\star\star}$ from a given class $\mathbb{J}$ by taking the polars with respect to a symmetric relation $\star$ on filters. We now investigate stability relationships between these classes under supremum. To begin, notice that an immediate consequence of the definitions is that a class of filters $\mathbb{J}$ is $(\mathbb{D}, \mathbb{M})$-*steady* if and only if $\mathbb{D}$ is $(\mathbb{J}, \mathbb{M})$-*steady*.

Proposition 7. *If $\mathbb{J}$ is a $\mathbb{D}$-steady class of filters, and if $\star$ and $\square$ are $\vee$-compatible relations, then $\mathbb{J}^\star$ and $\mathbb{J}^{\star\square}$ are both $\mathbb{D}$-steady.*

Proof. Let $\mathcal{F} \in \mathbb{J}^\star(X)$ and $\mathcal{H} \in \mathbb{D}(X)$ such that $\mathcal{F} \# \mathcal{H}$. We want to show that $\mathcal{F} \vee \mathcal{H} \in \mathbb{J}^\star$. Let $\mathcal{L}$ be a $\mathbb{J}$-filter such that $\mathcal{L} \# \mathcal{F} \vee \mathcal{H}$. As $\mathbb{J}$ is $\mathbb{D}$-steady, $\mathcal{L} \vee \mathcal{H} \in \mathbb{J}$. Thus, $\mathcal{F} \star (\mathcal{L} \vee \mathcal{H})$ because $\mathcal{F} \in \mathbb{J}^\star$. By (•), $(\mathcal{F} \vee \mathcal{H}) \star \mathcal{L}$. Hence, $\mathcal{F} \vee \mathcal{H} \in \mathbb{J}^\star$.

Let $\mathcal{F} \in \mathbb{J}^{\star\square}(X)$ and $\mathcal{H} \in \mathbb{D}(X)$ such that $\mathcal{F} \# \mathcal{H}$. We want to show that $\mathcal{F} \vee \mathcal{H} \in \mathbb{J}^{\star\square}$. Let $\mathcal{L}$ be a $\mathbb{J}^\star$-filter such that $\mathcal{L} \# \mathcal{F} \vee \mathcal{H}$. As we have proved that $\mathbb{J}^\star$ is $\mathbb{D}$-steady, $\mathcal{L} \vee \mathcal{H} \in \mathbb{J}^\star$. Thus, $\mathcal{F} \square (\mathcal{L} \vee \mathcal{H})$ because $\mathcal{F} \in \mathbb{J}^{\star\square}$. By (•), $(\mathcal{F} \vee \mathcal{H}) \square \mathcal{L}$. Hence, $\mathcal{F} \vee \mathcal{H} \in \mathbb{J}^{\star\square}$. □

Corollary 8. *If $\mathbb{J}$ is a steady class of filters and $\star$ and $\square$ are $\vee$-compatible relations, then $\mathbb{J}^\star$ is $(\mathbb{J}^{\star\square}, \mathbb{J}^\square)$-steady. In particular, $\mathbb{J}^\star$ is $\mathbb{J}^{\star\star}$-steady.*

Proof. Let $\mathcal{F} \in \mathbb{J}^\star(X)$ and $\mathcal{H} \in \mathbb{J}^{\star\square}(X)$ such that $\mathcal{F} \# \mathcal{H}$. We want to show that $\mathcal{F} \vee \mathcal{H} \in \mathbb{J}^\square$. Let $\mathcal{L}$ be a $\mathbb{J}$-filter such that $\mathcal{L} \# \mathcal{F} \vee \mathcal{H}$. As $\mathbb{J}^{\star\square}$ is $\mathbb{J}$-steady, $\mathcal{L} \vee \mathcal{H} \in \mathbb{J}^{\star\square}$, so that $(\mathcal{L} \vee \mathcal{H}) \square \mathcal{F}$. By (•), $(\mathcal{F} \vee \mathcal{H}) \square \mathcal{L}$. □

Corollary 9. *If $\mathbb{J}$ is a steady class of filters and $\star$, $\square$, and $\bigtriangledown$ are $\vee$-compatible relations, then $\mathbb{J}^{\square\bigtriangledown}$ is $(\mathbb{J}^{\star\square}, \mathbb{J}^{\star\bigtriangledown})$-steady. In particular, $\mathbb{J}^{\star\star}$ is steady.*

Proof. Let $\mathcal{F} \in \mathbb{J}^{\square\bigtriangledown}(X)$ and $\mathcal{H} \in \mathbb{J}^{\star\square}(X)$ such that $\mathcal{F} \# \mathcal{H}$. We want to show that $\mathcal{F} \vee \mathcal{H} \in \mathbb{J}^{\star\bigtriangledown}$. Let $\mathcal{L}$ be a $\mathbb{J}^\star$-filter such that $\mathcal{L} \# \mathcal{F} \vee \mathcal{H}$. As $\mathbb{J}^\star$ is $(\mathbb{J}^{\star\square}, \mathbb{J}^\square)$-steady, $\mathcal{L} \vee \mathcal{H} \in \mathbb{J}^\square$, so that $(\mathcal{L} \vee \mathcal{H}) \bigtriangledown \mathcal{F}$. By (•), $(\mathcal{F} \vee \mathcal{H}) \bigtriangledown \mathcal{L}$. □

Theorem 10. *Let $\star$ and $\square$ be two $\vee$-compatible relations on $\mathbb{F}(X)$ and let $\mathbb{J}$ be a steady class of filters containing $\mathbb{F}_1$.*

$$\begin{aligned}(1)\quad \mathcal{F}\in\mathbb{J}^{\star} &\iff \forall\mathcal{G}\in\mathbb{J},\ \mathcal{G}\#\mathcal{F}\implies \mathcal{F}\vee\mathcal{G}\in\mathbb{F}_1^{\star}\\ &\iff \forall\mathcal{G}\in\mathbb{J},\ \mathcal{G}\#\mathcal{F}\implies \mathcal{F}\vee\mathcal{G}\in\mathbb{J}^{\star}\\ &\iff \forall\mathcal{G}\in\mathbb{J}^{\star\star},\ \mathcal{G}\#\mathcal{F}\implies \mathcal{F}\vee\mathcal{G}\in\mathbb{J}^{\star}.\end{aligned}$$

$$\begin{aligned}(2)\quad \mathcal{F}\in\mathbb{J}^{\star\square} &\iff \forall\mathcal{G}\in\mathbb{J}^{\star},\ \mathcal{G}\#\mathcal{F}\implies \mathcal{F}\vee\mathcal{G}\in\mathbb{J}^{\star\star\square}\\ &\iff \forall\mathcal{G}\in\mathbb{J}^{\star},\ \mathcal{G}\#\mathcal{F}\implies \mathcal{F}\vee\mathcal{G}\in\mathbb{F}_1^{\star\star\square}\\ &\iff \forall\mathcal{G}\in\mathbb{J}^{\star},\ \mathcal{G}\#\mathcal{F}\implies \mathcal{F}\vee\mathcal{G}\in\mathbb{J}^{\square}\\ &\iff \forall\mathcal{G}\in\mathbb{J}^{\star},\ \mathcal{G}\#\mathcal{F}\implies \mathcal{F}\vee\mathcal{G}\in\mathbb{F}_1^{\square}.\end{aligned}$$

Proof. We show (1). If $\mathcal{F}\in\mathbb{J}^{\star}$ then

$$\forall\mathcal{G}\in\mathbb{J}^{\star\star},\ \mathcal{G}\#\mathcal{F}\implies \mathcal{F}\vee\mathcal{G}\in\mathbb{J}^{\star}$$

because $\mathbb{J}^{\star}$ is $\mathbb{J}^{\star\star}$-steady, by Corollary 8. The two other direct implications follow from $\mathbb{F}_1\subseteq\mathbb{J}\subseteq\mathbb{J}^{\star\star}$ (the last inclusion comes from Lemma 2).

Conversely, if $\mathcal{F}\notin\mathbb{J}^{\star}(X)$, then there exists $\mathcal{G}\in\mathbb{J}(X)$ such that $\mathcal{F}(\neg\star)\mathcal{G}$, or in other words, $\mathcal{F}(\neg\star)(\mathcal{G}\vee X)$. Therefore $(\mathcal{F}\vee\mathcal{G})(\neg\star)X$, so that $(\mathcal{F}\vee\mathcal{G})\notin\mathbb{F}_1^{\star}$.

We show (2). Let $\mathcal{F}\in\mathbb{J}^{\star\square}$ and $\mathcal{G}\in\mathbb{J}^{\star}$. Let $\mathcal{H}\in\mathbb{J}^{\star\star}$ and $\mathcal{H}\#(\mathcal{F}\vee\mathcal{G})$. Since $\mathcal{H}\#\mathcal{G}$, we have, by Corollary 8, $\mathcal{H}\vee\mathcal{G}\in\mathbb{J}^{\star}$. So, $\mathcal{F}\square(\mathcal{H}\vee\mathcal{G})$. By ($\bullet$), $(\mathcal{F}\vee\mathcal{G})\square\mathcal{H}$. So, $\mathcal{F}\vee\mathcal{G}\in\mathbb{J}^{\star\star\square}$. By the containments $\mathbb{J}^{\star\star\square}\subseteq\mathbb{J}^{\square}\subseteq\mathbb{F}_1^{\square}$ and $\mathbb{J}^{\star\star\square}\subseteq\mathbb{F}_1^{\star\star\square}\subseteq\mathbb{F}_1^{\square}$, $\mathcal{F}\in\mathbb{J}^{\star\square}$ implies any of the first three statements on the right and each of the first three statements on the right implies the last statement on the right.

Conversely, assume that $\mathcal{F}\notin\mathbb{J}^{\star\square}$. Then, there exists $\mathcal{H}\in\mathbb{J}^{\star}$ such that $\mathcal{H}\#\mathcal{F}$ but $\mathcal{H}(\neg\square)\mathcal{F}$. By ($\bullet$), $\mathcal{H}\vee X(\neg\square)\mathcal{F}$ and $\mathcal{F}\vee\mathcal{H}(\neg\square)X$. Thus, $\mathcal{F}\vee\mathcal{H}\notin\mathbb{F}_1^{\square}$. $\square$

A topological space is called $\mathbb{J}$-*based* if all its neighborhood filters are $\mathbb{J}$-filters. For instance, the $\mathbb{J}$-based spaces are respectively the *finitely generated* [19,22], first-countable, Fréchet, strongly Fréchet, countably tight, countably fan-tight spaces, when $\mathbb{J}$ is the class $\mathbb{F}_1$, $\mathbb{F}_\omega$, $\mathbb{F}_1^{\triangle}$, $\mathbb{F}_\omega^{\triangle}$, $\mathbb{F}_1^{\diamond}$ and $\mathbb{F}_1^{\dagger}$ respectively. We say that a class $\mathbb{J}$ of filters is called *pointable* if $\mathcal{F}\wedge\{x\}\in\mathbb{J}(X)$ whenever $\mathcal{F}\in\mathbb{J}(X)$, for every set X and every $x\in X$.

Corollary 11. *Let $\star$, $\bigtriangledown$, and $\square$ be $\vee$-compatible relations on filters. Let $\mathbb{J}$ be a pointable and steady class containing $\mathbb{F}_1$.*

(1) *Each of the following statements are equivalent*:
   (a) (X,τ) *is $\mathbb{J}^{\star}$-based,*
   (b) $(X,\tau\vee\xi)$ *is $\mathbb{J}^{\star}$-based for every $\mathbb{J}^{\star\star}$-based topology ξ on X,*
   (c) $(X,\tau\vee\xi)$ *is $\mathbb{J}^{\star}$-based for every $\mathbb{J}$-based topology ξ on X,*
   (d) $(X,\tau\vee\xi)$ *is $\mathbb{F}_1^{\star}$-based for every $\mathbb{J}$-based topology ξ on X.*
(2) *If $\mathbb{J}^{\star}$ is pointable then the following are equivalent*:
   (a) (X,τ) *is $\mathbb{J}^{\star\square}$-based,*

(b) $(X, \tau \vee \xi)$ *is* $\mathbb{J}^{\star\star\square}$*-based for every* $\mathbb{J}^{\star}$*-based topology* ξ *on* X,
(c) $(X, \tau \vee \xi)$ *is* $\mathbb{F}_1^{\star\star\square}$*-based for every* $\mathbb{J}^{\star}$*-based topology* ξ *on* X,
(d) $(X, \tau \vee \xi)$ *is* $\mathbb{J}^{\square}$*-based for every* $\mathbb{J}^{\star}$*-based topology* ξ *on* X,
(e) $(X, \tau \vee \xi)$ *is* $\mathbb{F}_1^{\square}$*-based for every* $\mathbb{J}^{\star}$*-based topology* ξ *on* X.

(3) *If* (X, τ) *is* $\mathbb{J}^{\square\triangledown}$*-based, then* $(X, \tau \vee \xi)$ *is* $\mathbb{J}^{\star\triangledown}$*-based for every* $\mathbb{J}^{\star\square}$*-based topology* ξ *on* X.

Proof. (1a) $\Rightarrow$ (1b) follows immediately from Theorem 10. (1b) $\Rightarrow$ (1c) $\Rightarrow$ (1d) follows from $\mathbb{J} \subseteq \mathbb{J}^{\star\star}$ and $\mathbb{J}^{\star} \subseteq \mathbb{F}_1^{\star}$ (because $\mathbb{F}_1 \subseteq \mathbb{J}$).

(1d) $\Rightarrow$ (1a). By way of contradiction, assuming that (X, τ) is not $\mathbb{J}^{\star}$-based, there exists $x \in X$ such that $\mathcal{N}_\tau(x) \notin \mathbb{J}^{\star}$. In view of the first part of Theorem 10, there exists $\mathcal{G} \in \mathbb{J}$ such that $\mathcal{G} \# \mathcal{N}_\tau(x)$ and $\mathcal{N}_\tau(x) \vee \mathcal{G} \notin \mathbb{F}_1^{\star}$. Let ξ be the topology on X with all points but x isolated with $\mathcal{N}_\xi(x) = \mathcal{G} \wedge \{x\}$. The space (X, ξ) is $\mathbb{J}$-based because $\mathbb{J}$ is pointable. So, by (1d), $\mathcal{N}_\tau(x) \vee (\mathcal{G} \wedge \{x\}) = \mathcal{N}_{\tau\vee\xi}(x) \in \mathbb{F}_1^{\star}$. We consider two cases.

If $x \notin \bigcap \mathcal{G}$, then $(\mathcal{N}_\tau(x) \vee (\mathcal{G} \wedge \{x\})) \# (X \setminus \{x\})$ and $\mathcal{G} = (\mathcal{G} \wedge \{x\}) \vee (X \setminus \{x\})$. So, $(\mathcal{N}_\tau\{x\} \vee (\mathcal{G} \wedge \{x\})) \star (X \setminus \{x\})$. Since $\star$ is $\vee$-compatible, $\mathcal{N}_\tau(x) \star (((\mathcal{G} \wedge \{x\}) \vee (X \setminus \{x\}))$. Thus, $\mathcal{N}_\tau\{x\} \star \mathcal{G}$, a contradiction.

If $x \in \bigcap \mathcal{G}$, then $\mathcal{G} \wedge \{x\} = \mathcal{G}$. In this case we have $\mathcal{N}_\tau(x) \vee \mathcal{G} \in \mathbb{F}_1^{\star}$ and $\mathcal{N}_\tau(x) \# \mathcal{G}$. Since $(\mathcal{N}_\tau(x) \vee \mathcal{G}) \# X$, $(\mathcal{N}_\tau(x) \vee \mathcal{G}) \star X$. By the $\vee$-compatibility of $\star$, $\mathcal{N}_\tau \star (\mathcal{G} \vee X)$. Thus, $\mathcal{N}_\tau(x) \star \mathcal{G}$, a contradiction.

Therefore, (X, τ) is $\mathbb{J}^{\star}$-based.

The second part is proved in a similar way.

The third part follows from Corollary 9. $\square$

A class $\mathbb{J}$ of filters is called $\mathbb{F}_1$-*composable* if the image of a $\mathbb{J}$-filter under a relation is a $\mathbb{J}$-filter.

Lemma 12.

(1) $\mathbb{F}_1$, $\mathbb{F}_\omega$, $\mathbb{F}_1^{\diamond}$, $\mathbb{F}_1^{\dagger}$, $\mathbb{F}_1^{\triangle}$, $\mathbb{F}_\omega^{\triangle}$ *are all* $\mathbb{F}_1$*-composable.*
(2) *An* $\mathbb{F}_1$*-composable class is pointable.*

Proof. For brevity we only show that $\mathbb{F}_1^{\dagger}$ is $\mathbb{F}_1$-composable, the other cases are similar and more straightforward. Let $\mathcal{F} \in \mathbb{F}_1^{\dagger}(X)$, Y be a set and $R \subseteq X \times Y$. Suppose $B \subseteq Y$ and $(A_n)_\omega \# (B \vee R\mathcal{F})$. Since $R^-(A_n)_\omega \# (\mathcal{F} \vee R^- B)$, there exist finite sets $K_n \subseteq R^- A_n$ such that $(\bigcup_\omega K_n) \# (\mathcal{F} \vee R^- B)$. For each n pick a finite $J_n \subseteq A_n \cap B$ such that $K_n \subseteq R^- J_n$. Let $F \in \mathcal{F}$. There is an n such that $K_n \cap F \neq \emptyset$. Since $K_n \subseteq R J_n$, $R J_n \# F$. So, $J_n \cap RF \neq \emptyset$. Thus, $(\bigcup_\omega J_n) \# (B \vee R\mathcal{F})$ and $R\mathcal{F} \dagger B$.

For the second part consider the relation $R = \{(x, x)\colon x \in X\} \cup (X \times \{p\})$. Then $\mathcal{F} \wedge (p) = R\mathcal{F}$. $\square$

We call a filter $\mathcal{F}$ *almost principal* if there exists $F_0 \in \mathcal{F}$ such that $|F_0 \setminus F| < \omega$ for every $F \in \mathcal{F}$. Principal filters and cofinite filters (of an infinite set) are almost principal. We use the same name for spaces based in such filters. Such spaces include sequences and one point compactifications of discrete sets. It is easily verified that every almost principal filter is Fréchet.

If $\mathcal{H} \in \mathbb{F}(X)$, we denote by $\mathcal{H}^\bullet$ the *principal part* $\bigcap \mathcal{H}$ of $\mathcal{H}$ and by $\mathcal{H}^\circ$ the *free part* $\mathcal{H} \vee (\mathcal{H}^\bullet)^c$. One or the other may be the degenerate filter $\{\emptyset\}^\uparrow$. With the convention, that $\mathcal{G} \wedge \{\emptyset\}^\uparrow = \mathcal{G}$ for any filter $\mathcal{G}$, we have

$$\mathcal{H} = \mathcal{H}^\circ \wedge \mathcal{H}^\bullet.$$

Theorem 13. $\mathbb{F}_1^{\triangle\triangle}$ *is exactly the class of almost principal filters.*

Proof. Let $\mathcal{F}$ be an almost principal filter and let $\mathcal{H} \# \mathcal{F}$ be a Fréchet filter. There exists $F_0 \in \mathcal{F}$ such that $|F_0 \setminus F| < \omega$ for every $F \in \mathcal{F}$. If $\mathcal{H}^\bullet (\neg\#) \mathcal{F}$, then $\mathcal{H}^\circ \# \mathcal{F}$. In particular, $\mathcal{H}^\circ \# F_0$. So, there exists a free sequence finer than $\mathcal{H} \vee F_0$, by Fréchetness of $\mathcal{H}$. This sequence is also finer than $\mathcal{F}$, hence finer than $\mathcal{F} \vee \mathcal{H}$. If $\mathcal{H}^\bullet \# \mathcal{F}$, then since $\mathcal{F}$ is Fréchet, there is a sequence finer than $\mathcal{H}^\bullet \vee \mathcal{F} \geqslant \mathcal{H} \vee \mathcal{F}$. Thus, $\mathcal{F} \in \mathbb{F}_1^{\triangle\triangle}$.

Conversely, if $\mathcal{F}$ is not almost principal, then for all $F \in \mathcal{F}$ there exists $H_F \in \mathcal{F}$ such that $|F \setminus H_F| \geqslant \omega$. Therefore, there exists a free sequence $(x_n^F)_n$ on $F \setminus H_F$. The filter $\bigwedge_{F \in \mathcal{F}} (x_n^F)_n$ is a Fréchet filter meshing with $\mathcal{F}$. If $(y_n)_n$ is finer than $\mathcal{F}$, then for every $F \in \mathcal{F}$, there exists k_F such that $\{y_n\colon n \geqslant k_F\} \subseteq H_F$. Therefore, there exists n_F such that $\{x_n^F\colon n \geqslant n_F\} \cap \{y_n\colon n \in \omega\} = \emptyset$. The set $\bigcup_{F \in \mathcal{F}} \{x_n^F\colon n \geqslant n_F\}$ is an element of $\bigwedge_{F \in \mathcal{F}} (x_n^F)_n$ disjoint from $\{y_n\colon n \in \omega\}$. Thus, $\mathcal{F} \notin \mathbb{F}_1^{\triangle\triangle}$. $\square$

When $\star = \square = \triangle$ and $\mathbb{J} = \mathbb{F}_1$, Corollary 11 rephrases as

Corollary 14.

(1) *The following are equivalent*:
   (a) (X, τ) *is Fréchet*;
   (b) $(X, \tau \vee \xi)$ *is Fréchet for every finitely generated topology* ξ *on* X;
   (c) $(X, \tau \vee \xi)$ *is Fréchet for every almost principal topology* ξ *on* X;
(2) *The following are equivalent*:
   (a) (X, τ) *is almost principal*;
   (b) $(X, \tau \vee \xi)$ *is Fréchet for every Fréchet topology* ξ *on* X.

No general condition was known to ensure that the supremum of two Fréchet topology is Fréchet, as noticed for instance in [7,6].

Recall that $\mathbb{F}_\omega^\triangle$ is the class of strongly Fréchet filters (or of neighborhood filters of strongly Fréchet spaces). We call *productively Fréchet* [16] the filters from the class $\mathbb{F}_\omega^{\triangle\triangle}$ and we use the same name for spaces based in such filters. When $\star = \square = \triangle$ and $\mathbb{J} = \mathbb{F}_\omega$, Corollary 11 rephrases as

Corollary 15.

(1) *The following are equivalent*:
   (a) (X, τ) *is strongly Fréchet*;

(b) $(X, \tau \vee \xi)$ *is Fréchet for every first-countable topology* ξ *on* X;
(c) $(X, \tau \vee \xi)$ *is strongly Fréchet for every first-countable topology* ξ *on* X;
(d) $(X, \tau \vee \xi)$ *is strongly Fréchet for every productively Fréchet topology* ξ *on* X;

(2) *The following are equivalent*:
(a) (X, τ) *is productively Fréchet*;
(b) $(X, \tau \vee \xi)$ *is Fréchet for every strongly Fréchet topology* ξ *on* X;
(c) $(X, \tau \vee \xi)$ *is strongly Fréchet for every strongly Fréchet topology* ξ *on* X.

Consider $\mathbb{J} = \mathbb{F}_1$ and $\star = \Diamond$. We call *steadily countably tight* filters of $\mathbb{F}_1^{\Diamond\Diamond}$ and we use the same name for spaces based in such filters. Notice that $\mathbb{F}_1^{\Diamond} = \mathbb{F}_\omega^{\Diamond}$.

Corollary 16.

(1) *The following are equivalent*:
(a) (X, τ) *is countably tight*;
(b) $(X, \tau \vee \xi)$ *is countably tight for every finitely generated topology* ξ *on* X;
(c) $(X, \tau \vee \xi)$ *is countably tight for every countably based topology* ξ *on* X;
(d) $(X, \tau \vee \xi)$ *is countably tight for every steadily countably tight topology* ξ *on* X;

(2) *The following are equivalent*:
(a) (X, τ) *is steadily countably tight*;
(b) $(X, \tau \vee \xi)$ *is countably tight for every countably tight topology* ξ *on* X.

Consider $\mathbb{J} = \mathbb{F}_1$ and $\star = \dagger$. We call *steadily countably fan-tight* filters of $\mathbb{F}_1^{\dagger\dagger}$ and we use the same name for spaces based in such filters. Recall from Lemma 6 that $\mathbb{F}_1^{\dagger} = \mathbb{F}_\omega^{\dagger}$.

Corollary 17.

(1) *The following are equivalent*:
(a) (X, τ) *is countably fan-tight*;
(b) $(X, \tau \vee \xi)$ *is countably fan-tight for every finitely generated topology* ξ *on* X;
(c) $(X, \tau \vee \xi)$ *is countably fan-tight for every countably based topology* ξ *on* X;
(d) $(X, \tau \vee \xi)$ *is countably fan-tight for every steadily countably fan-tight topology* ξ *on* X;

(2) *The following are equivalent*:
(a) (X, τ) *is steadily countably fan-tight*;
(b) $(X, \tau \vee \xi)$ *is countably fan-tight for every countably fan-tight topology* ξ *on* X.

To our knowledge, no general conditions ensuring that the supremum of two countably (fan) tight topologies is countably (fan) tight (as in Corollaries 16 and 17) was known.

As a sample example of what one may get from Corollary 11 by mixing relations we let $\mathbb{J} = \mathbb{F}_\omega$, $\star = \triangle$, $\Box = \dagger$, $\triangledown = \Diamond$.

Corollary 18.

(1) *The following are equivalent*:
  (a) (X,τ) *is based in* $\mathbb{F}_\omega^{\triangle\dagger}$;
  (b) $(X,\tau\vee\xi)$ *is based in* $\mathbb{F}_\omega^{\triangle\triangle\dagger}$ *for every strongly Fréchet topology* ξ *on* X;
  (c) $(X,\tau\vee\xi)$ *is based in* $\mathbb{F}_1^{\triangle\triangle\dagger}$ *for every strongly Fréchet topology* ξ *on* X;
  (d) $(X,\tau\vee\xi)$ *is countably fan tight for every strongly Fréchet topology* ξ *on* X;
(2) *If* (X,τ) *is based in* $\mathbb{F}_\omega^{\triangle\dagger}$, *then* $(X,\tau\vee\xi)$ *is based in* $\mathbb{F}_\omega^{\triangle\diamond}$ *for every* $\mathbb{F}_\omega^{\dagger\diamond}$*-based topology* ξ *on* X.

4. From steady to composable

If $\mathcal{F}$ is a filter on a set X, $\mathcal{G}$ is a filter on a set Y and $\mathcal{H}$ is a filter on $X\times Y$ such that $\mathcal{F}\times Y\,\#\,\mathcal{H}$ and $X\times\mathcal{G}\,\#\,\mathcal{H}$, we denote by $\mathcal{HF}$ the filter on Y generated by the sets

$$HF=\{y\colon\ \exists x\in F,\ (x,y)\in H\},$$

for $H\in\mathcal{H}$ and $F\in\mathcal{F}$ and by $\mathcal{H}^-\mathcal{G}$ the filter on X generated by the sets

$$H^-G=\{x\colon\ \exists y\in G,\ (x,y)\in H\},$$

for $H\in\mathcal{H}$ and $G\in\mathcal{G}$. Notice that

$$\mathcal{H}\,\#\,(\mathcal{F}\times\mathcal{G})\iff\mathcal{HF}\,\#\,\mathcal{G}\iff\mathcal{F}\,\#\,\mathcal{H}^-\mathcal{G}.$$

A class $\mathbb{J}$ of filters is $(\mathbb{D},\mathbb{M})$-*composable* if for every X and Y and every $\mathbb{J}$-filter $\mathcal{F}$ on X and every $\mathbb{D}$ filter $\mathcal{H}$ on $X\times Y$, the filter $\mathcal{HF}$ is an $\mathbb{M}$-filter on Y. A class $\mathbb{J}$ is called $\mathbb{D}$-*composable* [11,22] if it is $(\mathbb{D},\mathbb{J})$-composable. We say that a class of filters $\mathbb{J}$ is *projectable* provided that for every X and Y and every $\mathbb{J}$-filter $\mathcal{F}$ on $X\times Y$ we have $\pi_Y(\mathcal{F})\in\mathbb{J}$. Obviously, every $\mathbb{F}_1$-composable class of filters is projectable.

Given a class of filters $\mathbb{J}$, we define the *kernel* of $\mathbb{J}$ ($\ker(\mathbb{J})$) to be the class of $(\mathbb{F}_1,\mathbb{J})$-composable filters. Notice that $\ker(\mathbb{J})$ is the largest $\mathbb{F}_1$-composable subclass of $\mathbb{J}$.

Lemma 19. *If* $\mathbb{D}$ *is an* $\mathbb{F}_1$*-composable class of filters, then* $\mathbb{D}$ *is* $\mathbb{F}_1$*-steady, and* $\mathcal{D}\times A\in\mathbb{D}$ *for every* $\mathcal{D}\in\mathbb{D}$ *and every principal filter* A.

Proof. Let $A\subseteq X$ and $\mathcal{F}\in\mathbb{D}(X)$ be such that $\mathcal{F}\,\#\,A$. Since $\Delta_A=\{(x,x)\colon\ x\in A\}\in\mathbb{F}_1(X\times X)$ and $\mathbb{D}$ is $\mathbb{F}_1$-composable, $A\vee\mathcal{F}=\Delta_A\mathcal{F}\in\mathbb{D}$.

Now, if $\mathcal{D}\in\mathbb{D}(X)$ and $A\subset Y$, then $\mathcal{D}\times A=\pi_X^-\mathcal{D}\vee\pi_Y^-A\in\mathbb{D}$ because we already have shown that $\mathbb{D}$ is $\mathbb{F}_1$-steady. □

Theorem 20.

(1) *Let* $\mathbb{M}$ *be an* $\mathbb{F}_1$*-steady and projectable class. If* $\mathcal{J}\times\mathcal{D}\in\mathbb{M}$ *for every* $\mathcal{J}\in\mathbb{J}$ *and every* $\mathcal{D}\in\mathbb{D}$, *then* $\mathbb{J}$ *is* $(\mathbb{D},\mathbb{M})$*-steady.*
(2) *Let* $\mathbb{M}$ *be projectable and* $\mathbb{J}$ *be* $\mathbb{F}_1$*-composable. If* $\mathbb{J}$ *is* $(\mathbb{D},\mathbb{M})$*-steady then* $\mathbb{J}$ *is* $(\mathbb{D},\mathbb{M})$*-composable.*

(3) *Let $\mathbb{D}$ be an $\mathbb{F}_1$-composable class. If $\mathbb{J}$ is $(\mathbb{D},\mathbb{M})$-composable, then $\mathcal{J}\times\mathcal{D}\in\mathbb{M}$ for every $\mathcal{J}\in\mathbb{J}$ and every $\mathcal{D}\in\mathbb{D}$.*

(4) *Let $\mathbb{M}$ be an $\mathbb{F}_1$-steady and projectable class. If either $\mathbb{D}$ or $\mathbb{J}$ is $\mathbb{F}_1$-composable then $\mathcal{J}\times\mathcal{D}\in\mathbb{M}$ for every $\mathcal{J}\in\mathbb{J}$ and every $\mathcal{D}\in\mathbb{D}$ if and only if $\mathcal{J}\times\mathcal{D}\in\ker(\mathbb{M})$ for every $\mathcal{J}\in\mathbb{J}$ and every $\mathcal{D}\in\mathbb{D}$.*

Proof. Proof of (1). Let $\mathcal{D}\in\mathbb{D}(X)$ and $\mathcal{J}\in\mathbb{J}(X)$ such that $\mathcal{D}\#\mathcal{J}$. Then $\mathcal{D}\times\mathcal{J}\in\mathbb{M}(X\times X)$. Let Δ be the diagonal of $X\times X$. As $\mathbb{M}$ is $\mathbb{F}_1$-steady, $(\mathcal{D}\times\mathcal{J})\vee\Delta\in\mathbb{M}(X\times X)$. As $\mathbb{M}$ is projectable, the X-projection of $(\mathcal{D}\times\mathcal{J})\vee\Delta$ is an $\mathbb{M}$-filter. We conclude with the observation that $\pi_X((\mathcal{D}\times\mathcal{J})\vee\Delta)=\mathcal{D}\vee\mathcal{J}$.

Proof of (2). Let $\mathcal{J}\in\mathbb{J}(X)$ and let $\mathcal{H}\in\mathbb{D}(X\times Y)$. The filter $\mathcal{J}\times Y$ is a $\mathbb{J}$-filter because $\mathbb{J}$ is $\mathbb{F}_1$-composable. Therefore $(\mathcal{J}\times Y)\vee\mathcal{H}$ is an $\mathbb{M}$-filter. As $\mathbb{M}$ is projectable, $\pi_Y((\mathcal{J}\times Y)\vee\mathcal{H})\in\mathbb{M}(Y)$. We conclude with the observation that $\mathcal{H}(\mathcal{J})=\pi_Y((\mathcal{J}\times Y)\vee\mathcal{H})$.

Proof of (3). Let $\mathcal{J}\in\mathbb{J}(X)$ and $\mathcal{D}\in\mathbb{D}(Y)$. Let Δ_X be the diagonal of $X\times X$, and consider the filter $\Delta_X\times\mathcal{D}$ as a filter of relations from X to $X\times Y$, that is, a filter on $X\times(X\times Y)$. This is a $\mathbb{D}$-filter by Lemma 19, because $\mathbb{D}$ is $\mathbb{F}_1$-composable. Moreover, $\mathcal{J}\times\mathcal{D}=(\Delta_X\times\mathcal{D})(\mathcal{J})$, so that $\mathcal{J}\times\mathcal{D}\in\mathbb{M}(X\times Y)$.

Proof of (4). Let $\mathcal{J}\in\mathbb{J}(X)$, $\mathcal{D}\in\mathbb{D}(Y)$ and $A\subset X\times Y\times Z$. We want to show that $A(\mathcal{J}\times\mathcal{D})\in\mathbb{M}(Z)$. If either $\mathbb{D}$ or $\mathbb{J}$ is $\mathbb{F}_1$-composable, then either $\mathcal{J}\times Z\in\mathbb{J}(X\times Z)$ or $\mathcal{D}\times Z\in\mathbb{D}(Y\times Z)$. In any case, $\mathcal{J}\times\mathcal{D}\times Z\in\mathbb{M}(X\times Y\times Z)$. Moreover, $\pi_Z(A\vee(\mathcal{J}\times\mathcal{D}\times Z))\in\mathbb{M}(Z)$ because $\mathbb{M}$ is both $\mathbb{F}_1$-steady and projectable. We conclude with the observation that $A(\mathcal{J}\times\mathcal{D})=\pi_Z(A\vee(\mathcal{J}\times\mathcal{D}\times Z))$. $\square$

The following diagram summarizes these relationships between stability under product, composability and steadiness. Notice that $\ker(\mathbb{M})$ is $\mathbb{F}_1$-composable, hence $\mathbb{F}_1$-steady and projectable.

$$\begin{tikzcd}
 & \mathbb{D}(\mathbb{J})\subset\mathbb{M} \arrow[dl, "\mathbb{D}\text{ is }\mathbb{F}_1\text{-composable}"'] \\
\mathbb{D}\times\mathbb{J}\subset\mathbb{M} \arrow[r] \arrow[d, shift left, "\substack{\mathbb{M}\text{ is }\mathbb{F}_1\text{-steady and projectable}\\ \mathbb{D}\text{ or }\mathbb{J}\text{ is }\mathbb{F}_1\text{-composable}}"] & \mathbb{J}\vee\mathbb{D}\subset\mathbb{M} \arrow[u, "\substack{\mathbb{M}\text{ projectable}\\ \mathbb{J}\text{ is }\mathbb{F}_1\text{-composable}}"'] \\
\mathbb{D}\times\mathbb{J}\subset\ker(\mathbb{M}) \arrow[u, shift left] \arrow[r] & \mathbb{J}\vee\mathbb{D}\subset\ker(\mathbb{M}) \arrow[d, "\mathbb{J}\text{ is }\mathbb{F}_1\text{-composable}"] \\
 & \mathbb{D}(\mathbb{J})\subset\ker(\mathbb{M}) \arrow[ul, "\mathbb{D}\text{ is }\mathbb{F}_1\text{-composable}"]
\end{tikzcd}$$

Corollary 21. *Let $\mathbb{J}$ and $\mathbb{D}$ be two $\mathbb{F}_1$-composable classes of filters, and $\mathbb{M}$ be an $\mathbb{F}_1$-steady projectable class of filters. The following are equivalent*:

(1) $\mathbb{J}$ *is* $(\mathbb{D},\mathbb{M})$*-composable*;
(2) $\mathbb{J}$ *is* $(\mathbb{D},\ker(\mathbb{M}))$*-composable*;
(3) $\mathbb{J}$ *is* $(\mathbb{D},\ker(\mathbb{M}))$*-steady*;
(4) $\mathbb{J}$ *is* $(\mathbb{D},\mathbb{M})$*-steady*;
(5) $(\mathcal{F}\times\mathcal{L})\in\ker(\mathbb{M})$ *whenever* $\mathcal{L}\in\mathbb{D}$ *and* $\mathcal{F}\in\mathbb{J}$;
(6) $(\mathcal{F}\times\mathcal{L})\in\mathbb{M}$ *whenever* $\mathcal{L}\in\mathbb{D}$ *and* $\mathcal{F}\in\mathbb{J}$.

In particular, $\mathbb{F}_\omega, \mathbb{F}_\omega^{\triangle}, \mathbb{F}_\omega^{\triangle\triangle}$ are all $\mathbb{F}_1$-composable, so that steady and composable are the same for these classes. Therefore, Corollary 15 combines with Corollary 21 to the effect that

Proposition 22. [21,16] *The following are equivalent*:

(1) *X is strongly Fréchet*;
(2) *$X\times Y$ is strongly Fréchet for every productively Fréchet space Y*;
(3) *$X\times Y$ is Fréchet for every first-countable space Y.*

Proposition 23. [16] *X is productively Fréchet if and only if $X\times Y$ is Fréchet* (*equivalently strongly Fréchet*) *for every strongly Fréchet space Y.*

G. Gruenhage brought to our attention that another characterization of productively Fréchet spaces (under the name of $\mathcal{PP}$ spaces) was contained in [14], providing an earlier solution to the problem studied in [16].

Analogously, $\mathbb{F}_1^{\triangle}$ (the class of Fréchet filters) is $\mathbb{F}_1$-composable. From Theorem 10 applied with $\star=\triangle$ and $\mathbb{J}=\mathbb{F}_1$ we get

Proposition 24. [22] *A topological space is Fréchet if and only if its product with every finitely generated space is Fréchet.*

The classes $\mathbb{F}_1$, $\mathbb{F}_\omega$, and $\mathbb{F}_\omega^{\dagger}$, and $\mathbb{F}_1^{\diamond}$ are $\mathbb{F}_1$-composable, so that steady and composable are the same for these classes. Therefore, Corollaries 16 and 17 combine with Corollary 21 to the effect that:

Proposition 25. *X is countably tight if and only if $X\times Y$ is countably tight for every countably based Y.*

Proposition 26. *X is countably fan tight if and only if $X\times Y$ is countably fan tight for every countably based space Y.*

In contrast, we will see that the classes $\mathbb{F}_1^{\diamond\diamond}$, $\mathbb{F}_1^{\triangle\diamond}$, $\mathbb{F}_1^{\dagger\diamond}$, $\mathbb{F}_\omega^{\triangle\diamond}$ and $\mathbb{F}_1^{\triangle\triangle}$ are not $\mathbb{F}_1$-composable.[1] Hence, to investigate stability under product of the associated properties, we need to introduce more machinery.

[1] We do not know if $\mathbb{F}_\omega^{\triangle\dagger}$ and $\mathbb{F}_1^{\dagger\dagger}$ are $\mathbb{F}_1$-composable.

A $\vee$-compatible relation on filters (that is, defined on $\mathbb{F}(X)$ for any set X) is $\times$-*compatible* provided that

$$(\mathcal{F} \times \mathcal{G}) \star A \implies \mathcal{G} \star A\mathcal{F}. \tag{$\bullet\bullet$}$$

The proof of the following lemma is left to the reader.

Lemma 27. *$\triangle$, $\diamond$ and $\dagger$ are all $\times$-compatible.*

Lemma 28. *If $\mathbb{J}$ is an $\mathbb{F}_1$-composable class of filters and $\star$ is a $\times$-compatible relation, then $\mathbb{J}^\star$ is $\mathbb{F}_1$-steady and projectable.*

Proof. The fact that $\mathbb{J}^\star$ is $\mathbb{F}_1$-steady follows from Proposition 7. Let $\mathcal{F} \in \mathbb{J}^\star(X \times Y)$ and $\mathcal{G} \in \mathbb{J}(Y)$. Since $X \times \mathcal{G} \in \mathbb{J}(X \times Y)$, $(X \times \mathcal{G}) \star \mathcal{F}$. Since $\star$ is $\times$-compatible, $\mathcal{F}X \star \mathcal{G}$. However, $\mathcal{F}X = \pi_Y(\mathcal{F})$. Thus, $\pi_Y(\mathcal{F}) \in \mathbb{J}^\star$. $\square$

Theorem 29. *Let $\mathbb{J}$ be a steady $\mathbb{F}_1$-composable class of filters containing $\mathbb{F}_1$, and let $\star$, $\square$, and $\triangledown$ be $\times$-compatible relations.*

(1) *The following are equivalent*:
   (a) $\mathcal{F} \in \ker(\mathbb{J}^\star)$;
   (b) $(\mathcal{F} \times \mathcal{G}) \in \ker(\mathbb{J}^\star)$ *for every* $\mathcal{G} \in \ker(\mathbb{J}^{\star\star})$;
   (c) $(\mathcal{F} \times \mathcal{G}) \in \mathbb{J}^\star$ *for every* $\mathcal{G} \in \mathbb{J}$;
   (d) $(\mathcal{F} \times \mathcal{G}) \in \mathbb{F}_1^\star$ *for every* $\mathcal{G} \in \mathbb{J}$.
(2) *If $\mathbb{J}^\star$ is $\mathbb{F}_1$-composable and contains $\mathbb{F}_1$, then the following are equivalent*:
   (a) $\mathcal{F} \in \ker(\mathbb{J}^{\star\square})$;
   (b) $(\mathcal{F} \times \mathcal{G}) \in \ker((\ker(\mathbb{J}^{\star\star}))^\square)$ *for every* $\mathcal{G} \in \mathbb{J}^\star$;
   (c) $(\mathcal{F} \times \mathcal{G}) \in \mathbb{J}^\square$ *for every* $\mathcal{G} \in \mathbb{J}^\star$;
   (d) $(\mathcal{F} \times \mathcal{G}) \in \mathbb{F}_1^\square$ *for every* $\mathcal{G} \in \mathbb{J}^\star$.
(3) *Let $\mathbb{J}^\star$ be an $\mathbb{F}_1$-composable class of filters. If $\mathcal{F} \in \ker(\mathbb{J}^{\triangledown\square})$ then $(\mathcal{F} \times \mathcal{G}) \in \ker(\mathbb{J}^{\star\square})$ for every $\mathcal{G} \in \ker(\mathbb{J}^{\star\triangledown})$.*

Proof. We prove (1).

(1a) $\Rightarrow$ (1b). By Theorem 10 (1), $\mathcal{F} \vee \mathcal{G} \in \mathbb{J}^\star$ for every $\mathcal{G} \in \mathbb{J}^{\star\star}$, hence in particular for every $\mathcal{G} \in \ker(\mathbb{J}^{\star\star})$. Since $\mathbb{J}$ is $\mathbb{F}_1$-composable, Lemma 28 applies to the effect that $\mathbb{J}^\star$ is $\mathbb{F}_1$-steady and projectable. Moreover, $\ker(\mathbb{J}^\star)$ and $\ker(\mathbb{J}^{\star\star})$ are $\mathbb{F}_1$-composable by definition. In view of Corollary 21, (1b) follows.

(1b) $\Rightarrow$ (1c) because $\mathbb{J} \subset \ker(\mathbb{J}^{\star\star})$ and $\ker(\mathbb{J}^\star) \subset \mathbb{J}^\star$. Similarly, (1c) $\Rightarrow$ (1d) because $\mathbb{J}^\star \subset \mathbb{F}_1^\star$.

(1d) $\Rightarrow$ (1a). Let $A \subset X \times Y$. We want to show that $A\mathcal{F} \in \mathbb{J}^\star$. If $\mathcal{G} \in \mathbb{J}(Y)$ meshes with $A\mathcal{F}$ then $A \,\#\, (\mathcal{F} \times \mathcal{G})$. But $\mathcal{F} \times \mathcal{G} \in \mathbb{F}_1^\star$, so that $(\mathcal{F} \times \mathcal{G}) \star A$. By ($\bullet\bullet$), we have $\mathcal{G} \star A\mathcal{F}$, so that $A\mathcal{F} \in \mathbb{J}^\star$.

(2a) $\Rightarrow$ (2b) is proved from Theorem 10 (2), combined with Lemma 28 and Corollary 21 as (1a) $\Rightarrow$ (1b). (2b) $\Rightarrow$ (2c) $\Rightarrow$ (2d) follows from inclusions among the classes of filters considered and (2d) $\Rightarrow$ (2a) follows from ($\bullet\bullet$).

Finally (3) follows in a similar way from Corollary 9. $\square$

In the very same way that we deduced Corollary 11 from Theorem 10, we deduce the following from Theorem 29.

Corollary 30. *Let $\mathbb{J}$ be a steady $\mathbb{F}_1$-composable class containing $\mathbb{F}_1$ and let $\star$ and $\square$ be two $\times$-compatible relations.*

(1) *The following are equivalent*:
  (a) *X is $\ker(\mathbb{J}^{\star})$-based*;
  (b) *$X \times Y$ is $\ker(\mathbb{J}^{\star})$-based for every $\ker(\mathbb{J}^{\star\star})$-based Y*;
  (c) *$X \times Y$ is $\mathbb{J}^{\star}$-based for every $\mathbb{J}$-based Y*;
  (d) *$X \times Y$ is $\mathbb{F}_1^{\star}$-based for every $\mathbb{J}$-based Y.*
(2) *If $\mathbb{J}^{\star}$ is $\mathbb{F}_1$-composable and contains $\mathbb{F}_1$, then the following are equivalent*:
  (a) *X is $\ker(\mathbb{J}^{\star\square})$-based*;
  (b) *$X \times Y$ is $\ker((\ker(\mathbb{J}^{\star\star}))^{\square})$-based for every $\mathbb{J}^{\star}$-based Y*;
  (c) *$X \times Y$ is $\mathbb{J}^{\square}$-based for every $\mathbb{J}^{\star}$-based Y*;
  (d) *$X \times Y$ is $\mathbb{F}_1^{\square}$-based for every $\mathbb{J}^{\star}$-based Y.*
(3) *If $\mathbb{J}^{\star}$ is $\mathbb{F}_1$-composable and contains $\mathbb{F}_1$, and if X is $\ker(\mathbb{J}^{\triangledown\square})$-based, then $X \times Y$ is $\ker(\mathbb{J}^{\star\square})$-based for every $\ker(\mathbb{J}^{\star\triangledown})$-based Y.*

5. Characterizations of kernels

Now the missing point to apply the previous results to theorems of stability of local topological properties like tightness or fan-tightness under product is to characterize kernels of the corresponding classes of filters.

Theorem 31.

$$\ker\bigl(\mathbb{F}_1^{\triangle\triangle}\bigr) = \mathbb{F}_1.$$

Proof. $\mathbb{F}_1 \subseteq \ker(\mathbb{F}_1^{\triangle\triangle})$ is clear. Conversely, if X is a set and $\mathcal{F} \notin \mathbb{F}_1(X)$, then for every $F \in \mathcal{F}$, there exists $H_F \in \mathcal{F}$ and $x_F \in F \setminus H_F$. Let Y be an infinite set. Then $\mathcal{F} \times Y$ is not an almost principal filter because the sets $\{x_F\} \times Y$ are infinite. By Theorem 13, $\mathcal{F} \times Y \notin \mathbb{F}_1^{\triangle\triangle}$, and $\mathcal{F} \notin \ker(\mathbb{F}_1^{\triangle\triangle})$. $\square$

As a consequence, we obtain the dual statement to Proposition 24.

Corollary 32. [22] *A topological space is finitely generated if and only if its product with every Fréchet space is Fréchet.*

5.1. κ-tightness, productive κ-tightness, tight points, and absolute countable tightness

For our discussion of kernels for tightness we consider the following relation. Let κ be an infinite cardinal and define the relation $\diamondsuit_\kappa$ on $\mathbb{F}(X)$ by $\mathcal{F} \diamondsuit_\kappa \mathcal{H}$ if

$$\mathcal{H} \# \mathcal{F} \implies \exists A \in \mathbb{F}_1,\ |A| \leqslant \kappa\colon\ A \# \mathcal{F} \vee \mathcal{H}.$$

This is a $\vee$-compatible and $\times$-compatible relation. $\mathbb{F}_1^{\diamondsuit_\kappa}$ is the class of *κ-tight filters*. Recall that a topological space X has tightness κ [1] if for every $x \in X$ and $A \subseteq X$ such that $x \in \mathrm{cl}(A)$, there exists a subset $B \subseteq A$ such that $|B| \leqslant \kappa$ and $x \in \mathrm{cl}(B)$. It is easy to see that a space X has κ-tightness if and only if all its neighborhood filters are κ-tight.

Now we give a general result for kernels of $\diamondsuit_\kappa$-polars for classes of filters satisfying certain conditions.

Let I and J be sets. A function $\gamma : I \times J \to 2^X$ is called a *presentation on X*. We define

$$\gamma_* = \left\{ \bigcup_{\alpha \in I} \gamma(\alpha, \beta) \colon \beta \in J \right\}^{\uparrow}.$$

If $\mathcal{G}$ is a filter on X and $\mathcal{G} = \gamma_*$, then we say that γ is a *presentation of $\mathcal{G}$*. Of course, if γ_* is a filter, then γ is a presentation of γ_*. A presentation $\gamma : I \times J \to 2^X$ is called $\mathbb{J}$-*expandable* if the filter γ^* defined on $X \times I$ by

$$\gamma^* = \left\{ \left\{ \bigcup_{\alpha \in I} \gamma(\alpha, \beta) \times \{\alpha\} \right\} \colon \beta \in J \right\}^{\uparrow}$$

is a $\mathbb{J}$-filter on $X \times I$. Notice that $\gamma_* = \pi_X(\gamma^*)$. Therefore, if $\mathbb{J}$ is projectable and $\gamma : I \times J \to 2^X$ is a $\mathbb{J}$-expandable presentation on X, then $\gamma_* \in \mathbb{J}(X)$.

Theorem 33. *Let $\mathbb{J}$ be an $\mathbb{F}_1$-composable class of filters included in $\mathbb{F}_1^{\diamondsuit_\kappa}$. The following are equivalent*:

(1) $\mathcal{F} \in \ker(\mathbb{J}^{\diamondsuit_\kappa})$;
(2) $\mathcal{F} \in \mathbb{J}^{\diamondsuit_\kappa}$ *and* $A \vee \mathcal{F} \in \ker(\mathbb{J}^{\diamondsuit_\kappa})$ *for all A such that $A \# \mathcal{F}$ and $|A| \leqslant \kappa$*;
(3) *For any $\mathbb{J}$-expandable presentation $\gamma : I \times J \to 2^X$ such that $\gamma_* \# \mathcal{F}$, there exists a subset C of I of cardinality at most κ such that $\{\bigcup_{\alpha \in C} \gamma(\alpha, \beta) \colon \beta \in J\} \# \mathcal{F}$*;
(4) *For any $\mathbb{J}$-expandable presentation $\gamma : \mathcal{F} \times \mathcal{J} \to 2^X$ such that $\gamma_* \# \mathcal{F}$, where $\mathcal{J}$ is a $\mathbb{J}$-filter, each set $\gamma(F, J)$ is a subset of F of cardinality at most κ and each $\gamma(F, \mathcal{I})$ is a $\mathbb{J}$-filter, there exists a subfamily $\mathcal{K}$ of $\mathcal{F}$ of cardinality at most κ such that $\{\bigcup_{F \in \mathcal{K}} \gamma(F, J) \colon J \in \mathcal{J}\} \# \mathcal{F}$.*

Proof. (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (3). Let $\gamma : I \times J \to 2^X$ be a $\mathbb{J}$-expandable presentation such that $\gamma_* \# \mathcal{F}$. As $\mathcal{F} \in \mathbb{J}^{\diamondsuit_\kappa}$, there exists A of cardinality at most κ such that $A \# \mathcal{F} \vee \gamma_*$ or equivalently, $\mathcal{F} \vee A \# \gamma_*$. By $\mathbb{J}$-expandability of γ, the filter γ^* is a $\mathbb{J}$-filter on $X \times I$. Moreover $\pi_X^-(\mathcal{F} \vee A) \in \mathbb{J}^{\diamondsuit_\kappa}(X \times I)$ because $\mathcal{F} \vee A \in \ker(\mathbb{J}^{\diamondsuit_\kappa})$; and $\pi_X^-(\mathcal{F} \vee A) \# \gamma^*$. Thus, there exists a set D of cardinality at most κ such that $D \# \pi_X^-(\mathcal{F} \vee A) \vee \gamma^*$. Then $\{\bigcup_{\alpha \in \pi_I(D)} \gamma(\alpha, \beta) \colon \beta \in J\} \# \mathcal{F}$.

(3) $\Rightarrow$ (4) is obvious.

(4) $\Rightarrow$ (1). Let $A \subset X \times Y$ and let $\mathcal{G}$ be a $\mathbb{J}$-filter on Y such that $\mathcal{G} \# A\mathcal{F}$. For every $F \in \mathcal{F}$, $\mathcal{G} \# AF$ and $\mathcal{G} \in \mathbb{F}_1^{\diamondsuit_\kappa}$ so that there exists a subset B_F of AF of cardinality at most κ and meshing with $\mathcal{G}$. For each $F \in \mathcal{F}$, there exists a function $f_F : B_F \to F$ with graph included in A. Consider the function $\gamma : \mathcal{F} \times \mathcal{G} \to 2^X$ defined by $\gamma(F, G) = f_F(G \cap B_F)$. It is a $\mathbb{J}$-expandable presentation, because $\mathbb{J}$ is $\mathbb{F}_1$-composable and $\{\bigcup_{F \in \mathcal{F}} \gamma(F, G) \times \{F\} \colon G \in \mathcal{G}\}^{\uparrow}$ is the image of the $\mathbb{J}$-filter $\mathcal{G}$ under the multivalued map $R : Y \rightrightarrows X \times \mathcal{F}$ with

graph $\{(y,x,F)\colon\ f_F(x)=y\}$. By (4), there is a subfamily $\mathcal{K}$ of $\mathcal{F}$ of cardinality at most κ such that $\{\bigcup_{F\in\mathcal{K}}\gamma(F,G)\colon\ G\in\mathcal{G}\}\,\#\,\mathcal{F}$. The set $C=\bigcup_{F\in\mathcal{K}}B_F$ is of cardinality at most κ and $C\,\#\,\mathcal{G}\vee A\mathcal{F}$. $\square$

Let $\bigwedge(\mathbb{D})$ be the class of filters than can be represented as the infimum of a family of $\mathbb{D}$-filters.

Corollary 34. *If $\mathbb{J}$ is an $\mathbb{F}_1$-composable class included in the class $\mathbb{F}_1^{\Diamond_\kappa}$ such that $\mathbb{J}=\bigwedge(\mathbb{D})$ where $\mathbb{D}$ contains $\mathbb{F}_1$, then the following are equivalent*:

(1) $\mathcal{F}\in\ker(\mathbb{J}^{\Diamond_\kappa})$;
(2) $\mathcal{F}\in\mathbb{J}^{\Diamond_\kappa}$ *and* $A\vee\mathcal{F}\in\ker(\mathbb{J}^{\Diamond_\kappa})$ *for all A such that $A\,\#\,\mathcal{F}$ and $|A|\leqslant\kappa$*;
(3) *for every family $(\mathcal{G}_\alpha)_{\alpha\in I}$ of $\mathbb{D}$-filters such that $\bigwedge_{\alpha\in I}\mathcal{G}_\alpha\,\#\,\mathcal{F}$ there exists $J\subseteq I$ such that $|J|\leqslant\kappa$ and $\bigwedge_{\alpha\in J}\mathcal{G}_\alpha\,\#\,\mathcal{F}$*;
(4) *for every family $(\mathcal{G}_\alpha)_{\alpha\in I}$ of $\mathbb{D}$-filters*: *if $\forall F\in\mathcal{F}$, $\exists\alpha\in I$ and $C_F\subseteq F$: $C_F\in\mathcal{G}_\alpha$ and $|C_F|\leqslant\kappa$, then there exists $J\subseteq I$ such that $|J|\leqslant\kappa$ and $\bigwedge_{\alpha\in J}\mathcal{G}_\alpha\,\#\,\mathcal{F}$*;
(5) *for every family $(\mathcal{G}_\alpha)_{\alpha\in I}$ of $\mathbb{D}$-filters, each of which is either free or the principal filter of a set of cardinality at most κ*: *if $\forall F\in\mathcal{F}$, $\exists\alpha\in I$ and $C_F\subseteq F$: $C_F\in\mathcal{G}_\alpha$ and $|C_F|\leqslant\kappa$, then there exists $J\subseteq I$ such that $|J|\leqslant\kappa$ and $\bigwedge_{\alpha\in J}\mathcal{G}_\alpha\,\#\,\mathcal{F}$.*

Proof. (1) $\Leftrightarrow$ (2) follows from Theorem 33 and (3) $\Rightarrow$ (4) $\Rightarrow$ (5) by definition.

(1) $\Rightarrow$ (3). Let $(\mathcal{G}_\alpha)_{\alpha\in I}$ be a family of $\mathbb{D}$-filters such that $\bigwedge_{\alpha\in I}\mathcal{G}_\alpha\,\#\,\mathcal{F}$. Consider the presentation $\gamma: I\times\prod_{\alpha\in I}\mathcal{G}_\alpha\to 2^X$ defined by $\gamma(\alpha,(G_\beta)_{\beta\in I})=G_\alpha$. It is a $\mathbb{J}$-expandable presentation of $\bigwedge_{\alpha\in I}\mathcal{G}_\alpha$. Indeed, $\mathbb{J}$ is $\mathbb{F}_1$-composable so that $\mathcal{G}_\alpha\times\{\alpha\}\in\mathbb{J}$ for all $\alpha\in I$, and $\mathbb{J}$ is stable by infimum so that $\bigwedge_{\alpha\in I}(\mathcal{G}_\alpha\times\{\alpha\})$ is a $\mathbb{J}$-filter on $X\times I$. In view of Theorem 33, there exists $C\subset I$ such that $|C|\leqslant\kappa$ and $\bigwedge_{\alpha\in C}\mathcal{G}_\alpha\,\#\,\mathcal{F}$.

(5) $\Rightarrow$ (1). In view of Theorem 33, we only need to show that for any $\mathbb{J}$-filter $\mathcal{G}$ such that $\mathcal{G}\,\#\,\mathcal{F}$ and every $\mathbb{J}$-expandable presentation $\gamma:\mathcal{F}\times\mathcal{J}\to 2^X$ of $\mathcal{G}$ such that $\mathcal{J}$ is a $\mathbb{J}$-filter each $\gamma(F,J)$ is a subset of F of cardinality at most κ and each $\gamma(F,\mathcal{I})$ is a $\mathbb{J}$-filter, there exists a subfamily $\mathcal{K}$ of $\mathcal{F}$ of cardinality at most κ such that $\{\bigcup_{F\in\mathcal{K}}\gamma(F,J)\colon\ J\in\mathcal{J}\}\,\#\,\mathcal{F}$. For each $F\in\mathcal{F}$, consider the $\mathbb{J}$-filter $\mathcal{J}_F=\gamma(F,\mathcal{J})$. As $\mathbb{J}=\bigwedge(\mathbb{D})$, there exists a $\mathbb{D}$-filter $\mathcal{L}_F\geqslant\mathcal{J}_F$ that is either free or principal. Indeed, if $\mathcal{J}_F^{\bullet}$ is non-degenerate, then it is a $\mathbb{D}$-filter finer than $\mathcal{J}_F$. Otherwise, $\mathcal{J}_F$ is free and therefore admits finer free $\mathbb{D}$-filters. $\bigwedge_{F\in\mathcal{F}}\mathcal{L}_F\,\#\,\mathcal{F}$ and each $\mathcal{L}_F$ contains a subset a cardinality at most κ of F. Therefore, there exists a subfamily $\mathcal{K}$ of $\mathcal{F}$ of cardinality at most κ such that $\bigwedge_{F\in\mathcal{K}}\mathcal{L}_F\,\#\,\mathcal{F}$. Moreover, $\{\bigcup_{F\in\mathcal{K}}\gamma(F,J)\colon\ J\in\mathcal{J}\}^{\uparrow}\leqslant\bigwedge_{F\in\mathcal{K}}\mathcal{J}_F\leqslant\bigwedge_{F\in\mathcal{K}}\mathcal{L}_F$. Therefore, $\{\bigcup_{F\in\mathcal{K}}\gamma(F,J)\colon\ J\in\mathcal{J}\}^{\uparrow}\,\#\,\mathcal{F}$. $\square$

5.1.1. Productive κ-tightness

Corollary 35. *The following are equivalent*:

(1) $\mathcal{F}\in\ker(\mathbb{F}_1^{\Diamond_\kappa\Diamond_\kappa})$;
(2) $\mathcal{F}\in\mathbb{F}^{\Diamond_\kappa\Diamond_\kappa}$ *and* $A\vee\mathcal{F}\in\ker(\mathbb{F}_1^{\Diamond_\kappa\Diamond_\kappa})$ *for all A such that $A\,\#\,\mathcal{F}$ and $|A|\leqslant\kappa$*;
(3) *for every collection $\{\mathcal{G}_\alpha\colon\ \alpha\in I\}\subseteq\mathbb{F}_1^{\Diamond_\kappa}$*: *if $\bigwedge_{\alpha\in I}\mathcal{G}_\alpha\,\#\,\mathcal{F}$, then $\exists J\subseteq I$ such that $|J|\leqslant\kappa$ and $\bigwedge_{\alpha\in\mathcal{J}}\mathcal{G}_\alpha\,\#\,\mathcal{F}$*;

(4) *for every collection* $\{\mathcal{G}_\alpha\colon \alpha \in I\} \subseteq \mathbb{F}_1^{\diamondsuit_\kappa}$: *if* $\forall F \in \mathcal{F}$, $\exists \alpha \in I$ *and* $C_F \subseteq F$ *such that* $C_F \in \mathcal{G}_\alpha$ *and* $|C_F| \leqslant \kappa$, *then* $\exists J \subseteq I$ *such that* $|J| \leqslant \kappa$ *and* $\bigwedge_{\alpha\in J} \mathcal{G}_\alpha \# \mathcal{F}$;
(5) *for every collection* $\{\mathcal{G}_\alpha\colon \alpha \in I\} \subseteq \mathbb{F}_1^{\diamondsuit_\kappa}$ *each of which is either free or the principal filter of a set of cardinality at most* κ: *if* $\forall F \in \mathcal{F}$, $\exists \alpha \in I$ *and* $C_F \subseteq F$ *such that* $C_F \in \mathcal{G}_\alpha$ *and* $|C_F| \leqslant \kappa$, *then there exists* $J \subseteq I$ *such that* $|J| \leqslant \kappa$ *and* $\bigwedge_{\alpha\in J} \mathcal{G}_\alpha \# \mathcal{F}$.

Proof. Since $\mathbb{F}_1^{\diamondsuit_\kappa} = \bigwedge(\mathbb{F}_1^{\diamondsuit_\kappa})$, the corollary is just a restatement of Corollary 34 with $\mathbb{D} = \mathbb{J} = \mathbb{F}_1^{\diamondsuit_\kappa}$. □

A point x of a topological space X is *a productively κ-tight point* if $\mathcal{N}_X(x) \in \ker(\mathbb{F}_1^{\diamondsuit_\kappa \diamondsuit_\kappa})$. A topological space is *productively κ-tight* if all its points are productively κ-tight.

The interest of such an explicit description of a kernel lies in its combination with the corresponding instance of Corollary 30. In the present case,

Corollary 36. *$X \times Y$ is κ-tight for every κ-tight space Y if and only if X is productively κ-tight.*

Notice that the fourth condition in Corollary 35 corresponds to the non-existence of *singular families* in the sense of Arhangel'skii [1]. Hence, the combination of Corollaries 35 and 36 extends the equivalence between (b) and (c) in [1, Theorem 3.6], and provides a shorter proof. Of all the conditions of Corollary 35 the third is probably the most straightforward and easy to use, but was apparently not known to be equivalent to productive κ-tightness.

5.1.2. Tight points

We now characterize $\ker(\mathbb{F}_1^{\triangle\diamondsuit})$. For a cardinal κ we let S_κ denote the space obtained as a quotient of κ-many mutually disjoint convergent sequences $(z_n^\xi)_{n\in\omega}$ by identifying their limit points to a single point w. S_κ is endowed with the quotient topology. We denote the neighborhood of w by $\mathcal{S}_\kappa$.

Theorem 37. *The following are equivalent for a filter on X*:

(1) $\mathcal{F} \in \ker(\mathbb{F}_1^{\triangle\diamondsuit})$;
(2) $\mathcal{F} \in \mathbb{F}_1^{\triangle\diamondsuit}$ *and* $\mathcal{F} \vee A \in \ker(\mathbb{F}_1^{\triangle\diamondsuit})$ *for any countable set A meshing with* $\mathcal{F}$;
(3) *If* $\bigwedge_{\alpha\in I}(x_n^\alpha)_{n\in\omega} \# \mathcal{F}$, *then there exists a countable* $J \subseteq I$ *such that* $\bigwedge_{\alpha\in J}(x_n^\alpha)_{n\in\omega} \# \mathcal{F}$;
(4) *If* $\bigwedge_{\alpha\in I}(x_n^\alpha)_{n\in\omega} \# \mathcal{F}$, *where each sequence* $(x_n^\alpha)_{n\in\omega}$ *is either free or principal, then there exists a countable* $J \subseteq I$ *such that* $\bigwedge_{\alpha\in J}(x_n^\alpha)_{n\in\omega} \# \mathcal{F}$;
(5) *If* $\{\mathcal{G}_\alpha\colon \alpha \in I\} \subseteq \mathbb{F}_1^{\triangle}$ *and* $\bigwedge_{\alpha\in I} \mathcal{G}_\alpha \# \mathcal{F}$, *then there exists a countable* $J \subseteq I$ *such that* $\bigwedge_{\alpha\in J} \mathcal{G}_\alpha \# \mathcal{F}$;
(6) $\mathcal{F} \times \mathcal{S}_{|X^\omega|} \in \mathbb{F}_1^{\diamondsuit}$.

Proof. (1) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (2) follows from Corollary 34 with $\mathbb{J} = \mathbb{F}_1^{\triangle}$ and $\mathbb{D}$ equal to the class of sequences and the observation that $\mathbb{F}_1^{\triangle} = \bigwedge(\mathbb{D})$ (e.g., [10]).

(1) $\Leftrightarrow$ (5) follows from Corollary 34 with $\mathbb{J} = \mathbb{F}_1^{\triangle}$ and $\mathbb{D} = \mathbb{F}_1^{\triangle}$ and the observation that $\mathbb{F}_1^{\triangle} = \bigwedge(\mathbb{F}_1^{\triangle})$.

(1) $\Rightarrow$ (6). By Theorem 20, $\mathcal{F} \times \mathcal{G} \in \mathbb{F}_1^{\diamond}(X \times Y)$ for any Fréchet filter $\mathcal{G}$ on any set Y. In particular, $\mathcal{F} \times \mathcal{S}_{|X^\omega|} \in \mathbb{F}_1^{\diamond}(X \times S_{|X^\omega|})$.

(6) $\Rightarrow$ (3). Let $\{(x_n^\alpha)_{n\in\omega}\colon \alpha \in I\}$ be a collection of sequences such that $(\bigwedge_{\alpha\in I}(x_n^\alpha)_{n\in\omega}) \# \mathcal{F}$. Let $\kappa = |X^\omega|$ and $K = \{(x_n^\xi)_{n\in\omega}\colon \xi \in \kappa\}$ be an enumeration of the set $\{(x_n^\alpha)_{n\in\omega}\colon \alpha \in I\}$ possibly with repetitions. Let $R \subseteq X \times S_\kappa$ be defined by $R = \bigcup_{\xi\in\kappa}\{(x_n^\xi, z_n^\xi)\colon n \in \omega\}$. Notice that $R \# (\mathcal{F} \times \mathcal{S}_\kappa)$. By (6), there is a countable set $T \# (R \vee (\mathcal{F} \times \mathcal{S}_\kappa))$. Let $Q = \{\xi \in \kappa\colon (\exists n \in \omega)(x_n^\xi, z_n^\xi) \in T\}$. Let $F \in \mathcal{F}$. Since $T \# (R \vee (F \times \mathcal{S}_\kappa))$, there is $\xi \in Q$ such that $\pi_X^{-1}(F) \# (x_n^\xi, z_n^\xi)_{n\in\omega}$. So, for some $\xi \in Q$ we have $F \# (x_n^\xi)_{n\in\omega}$. Thus, $\mathcal{F} \# \bigwedge_{\xi\in Q}(x_n^\xi)_{n\in\omega}$ and Q is countable. $\square$

In [5] the notion of a tight point is introduced. A collection of sets $\mathcal{E}$ is said to *cluster at a point* p provided that for every neighborhood U of x there is an $E \in \mathcal{E}$ such that $U \cap E$ is infinite. A point x in a space X is said to be *tight* provided that for any collection of sets $\mathcal{E}$ that clusters at p one can find a countable subcollection $\mathcal{E}^* \subseteq \mathcal{E}$ such that $\mathcal{E}^*$ clusters at p.

Theorem 38. *A point p of a space X is tight if and only if $\mathcal{N}_X(p) \in \ker(\mathbb{F}_1^{\triangle\diamond})$.*

Proof. Suppose that the neighborhood filter $\mathcal{N}(p)$ is in $\ker(\mathbb{F}_1^{\triangle\diamond})$. Let $\mathcal{E}$ be a collection of sets which clusters at p. For each $F \in \mathcal{N}(p)$ we can find an $E^F \in \mathcal{E}$ such that $E^F \cap F$ is infinite. Let $(x_n^F)_{n\in\omega}$ be a free sequence on $E^F \cap F$. Clearly, $\mathcal{N}(p) \# \bigwedge_{F\in\mathcal{N}(x)}(x_n^F)_{n\in\omega}$. There exists $\{F_k\colon k \in \omega\} \subseteq \mathcal{F}$ such that $\mathcal{N}(p) \# \bigwedge_{k\in\omega}(x_n^{F_k})_{n\in\omega}$. Let $\mathcal{E}^* = \{E^{F_k}\colon k \in \omega\}$. It is easily verified that $\mathcal{E}^*$ clusters at p.

Suppose that p is a tight point of X. Let $\mathcal{N}(p) \# \bigwedge_{\alpha\in I}(x_n^\alpha)_{n\in\omega}$, where each sequence $(x_n^\alpha)_{n\in\omega}$ is either free or principal. In other words, we can write $\mathcal{N}(p) \# A(\bigwedge_{\alpha\in J}(x_n^\alpha)_{n\in\omega})$, where $A = \bigwedge\{(x_n^\alpha)_{n\in\omega}\colon (x_n^\alpha)_{n\in\omega} \in \mathbb{F}_1\}$ and every sequence $(x_n^\alpha)_{n\in\omega}$ with $\alpha \in J$ is free. If $A \# \mathcal{N}(p)$, either $A \# \mathcal{N}(p)^\bullet$ or $A \# \mathcal{N}(p)^\circ$. In the first case, there is a fixed sequence of the original collection whose defining point is in $\mathcal{N}(p)^\bullet$ and is therefore finer than $\mathcal{N}(p)$. In the second case, $A \cap N$ is infinite for every $N \in \mathcal{N}(p)$ and there exists a countably infinite set $A_N \subset A \cap N$. The collection $\{A_N\colon N \in \mathcal{N}(p)\}$ clusters at p. Since p is a tight point, there exists $\{A_{N_k}\colon k \in \omega\}$ which clusters at p. Then $\bigcup_{k\in\omega} A_{N_k}$ is a countable subset of A meshing with $\mathcal{N}(p)$, and therefore defines a countable collection of (fixed) sequences from the original collection whose infimum is meshing with $\mathcal{N}(p)$.

If $A(\neg\#)\mathcal{N}(p)$ then $\mathcal{N}(p) \# \bigwedge_{\alpha\in J}(x_n^\alpha)_{n\in\omega}$. Each $N \in \mathcal{N}(p)$ is meshing with one of the sequences $(x_n^\alpha)_{n\in\omega}$ for $\alpha \in J$, so that the family $\{\{x_n^\alpha\colon n \in \omega\}\colon \alpha \in J\}$ clusters at p. Since p is tight, there is $(\alpha_k)_{k\in\omega}$ in J such that $\{\{x_n^{\alpha_k}\colon n \in \omega\}\colon k \in \omega\}$ clusters at p. Then $\mathcal{N}(p) \# \bigwedge_{k\in\omega}(x_n^{\alpha_k})_{n\in\omega}$. $\square$

From Theorems 37 and 38, we have the following improvement of [5, Theorem 3.1] which states the equivalence between (1) and (2), but only for a *countable* space X.

Corollary 39. *The following are equivalent*:

(1) *every point of X is tight*;
(2) *$X \times S_{|X^\omega|}$ is countably tight*;
(3) *$X \times Y$ is countably tight for every Fréchet space Y.*

By [1, Example 1.3], $S_\omega \times S_{\omega^\omega}$ is not countably tight. Since S_ω is a countable space, $\mathcal{S}_\omega \in \mathbb{F}_1^{\Diamond\Diamond}$. As $\mathcal{S}_{\omega^\omega} \in \mathbb{F}_1^{\triangle}$, we have $\mathbb{F}_1^{\Diamond\Diamond} \setminus \ker(\mathbb{F}_1^{\triangle\Diamond}) \neq \emptyset$. Hence, $\mathbb{F}_1^{\Diamond\Diamond} \neq \ker(\mathbb{F}_1^{\Diamond\Diamond})$ and $\mathbb{F}_1^{\triangle\Diamond} \neq \ker(\mathbb{F}_1^{\triangle\Diamond})$.

In view of Theorem 38 and considering that $\mathbb{F}_1^{\triangle} \subset \mathbb{F}_1^{\Diamond}$, hence $\ker(\mathbb{F}_1^{\Diamond\Diamond}) \subset \ker(\mathbb{F}_1^{\triangle\Diamond})$, we obtain:

Corollary 40. *Every productively countably tight point is a tight point.*

5.1.3. Absolute tightness

A notion related to tightness, productive tightness and tight points and introduced in [1] is that of absolute tightness. A point of a topological space X is *absolutely tight* if it is a point of countable tightness in a compactification of X. Absolute tightness can be characterized in a way that is similar to the characterization of productive tightness in Corollary 35.

Theorem 41. *Let X be a completely regular topological space and let $x \in X$. The following are equivalent*:

(1) *x is an absolutely tight point*;
(2) *for any family $\{\mathcal{F}_\alpha\colon \alpha \in I\}$ of filters such that $\bigwedge_{\alpha\in I} \mathcal{F}_\alpha \# \mathcal{N}(x)$ there exists $(\alpha_i)_{i\in\omega}$ in I such that $\bigwedge_{i\in\omega} \mathcal{F}_{\alpha_i} \# \mathcal{N}(x)$*;
(3) *for any family $\{\mathcal{U}_\alpha\colon \alpha \in I\}$ of ultrafilters such that $\bigwedge_{\alpha\in I} \mathcal{U}_\alpha \# \mathcal{N}(x)$ there exists $(\alpha_i)_{i\in\omega}$ in I such that $\bigwedge_{i\in\omega} \mathcal{U}_{\alpha_i} \# \mathcal{N}(x)$.*

Proof. (1) $\Rightarrow$ (2). Let $\{\mathcal{F}_\alpha\colon \alpha \in I\}$ be a family of filters such that $\bigwedge_{\alpha\in I} \mathcal{F}_\alpha \# \mathcal{N}(x)$. Since each $\mathcal{F}_\alpha = \bigwedge_{\mathcal{U}\in\beta(\mathcal{F}_\alpha)} \mathcal{U}$, where $\beta(\mathcal{F})$ denotes the set of ultrafilters finer than $\mathcal{F}$, we have $\bigwedge_{\alpha\in I, \mathcal{U}\in\beta(\mathcal{F}_\alpha)} \mathcal{U} \# \mathcal{N}(x)$. Let bX denote a compactification of X and let $A = \bigcup_{\alpha\in I, \mathcal{U}\in\beta(\mathcal{F}_\alpha)} \lim_{bX} \mathcal{U}$. Let $W \in \mathcal{N}_{bX}(x)$. By regularity of bX, there exists $V \in \mathcal{N}_{bX}(x)$ such that $\mathrm{cl}_{bX} V \subset W$. Since $V \cap X \in \mathcal{N}_X(x)$, there exists $\alpha \in I$, $\mathcal{U} \in \beta(\mathcal{F}_\alpha)$ and $U \in \mathcal{U}$ such that $U \subset V \cap X$. Then $\lim_{bX} \mathcal{U} \subset \mathrm{cl}_{bX} V \subset W$. Therefore, $x \in \mathrm{cl}_{bX} A$. Since x is an absolutely tight point, there exist points a_n in A such that $x \in \mathrm{cl}_{bX}(\{a_n\colon n \in \omega\})$. For each n, pick $\mathcal{U}_n$ in $\bigcup_{\alpha\in I} \beta(\mathcal{F}_\alpha)$ such that $a_n \in \lim_{bX} \mathcal{U}_n$, and pick α_n such that $\mathcal{U}_n \in \beta(\mathcal{F}_{\alpha_n})$. We claim that $\mathcal{N}(x) \# \bigwedge_{n\in\omega} \mathcal{F}_{\alpha_n}$. Indeed, for each open $B \in \mathcal{N}_X(x)$, there is an open $B_1 \in \mathcal{N}_{bX}(x)$ such that $B_1 \cap X = B$. There is an $n \in \omega$ such that $a_n \in B_1$. Therefore, there exists $U \in \mathcal{U}_n$ such that $U \subset B_1 \cap X = B$. Hence $\mathcal{N}(x) \# \bigwedge_{n\in\omega} \mathcal{U}_n$ so that $\mathcal{N}(x) \# \bigwedge_{n\in\omega} \mathcal{F}_{\alpha_n}$.

(2) $\Rightarrow$ (3) is obvious.

(3) $\Rightarrow$ (1). Let bX denote a compactification of X and let $A \subset bX$ be such that $x \in \mathrm{cl}_{bX} A$. Let $W \in \mathcal{N}_X(x)$ be open. There is a bX-open set $V \in \mathcal{N}_{bX}(x)$ such that $W = V \cap X$. There exists $a \in A \cap V$. We can find an ultrafilter $\mathcal{U}_a$ on X such that $\lim_{bX} \mathcal{U}_a = a$. As V is open, there exists $U \in \mathcal{U}_a$ such that $U \subset V \cap X = W$. Hence, $\bigwedge_{a\in A} \mathcal{U}_a \# \mathcal{N}_X(x)$. By (3), there exists $(a_n)_{n\in\omega}$ in A such that $\bigwedge_{n\in\omega} \mathcal{U}_{a_n} \# \mathcal{N}_X(x)$. We claim that $x \in \mathrm{cl}_{bX}\{a_n\colon n \in \omega\}$. Indeed, for each $V \in \mathcal{N}_{bX}(x)$, we can find $V_1 \in \mathcal{N}_{bX}(x)$ such that $\mathrm{cl}_{bX} V_1 \subset V$. As $X \cap V_1 \in \mathcal{N}_X(x)$ and $\mathcal{N}_X(x) \# \bigwedge_{n\in\omega} \mathcal{U}_{a_n}$, there is an n such that $(X \cap V_1) \in \mathcal{U}_{a_n}$. Now, $a_n \in \mathrm{cl}_{bX} V_1 \subset V$ and x is an absolutely tight point. □

In view of Corollary 35, we obtain:

Corollary 42. [1] *An absolutely tight point is productively countably tight.*

The converse is not true in general. Indeed, as observed in [1], a Σ-product of uncountably many copies of the discrete two point space $\{0, 1\}$ (or of the real line) is an example of a productively countably tight space (all points of which are, in particular, tight points) which is not absolutely countably tight. However, the converse is true for countable spaces because every filter on a countable set is countably tight. More precisely:

Proposition 43. *Let X be a topological space. The following are equivalent*:

(1) *X is productively countably tight*;
(2) *X is steadily countably tight and every countable subset of X is productively countably tight*;
(3) *X is steadily countably tight and every countable subset of X is absolutely tight.*

Proof. The equivalence between (1) and (2) follows from (1) $\Leftrightarrow$ (2) in Corollary 35. Moreover, every filter on a countable set is countably tight, so that productive countable tightness and absolute tightness coincide on countable sets; and (2) $\Leftrightarrow$ (3) follows. □

Bella and Malykhin [4, Example 1] is an example under (CH) of a tight point which does not have countable absolute tightness. The space considered is countable, so that it provides an example of a tight point which is not productively countably tight. The associated filter is in $\ker(\mathbb{F}_1^{\triangle\diamond}) \setminus \ker(\mathbb{F}_1^{\diamond\diamond})$. It shows that, at least under (CH), the converse of Corollary 40 is not true.

5.2. Productively Fréchet and $\aleph_0$-bisequential spaces

In [1] a regular space X is called $\aleph_0$-bisequential provided that ω is in the frequency spectrum of X (by [1, Theorem 3.6] and Corollary 36, it means that X is productively countably tight) and every countable subset of X is bisequential.[2] [1, Proposition 6.27]

[2] A filter $\mathcal{F}$ is *bisequential* if for every filter $\mathcal{G} \# \mathcal{F}$ there exists a countably based filter $\mathcal{H} \# \mathcal{G}$ such that $\mathcal{H} \geqslant \mathcal{F}$. A topological space is *bisequential* if all its neighborhood filters are bisequential.

states that the product of an $\aleph_0$-bisequential space with a strongly Fréchet space is strongly Fréchet. In view of Proposition 23, every $\aleph_0$-bisequential space is productively Fréchet. We can also give a direct proof of this fact by characterizing productive Fréchetness in terms comparable to $\aleph_0$-bisequentiality.

Theorem 44. *$\mathcal{F} \in \mathbb{F}_\omega^{\triangle\triangle}(X)$ if and only if $\mathcal{F} \in \mathbb{F}_\omega^{\triangle\diamond}(X)$ and $\mathcal{F} \vee A \in \mathbb{F}_\omega^{\triangle\triangle}(X)$ for every countable $A \subseteq X$ such that $A \# \mathcal{F}$.*

Proof. Suppose $\mathcal{F} \in \mathbb{F}_\omega^{\triangle\diamond}(X)$ and $\mathcal{F} \vee A \in \mathbb{F}_\omega^{\triangle\triangle}$ for every countable $A \subseteq X$ such that $A \# \mathcal{F}$. Let $\mathcal{G} \in \mathbb{F}_\omega^{\triangle}$ and $\mathcal{G} \# \mathcal{F}$. Since $\mathcal{F} \in \mathbb{F}_\omega^{\triangle\diamond}(X)$, there is a countable set A such that $A \# \mathcal{G} \vee \mathcal{F}$. Since $\mathcal{F} \vee A \in \mathbb{F}_\omega^{\triangle\triangle}$, and $\mathcal{G} \# (\mathcal{F} \vee A)$, there is countably based filter $\mathcal{C} \geqslant \mathcal{G} \vee (\mathcal{F} \vee A) \geqslant \mathcal{G} \vee \mathcal{F}$. Thus, $\mathcal{F} \triangle \mathcal{G}$. Therefore, $\mathcal{F} \in \mathbb{F}_\omega^{\triangle\triangle}$.

The opposite implication is trivial. □

We now deduce

Corollary 45. [1, Proposition 6.27] *If X is $\aleph_0$-bisequential, then X is productively Fréchet.*

Proof. Let $\mathcal{F}$ be the neighborhood filter of a point in X. By [1, Theorem 3.6], $\mathcal{F} \in \ker(\mathbb{F}_1^{\diamond\diamond}) \subseteq \mathbb{F}_1^{\diamond\diamond}$. In particular, $\mathcal{F} \in \mathbb{F}_1^{\diamond\diamond} \subseteq \mathbb{F}_\omega^{\triangle\diamond}$. Also, for every countable set A meshing $\mathcal{F}$ we have $\mathcal{F} \vee A$ bisequential and hence $\mathcal{F} \vee A$ is in $\mathbb{F}_\omega^{\triangle\triangle}$. So by Theorem 44, $\mathcal{F} \in \mathbb{F}_\omega^{\triangle\triangle}$. □

There are models of set theory in which the converse of Corollary 45 is false [16]. It is unknown if there is a ZFC example of a productively Fréchet space which is not $\aleph_0$-bisequential.

5.3. $\mathbb{J}$-expandable presentations and kernels of $\diamond$-polars

To apply Theorem 33 for a class $\mathbb{J}$ that is not stable under infima, we need to use general presentations of filters and not only infima as in Corollary 34. Practically speaking, to apply Theorem 33 in such cases, we need to characterize $\mathbb{J}$-expandability of a presentation.

A *crossing of a presentation* $\gamma : I \times J \to 2^X$ is a family $\{D_{\rho,l}\colon (\rho, l) \in I \times K\}$ such that for every $\beta \in J$ and $l \in K$ there exists $\alpha \in I$ such that $\gamma(\alpha, \beta) \cap D_{\alpha,l} \neq \emptyset$. A crossing $\{D_{\rho,l}\colon (\rho, l) \in I \times \omega\}$ satisfying $D_{\rho,l+1} \subseteq D_{\rho,l}$ for every $(\rho, l) \in I \times \omega$ is called *ω-decreasing*.

Lemma 46. *A presentation $\gamma : I \times J \to 2^X$ is $\mathbb{F}_\omega^{\triangle}$-expandable if and only if for every ω-decreasing crossing $\{D_{\rho,l}\colon (\rho, l) \in I \times \omega\}$ of γ, there exist $e_k \in X$ and $\alpha_k \in I$, such that for every $\beta \in J$ and $n \in \omega$ we have $e_k \in \gamma(\alpha_k, \beta) \cap D_{\alpha_k,n}$ for all k sufficiently large.*

Proof. Suppose $\gamma : I \times J \to X$ is not $\mathbb{F}_\omega^{\triangle}$-expandable. There is a decreasing countably based filter $\mathcal{E} = \{E_k\}_{k\in\omega}$ on $X \times I$ such that $\mathcal{E} \# \gamma^*$ and there is no sequence $(x_n, \alpha_n) \geqslant \mathcal{E} \vee \gamma^*$. Since $\mathcal{E} \# \gamma^*$, for every $\beta \in J$ and $k \in \omega$ there is an $\alpha \in I$ such that

$$\gamma(\alpha,\beta)\times\{\alpha\}\cap E_k\neq\emptyset. \tag{2}$$

For each $k\in\omega$ and $\alpha\in I$, let $D_{\alpha,k}=\{x\in X\colon (x,\alpha)\in E_k\}$. By (2), $\{D_{\alpha,k}\colon (\alpha,k)\in I\times\omega\}$ is a crossing of γ. It is ω-decreasing because $\{E_k\colon k\in\omega\}$ is decreasing. By way of contradiction, assume that there exist points $e_l\in X$ and $\alpha_k\in I$, such that for every $\beta\in J$ and $k\in\omega$ we have $e_l\in\gamma(\alpha_l,\beta)\cap D_{\alpha_l,k}$ for all l sufficiently large. For each $l\in\omega$ let $y_l=(e_l,\alpha_l)$. We show that $(y_l)_{l\in\omega}\geqslant\gamma^*\vee\mathcal{E}$. Let $\beta\in J$ and $k\in\omega$. There is $n\in\omega$ such that $e_l\in\gamma(\alpha_k,\beta)\cap D_{\alpha_k,k}$ for all $l\geqslant n$. Now $y_l=(e_l,\alpha_l)\in(\gamma(\alpha_l,\beta)\cap D_{\alpha_l,k})\times\{\alpha_l\}=(\gamma(\alpha_l,\beta)\times\{\alpha_l\})\cap E_k$ for all $l\geqslant n$. So, $(y_l)_{l\in\omega}\geqslant\gamma^*\vee\mathcal{E}$, contradicting our choice of $\mathcal{E}$.

Suppose $\gamma: I\times J\to 2^X$ is an $\mathbb{F}_\omega^{\triangle}$-expandable presentation. Let $\{D_{\alpha,k}\colon (\alpha,k)\in I\times\omega\}$ be an ω-decreasing crossing of γ. For each $k\in\omega$ define $E_k=\{(x,\alpha)\colon x\in D_{k,\alpha}\}$. It is easily verified that $(E_k)_{k\in\omega}$ is a countably based filter meshing with γ^*. So, there exists $(y_k)_\omega\geqslant(E_k)_{k\in\omega}\vee\gamma^*$. For each k let $e_k=\pi_X(y_k)$ and $\alpha_k=\pi_I(y_k)$. Let $\beta\in J$ and $n\in\omega$. There is $l\in\omega$ such that $y_k\in E_n\cap\gamma(\alpha_k,\beta)\times\alpha_k$ for all $k\geqslant l$. So, $e_k\in D_{\alpha_k,n}\cap\gamma(\alpha_k,\beta)$ for all $k\geqslant l$. □

Lemma 47. *A presentation $\gamma: I\times J\to 2^X$ is $\mathbb{F}_1^{\dagger}$-expandable if and only if for any crossing $\{D_{\rho,l}\colon (\rho,l)\in I\times\omega\}$ of γ, we may find finite sets $E_k\subseteq X$ and $H_k\subseteq I$, such that for every $\beta\in J$ there is $k\in\omega$ and $\alpha\in H_k$ such that we have $E_k\cap\gamma(\alpha,\beta)\cap D_{\alpha,k}\neq\emptyset$.*

Proof. Suppose $\gamma: I\times J\to 2^X$ is not $\mathbb{F}_1^{\dagger}$-expandable. There is a countable collection of sets $\{B_k\}_{k\in\omega}$ on $X\times I$ meshing with γ^* such that for any selection of finite sets $G_k\subseteq B_k$, $\bigcup_{k\in\omega}G_k$ does not mesh with γ^*. Since $B_k\,\#\,\gamma^*$, for every $\beta\in J$ and $k\in\omega$ there is an $\alpha\in I$ such that

$$\gamma(\alpha,\beta)\times\{\alpha\}\cap B_k\neq\emptyset. \tag{3}$$

For each $k\in\omega$ and $\alpha\in I$ let $D_{\alpha,k}=\{x\in X\colon (x,\alpha)\in B_k\}$. By (3), it is a crossing of γ. By way of contradiction, assume that there exist finite sets $E_k\in X$ and $H_k\in I$, such that for every $\beta\in J$ there is $k\in\omega$ and $\alpha\in H_k$ such that $E_k\cap\gamma(\alpha,\beta)\cap D_{\alpha,k}\neq\emptyset$. For each $k\in\omega$ let $G_k=\{(x,\alpha)\colon \alpha\in H_k \text{ and } x\in D_{\alpha,k}\cap E_k\}$. Notice that G_k is a finite subset of B_k. Let $\beta\in J$. There is $k\in\omega$ and $\alpha\in H_k$ such that $E_k\cap\gamma(\alpha,\beta)\cap D_{\alpha,k}\neq\emptyset$. Now $G_k\cap(\gamma(\alpha,\beta)\times\{\alpha\})\neq\emptyset$. So, $\bigcup_{k\in\omega}G_k\,\#\,\gamma^*$, contradicting our choice of $\{B_k\}_k$.

Suppose that γ is $\mathbb{F}_1^{\dagger}$-expandable. Let $\{D_{\alpha,k}\colon (\alpha,k)\in I\times\omega\}$ be a crossing of γ. For each $k\in\omega$ define $B_k=\{(x,\alpha)\colon x\in D_{\alpha,k}\}$. It is easily verified that $B_k\,\#\,\gamma^*$ for every $k\in\omega$. So, there exist finite sets $G_k\subseteq B_k$ such that $\bigcup_{k\in\omega}G_k\,\#\,\gamma^*$. For each k let $E_k=\pi_X(G_k)$ and $H_k=\pi_I(G_k)$. Let $\beta\in J$. There is $k\in\omega$ and $\alpha\in I$ such that $G_k\cap(\gamma(\alpha,\beta)\times\{\alpha\})\neq\emptyset$. Let $(x,\alpha)\in G_k\cap(\gamma(\alpha,\beta)\times\{\alpha\})$. Notice that $\alpha\in H_k$ and $x\in E_k$. Thus, $E_k\cap D_{\alpha,k}\cap\gamma(\alpha,\beta)\neq\emptyset$ for some $\alpha\in H_k$. □

Lemma 46 combined with Theorem 33 leads to:

Theorem 48. *The following are equivalent for a filter $\mathcal{F}\in\mathbb{F}(X)$:*

(1) $\mathcal{F}\in\ker(\mathbb{F}_\omega^{\triangle\diamond})$;
(2) $\mathcal{F}\in\mathbb{F}_\omega^{\triangle\diamond}$ *and* $A\vee\mathcal{F}\in\ker(\mathbb{F}_\omega^{\triangle\diamond})$ *for all countable set A meshing with $\mathcal{F}$;*

(3) *for every presentation $\gamma : I \times J \to 2^X$ for which there exists points e_k in X and indices $\alpha_k \in I$ such that for every $\beta \in J$ and every $n \in \omega$ we have $e_k \in \gamma(\alpha_k, \beta) \cap D_{\alpha_k,n}$ for all k sufficiently large, whenever $\{D_{\rho,l}\colon (\rho, l) \in I \times \omega\}$ is an ω-decreasing crossing of γ: if $\gamma_* \# \mathcal{F}$, then there exists a countable subset C of I such that $\{\bigcup_{\alpha\in C} \gamma(\alpha, \beta)\colon \beta \in J\} \# \mathcal{F}$.*

Theorem 48 gives a concrete characterization of the second property in the following instance of Corollary 30.

Corollary 49. *The following are equivalent for a topological space X:*

(1) *the product of X with every strongly Fréchet space is countably tight*;
(2) *X is $\ker(\mathbb{F}_\omega^{\triangle\diamondsuit})$-based.*

Lemma 47 combined with Theorem 33 leads to:

Theorem 50. *The following are equivalent for a filter $\mathcal{F} \in \mathbb{F}(X)$.*

(1) $\mathcal{F} \in \ker(\mathbb{F}_1^{\dagger\diamondsuit})$;
(2) $\mathcal{F} \in \mathbb{F}_1^{\dagger\diamondsuit}$ *and* $A \vee \mathcal{F} \in \ker(\mathbb{F}_1^{\dagger\diamondsuit})$ *for all countable set A meshing with $\mathcal{F}$*;
(3) *for any presentation $\gamma : I \times J \to 2^X$ for which there exist finite sets $E_k \subseteq X$ and $H_k \subseteq I$, such that for every $\beta \in J$ there is $k \in \omega$ and $\alpha \in H_k$ verifying $E_k \cap \gamma(\alpha_k, \beta) \cap D_{\alpha,k} \neq \emptyset$ whenever $\{D_{\rho,l}\colon (\rho, l) \in I \times \omega\}$ is a crossing of γ: if $\gamma_* \# \mathcal{F}$, then there is a countable subset C of I such that $\{\bigcup_{\alpha\in C} \gamma(\alpha, \beta)\colon \beta \in J\} \# \mathcal{F}$.*

Theorem 50 gives a concrete characterization of the second property in the following instance of Corollary 30.

Corollary 51. *The following are equivalent for a topological space X:*

(1) *the product of X with every countably fan-tight space is countably tight*;
(2) *X is $\ker(\mathbb{F}_1^{\dagger\diamondsuit})$-based.*

Characterizing kernels of †-polars internally seems to require even more machinery. While we do not have concrete characterizations of these kernels, we can give some information (in the next section) on how they relate to kernels of $\diamondsuit$-polars that we have characterized before.

6. Inclusions

Theorem 52. $\ker(\mathbb{F}_1^{\triangle\diamondsuit}) \subseteq \ker(\mathbb{F}_\omega^{\triangle\dagger})$.

Proof. Let $\mathcal{F} \in \ker(\mathbb{F}_1^{\triangle\diamondsuit})$ on X, Y be a set, and $A \subseteq X \times Y$. We show that $A\mathcal{F} \in \mathbb{F}_\omega^{\triangle\dagger}(Y)$, using Lemma 6. Let $\mathcal{G} \in \mathbb{F}_\omega^{\triangle}(Y)$ and $\mathcal{B} = \{B_k\}_{k\in\omega}$ be a countably based filter such that $B_k \# (\mathcal{G} \vee A\mathcal{F})$ for every $k \in \omega$.

Fix $F \in \mathcal{F}$. Since $\mathcal{G} \in \mathbb{F}_{\omega}^{\triangle}(Y)$ and $\mathcal{G} \# (AF \vee \mathcal{B})$, we may find a sequence $(y_k^F)_{k\in\omega}$ such that $(y_k^F)_{k\in\omega} \geqslant AF \vee \mathcal{G}$ and $y_k^F \in B_k$ for every $k \in \omega$. For each $k \in \omega$ define $x_k^F \in F$ so that $(x_k^F, y_k^F) \in A$.

Since $\mathcal{F} \in \ker(\mathbb{F}_1^{\triangle\diamond})$ on X, and $\mathcal{F} \# \bigwedge_{F\in\mathcal{F}} (x_k^F)_{k\in\omega}$, we may find $\{F_n\colon \in \omega\} \subseteq \mathcal{F}$ such that $\mathcal{F} \# \bigwedge_{n\in\omega} (x_k^{F_n})_{k\in\omega}$.

For every $k \in \omega$ let $T_k = \{y_k^{F_n}\colon n \leqslant k\}$. Notice that $T_k \subseteq B_k$ and T_k is finite for all $k \in \omega$. Let $F \in \mathcal{F}$ and $G \in \mathcal{G}$. There is $n \in \omega$ such that $(x_k^{F_n})_{k\in\omega} \# F$. Since $(y_k^{F_n})_{k\in\omega} \geqslant \mathcal{G}$ and $x_k^{F_n} \in F$ for infinitely many $k \in \omega$, there is $p \geqslant n$ such that $x_p^{F_n} \in F$ and $y_p^{F_n} \in G$. Hence, $y_p^{F_n} \in AF \cap G$. Since $n \leqslant p$, we have $T_p \cap AF \cap G \neq \emptyset$. So, $\bigcup_{k\in\omega} T_k \# (\mathcal{G} \vee A\mathcal{F})$. Thus, $A\mathcal{F} \in \mathbb{F}_{\omega}^{\triangle\dagger}(Y)$. $\square$

This improves significantly [5, Proposition 2.1] that states the weaker inclusion $\ker(\mathbb{F}_1^{\triangle\diamond}) \subset \mathbb{F}_1^{\dagger}$ in topological terms (i.e., every tight point has countable fan-tightness) under the assumption of T_1.

Corollary 53. *The product of a space whose every point is tight with a strongly Fréchet space has countable fan-tightness.*

Theorem 54. $\ker(\mathbb{F}_1^{\diamond\diamond}) \subseteq \ker(\mathbb{F}_1^{\dagger\dagger})$.

Proof. Let $\mathcal{F} \in \ker(\mathbb{F}_1^{\diamond\diamond})$ on X, Y be a set, and $A \subseteq X \times Y$. We show that $A\mathcal{F} \in \mathbb{F}_1^{\dagger\dagger}(Y)$. Let $\mathcal{G} \in \mathbb{F}_1^{\dagger}(Y)$ and $\mathcal{B} = \{B_k\}_{k\in\omega}$ be a countably based filter such that $B_k \# (\mathcal{G} \vee A\mathcal{F})$ for every $k \in \omega$.

Fix $F \in \mathcal{F}$. Since $\mathcal{G} \in \mathbb{F}_1^{\dagger}(Y)$ and $\mathcal{G} \# (AF \vee \mathcal{B})$, we may find finite sets $C_k^F \subseteq B_k \cap AF$ such that $(\bigcup_{k\geqslant n} C_k) \# AF \vee \mathcal{G}$ for every $n \in \omega$. Let $\mathcal{H}_F = (\bigcup_{k\geqslant n} C_k^F)_{n\in\omega} \vee \mathcal{G}$. In view of Corollary 17, $\mathcal{H}_F \in \mathbb{F}_1^{\dagger}(Y)$. Define a function $f^F : \bigcup_{k\in\omega} C_k^F \to F$ such that $f^F \subseteq A$ and define $\mathcal{G}_F = f^F(\mathcal{H}_F)$.

Since $\mathcal{F} \in \ker(\mathbb{F}_1^{\diamond\diamond})$ on X, $\mathcal{G}_F \in \mathbb{F}_1^{\diamond}$ for all $F \in \mathcal{F}$, and $\mathcal{F} \# \bigwedge_{F\in\mathcal{F}} \mathcal{G}_F$, we may find $\{F_n\colon \in \omega\} \subseteq \mathcal{F}$ such that $\mathcal{F} \# \bigwedge_{n\in\omega} \mathcal{G}_{F_n}$.

For every $k \in \omega$ let $T_k = \bigcup\{C_k^{F_n}\colon n \leqslant k\}$. Notice that $T_k \subseteq B_k$ and T_k is finite for all $k \in \omega$. Let $F \in \mathcal{F}$ and $G \in \mathcal{G}$. There is $n \in \omega$ such that $\mathcal{G}_{F_n} \# F$. Thus, $f^{F_n}[G \cap \bigcup_{k\geqslant n} C_k^{F_n}] \cap F \neq \emptyset$. So, there is $k \geqslant n$ such that $f^{F_n}[G \cap C_k^{F_n}] \cap F \neq \emptyset$. Since $f^{F_n} \subseteq A$, we have $G \cap C_k^F \cap AF \neq \emptyset$. So, $\bigcup_{k\in\omega} T_k \# (\mathcal{G} \vee A\mathcal{F})$. Thus, $A\mathcal{F} \in \mathbb{F}_1^{\dagger\dagger}(Y)$. $\square$

Arhangel'skii [2, Theorem 5] proves the much weaker inclusion $\ker(\mathbb{F}_1^{\diamond\diamond}) \subset \mathbb{F}_1^{\dagger}$ in topological terms (i.e., if the product of X with every space of countable tightness has countable tightness, then X has countable fan-tightness) among Tychonoff spaces. From the theorem above, we conclude:

Corollary 55. *If X is productively countably tight* (*in particular if it has countable absolute tightness*)*, then its product with every space of countable fan-tightness has countable fan-tightness.*

This result generalizes [2, Corollary 6] that states the same conclusion (among regular spaces) under the assumption that X is a compact space of countable tightness, which, of course, implies that X has countable absolute tightness.

By definition, $\mathbb{F}_1^{\diamond\diamond} \subseteq \mathbb{F}_1^{\triangle\diamond}$, so that $\ker(\mathbb{F}_1^{\diamond\diamond}) \subseteq \ker(\mathbb{F}_1^{\triangle\diamond})$. By Theorem 52, $\ker(\mathbb{F}_1^{\triangle\diamond}) \subseteq \mathbb{F}_1^{\dagger}$. Therefore, $\ker(\mathbb{F}_1^{\diamond\diamond}) \subseteq \ker(\mathbb{F}_1^{\triangle\diamond}) \subseteq \mathbb{F}_1^{\dagger} \subseteq \mathbb{F}_1^{\diamond}$. All these inclusions are strict in general. For instance, $\mathcal{S}_\omega \in \mathbb{F}_1^{\diamond} \setminus \mathbb{F}_1^{\dagger}$. Moreover, we already have observed that [4, Example 1] shows that $\ker(\mathbb{F}_1^{\triangle\diamond}) \setminus \ker(\mathbb{F}_1^{\diamond\diamond}) \neq \emptyset$ under (CH). Finally, [4, Example 3] is a ZFC example of a point of countable fan-tightness which is not tight. The neighborhood filter is in $\mathbb{F}_1^{\dagger} \setminus \ker(\mathbb{F}_1^{\triangle\diamond})$. Generally, we have the following picture showing the containments that we know:

$$\begin{tikzcd}
 & \mathbb{F}_\omega^{\triangle\triangle} \arrow[r] \arrow[d] & \mathbb{F}_\omega^{\triangle} \arrow[d] \\
\ker(\mathbb{F}_1^{\triangle\diamond}) \arrow[r] & \ker(\mathbb{F}_\omega^{\triangle\dagger}) \arrow[r] \arrow[d] & \mathbb{F}_1^{\dagger} \arrow[d] \\
\ker(\mathbb{F}_1^{\diamond\diamond}) \arrow[u] \arrow[d] & \ker(\mathbb{F}_\omega^{\triangle\diamond}) \arrow[r] & \mathbb{F}_1^{\diamond} \\
\ker(\mathbb{F}_1^{\dagger\dagger}) \arrow[r] \arrow[uur] & \ker(\mathbb{F}_1^{\dagger\diamond}) \arrow[u] &
\end{tikzcd}$$

Bella and van Mill [5, Theorem 2.3] states that in a regular countably compact space, points of countable tightness are tight. This result can be stated at the level of filters via (4). If X is a topological space, we denote by $\mathcal{O}_X$ the class of filters admitting a base composed of open sets and by $\mathcal{K}_\omega$ the class of filters admitting a base of countably compact sets. The result quoted above follows from

$$\mathcal{K}_\omega \cap \mathcal{O} \cap \ker\bigl(\mathbb{F}_1^{\triangle\diamond}\bigr) = \mathcal{K}_\omega \cap \mathcal{O} \cap \mathbb{F}_1^{\diamond}, \tag{4}$$

by considering neighborhood filters.

We show that the coincidence of these classes is true not only for filters of $\mathcal{K}_\omega$, but for the broader class of strongly q-regular filters. A filter $\mathcal{F}$ is *strongly q-regular* if for every $F \in \mathcal{F}$, there exists a sequence $(Q_n^F)_{n\in\omega}$ in $\mathcal{F}$ such that $\operatorname{adh}_X(\bigwedge_{i\in\omega}(x_n^i)_{n\in\omega}) \cap F \neq \emptyset$ whenever $\bigwedge_{i\in\omega}(x_n^i)_{n\in\omega} \# (Q_n^F)_{n\in\omega}$. Let $\mathcal{Q}_\omega$ denote the class of strongly q-regular filters. We say that a point is strongly q-regular if its neighborhood filter is strongly q-regular.

This notion is a little bit stronger than that of a q-regular point in the sense of [12], which is a variant of a regular q-point in the sense of [21]. All points of countable character and all points of a regular locally countably compact space are strongly q-regular. Dolecki and Nogura [12, Proposition 13] gives an example of a non-first-countable and non-locally countably compact strongly q-regular space.

Theorem 56.

$$\mathcal{Q}_\omega \cap \mathcal{O} \cap \ker\bigl(\mathbb{F}_1^{\triangle\diamond}\bigr) = \mathcal{Q}_\omega \cap \mathcal{O} \cap \mathbb{F}_1^{\diamond}.$$

Proof. Let $\mathcal{F} \in \mathcal{Q}_\omega \cap \mathcal{O} \cap \mathbb{F}_1^{\diamond}$. For every $F \in \mathcal{F}$ there exists $(Q_n^F)_{n\in\omega}$ witnessing the definition of a strongly q-regular filter. Let $\bigwedge_{\alpha\in I}(x_k^\alpha)_{k\in\omega} \# \mathcal{F}$. In particular, $\bigwedge_{\alpha\in I}(x_k^\alpha)_{k\in\omega} \# Q_n^F$ for every $n \in \omega$ and $F \in \mathcal{F}$. Thus, there exists $(x_k^{\alpha_{n,F}})_{k\in\omega} \# Q_n^F$. Hence $\bigwedge_{n\in\omega}(x_k^{\alpha_{n,F}})_{k\in\omega} \# (Q_n^F)_{n\in\omega}$. Therefore, there exists $x_F \in F \cap \operatorname{adh} \bigwedge_{n\in\omega}(x_k^{\alpha_{n,F}})_{k\in\omega}$. Now $\{x_F\colon F \in \mathcal{F}\} \# \mathcal{F}$ and $\mathcal{F} \in \mathbb{F}_1^{\diamond}$, so that there exists a sequence $(F_i)_{i\in\omega}$ in $\mathcal{F}$ such that $\{x_{F_i}\colon i \in \omega\} \# \mathcal{F}$. Then $\bigwedge_{n\in\omega, i\in\omega}(x_k^{\alpha_{n,F_i}})_{k\in\omega} \# \mathcal{F}$. Indeed, for every open $U \in \mathcal{F}$, there exists $i \in \omega$ such that $x_{F_i} \in U$. But $x_{F_i} \in \operatorname{adh} \bigwedge_{n\in\omega}(x_k^{\alpha_{n,F_i}})_{k\in\omega}$. As U is open, $\bigwedge_{n\in\omega}(x_k^{\alpha_{n,F_i}})_{k\in\omega} \# U$. □

The following generalizes [5, Theorem 2.3]:

Corollary 57. *A strongly q-regular point of countable tightness is tight.*

A filter $\mathcal{F}$ is called *regularly of pointwise countable type* if for every $F \in \mathcal{F}$, there exists a sequence $(Q_n^F)_{n\in\omega}$ in $\mathcal{F}$ such that $\operatorname{adh}_X \mathcal{H} \cap F \neq \emptyset$ whenever $\mathcal{H} \# (Q_n^F)_{n\in\omega}$. Let $\mathcal{Q}$ denote the class of regularly of pointwise countable type filters.

Adapting the proof of Theorem 56 and letting $\mathbb{A}$ stand for the class of absolutely countably tight filters, we get:

Theorem 58.

$$\mathcal{Q} \cap \mathcal{O} \cap \mathbb{A} = \mathcal{Q} \cap \mathcal{O} \cap \ker\big(\mathbb{F}_1^{\diamond\diamond}\big) = \mathcal{Q} \cap \mathcal{O} \cap \mathbb{F}_1^{\diamond}.$$

Notice that a regular point of pointwise countable type in the sense of [21] has a neighborhood filter in $\mathcal{Q}$.

Corollary 59. *A point of pointwise countable type and of countable tightness of a completely regular space is absolutely countably tight.*

Combined with Corollary 55, we obtain another improvement of [2, Corollary 6]:

Corollary 60. *If X is regular, of pointwise countable type and of countable tightness, then its product with every space of countable fan-tightness has countable fan-tightness.*

Corollary 59 has other interesting consequences. For instance, it can be combined with Arhangel'skii 's theorem [3, Theorem 1] stating that a topological group with an everywhere dense subset of countable absolute tightness is metrizable to the effect that

Theorem 61. *If a (completely regular) topological group has an everywhere dense subset that has countable tightness and is of pointwise countable type, then this group is metrizable.*

Acknowledgements

The authors want to thank S. Dolecki for comments on [16] which are at the origin of the approach in terms of relations developed in this paper.

References

[1] A.V. Arhangel'skii, The frequency spectrum of a topological space and the product operation, Trans. Moscow Math. Soc. 40 (1981) 163–200.
[2] A.V. Arhangel'skii, A. Bella, Countable fan-tightness versus countable tightness, Comment. Math. Univ. Carolin. 37 (3) (1996) 567–578.
[3] A.V. Arhangel'skii, V.I. Malykhin, Metrizability of topological groups, Moscow Univ. Math. Bull. 51 (3) (1996) 9–11.
[4] A. Bella, V. Malykhin, Around tight points, Topology Appl. 87 (1998) 97–104.
[5] A. Bella, J. van Mill, Tight points and countable fan-tightness, Topology Appl. 76 (1997) 1–7.
[6] C. Costantini, L. Holá, P. Vitolo, Tightness, character and related properties of hyperspace topologies, Topology Appl. 142 (1–3) (2004) 245–292.
[7] C. Costantini, P. Vitolo, On the infimum of the Hausdorff metric topologies, Proc. London Math. Soc. 70 (1995) 441–480.
[8] C. Costantini, P. Simon, An α_4, not Fréchet product of α_4 Fréchet spaces, Topology Appl. 108 (1) (2000) 43–52.
[9] S. Dolecki, Active boundaries of upper semi-continuous and compactoid relations; closed and inductively perfect maps, Rostock Math. Kolloq. 54 (2000) 3–20.
[10] S. Dolecki, Sequential and related filters, Preprint.
[11] S. Dolecki, F. Mynard, Convergence-theoretic mechanisms behind product theorems, Topology Appl. 104 (2000) 67–99.
[12] S. Dolecki, T. Nogura, Two-fold theorem on Fréchetness of products, Czech. Math. J. 49 (2) (1999) 421–429.
[13] E. van Douwen, The product of a Fréchet space and a metrizable space, Topology Appl. 47 (3) (1992) 163–164.
[14] J. Gerlits, Z. Nagy, On Fréchet spaces, Rend. Circ. Mat. Palermo (2) Suppl. 18 (1988) 51–71.
[15] G. Gruenhage, A note on the product of Fréchet spaces, Topology Proc. 3 (1) (1979) 109–115.
[16] F. Jordan, F. Mynard, Productively Fréchet spaces, Czech. Math. J. 54 (4) (2004) 981–990.
[17] F. Jordan, F. Mynard, Espaces productivement de Fréchet, C. R. Acad. Sci. Paris Ser. I 335 (2002) 259–262.
[18] C. Kendrick, On product of Fréchet spaces, Math. Nachr. 65 (1975) 117–123.
[19] E. Lowen-Colebunders, G. Sonck, Exponential objects and Cartesian closedness in the construct PRTOP, Appl. Cat. Struct. 1 (1993) 345–360.
[20] V.I. Malyhin, On countable spaces having no bicompactification of countable tightness, Dokl. Akad. Nauk SSR 206 (6) (1972) 1407–1411.
[21] E. Michael, A quintuple quotient quest, Gen. Topology Appl. 2 (1972) 91–138.
[22] F. Mynard, Coreflectively modified continuous duality applied to classical product theorems, Appl. Gen. Topology 2 (2) (2002) 119–154.
[23] T. Nogura, Product of Fréchet spaces, in: General Topology and Its Relations to Modern Analysis and Algebra VI, Proc. Prague Topological Sympos., vol. 86, 1988, pp. 371–378.
[24] T. Nogura, A counterexample for a problem of Arhangel'skiĭ concerning the Product of Fréchet spaces, Topology Appl. 25 (1987) 75–80.
[25] J. Novák, Double convergence and products of Fréchet spaces, Czech. Math. J. 48 (2) (1998) 207–227.
[26] J. Novák, A note on product of Fréchet spaces, Czech. Math. J. 47 (2) (1997) 337–340.
[27] J. Novák, Concerning the topological product of two Fréchet spaces, in: General Topology and Its Relations to Modern Analysis and Algebra IV, Proc. Fourth Prague Topological Sympos., 1977, pp. 342–343.
[28] P. Nyikos, Classes of compact sequential spaces, in: Set Theory and its Applications, Toronto, ON, 1987, in: Lecture Notes in Math., vol. 1401, Springer, Berlin, 1989, pp. 135–159.
[29] R.C. Olson, Biquotient maps, countably bisequential spaces and related topics, Topology Appl. 4 (1974) 1–28.
[30] P. Simon, A compact Fréchet space whose square is not Fréchet, Comment. Math. Univ. Carolin. 21 (1980) 749–753.
[31] K. Tamano, Product of compact Fréchet spaces, Proc. Japan Acad. Ser. A. Math. Sci. 62 (8) (1986) 304–307.