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A UNIQUENESS THEOREM FOR A GOURSAT-TYPE PROBLEM

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Abstract—In this paper, a q uniqueness theorem is proved for a hyperbolic boundary problem with data on the characteristic cone.

1. INTRODUCTION AND STATEMENT OF THE RESULT

Consider the problem

$$u_{tt} - \Delta u + q(x)u = 0, \qquad x \in \mathbf{R}^3, \quad -r \le t \le r, \quad r = |x|, \tag{1}$$

$$u = 0 \qquad \text{at } t = \pm r, \tag{2}$$

$$\lim_{R\to\infty} \int_{|s|=R} \int_{-R}^{R} |u_r \mp u_t|^2 \, dt\, ds = 0. \tag{3}$$

The boundary data are given on the characteristic cone $t = \pm r$. Condition (3) is a condition at infinity. Assume that $q(x) \in L^2_{\text{loc}}(\mathbf{R}^3)$. No assumptions are made concerning the behavior of $q(x)$ at infinity. The main result of this paper is the following uniqueness theorem.

THEOREM 1. *Problem (1)–(3) has only the trivial solution $u(x,t) = 0$ in the class of functions $u(x,t)$ which are locally $C^3_{x,t}$ and such that*

$$\lim_{R\to\infty} \int_{|s|=R} \int_{-R}^{R} |u_{tr} \pm u_{tt}|^2 \, dt\, ds = 0. \tag{3'}$$

Theorem 1 is of importance in inverse scattering theory [1–3]. In [2] a uniqueness theorem similar to Theorem 1 is formulated. However, the proof in [2] requires that the solution $u(x,t)$ be infinitely differentiable in t, that u and all its derivatives go to zero as $r \to \infty$, and that $q(x)$ and ∇q vanish at a certain rate as $r \to \infty$.

Moreover, the proof uses some formal arguments which need a justification (the class which belongs to the kernel and the solution of equation (4) in [2] is not defined, function (7) is a distribution, and the meaning of the integral in (8) is not explained; also, the argument below formula (3) is not clear).

The purpose of this paper is to give a simple proof of Theorem 1 based on the result from [1]. More general results than in [1] are given in [4]. The results in [4] allow one to formulate an analogue for Theorem 1 for operator equations. In Section 2, a proof of Theorem 1 is given.

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2. PROOF OF THEOREM 1

In [1] the following result is proved: if u solves (1) and satisfies conditions

$$u = 0 \qquad \text{at } t = 0 \quad \text{and} \quad t = r \tag{4}$$

and (3) (with the minus sign) holds, then $u(x,t) = 0$, $x \in \mathbb{R}^3$, $0 \le t \le r$.

If $u(x,t)$ solves (1)–(3), then its even and odd parts $\frac{u(x,t)+u(x,-t)}{2}$ and $\frac{u(x,t)-u(x,-t)}{2}$ solve (1)–(3). The odd part of $u(x,t)$ solves also (4) and, by the cited result from [1], vanishes, so one can assume that

$$u(x,t) = u(x,-t). \tag{*}$$

Define $w(x,t) := u_t(x,t)$. From (*), it follows that $w(x,t) = -w(x,-t)$. Therefore,

$$w(x,t) = 0 \qquad \text{at } t = 0. \tag{5}$$

Clearly, $w(x,t)$ solves (1) since $q(x)$ does not depend on t. Moreover, w satisfies (3) by the assumption (3′). If one proves that

$$w = 0 \qquad \text{at } t = r, \tag{6}$$

then, by the uniqueness theorem from [1], it follows that $w = 0$ for all $x \in \mathbb{R}^3$, $0 \le t \le r$. This implies that $u(x,0) = \int_r^0 w(x,\tau)\,d\tau = 0$. Using again the uniqueness result from [1], one concludes that $u(x,t) = 0$ for all $x \in \mathbb{R}^3$ and all $0 \le t \le r$. By (*) it follows that $u(x,t) = 0$ for all $x \in \mathbb{R}^3$ and $-r \le t \le r$. This completes the proof of Theorem 1 as long as the following lemma is proved.

LEMMA 1. *If u solves (1), $u(x,t) = u(x,-t)$, $u(x,t) \in C^3_{\text{loc}}$, $u(x,\pm r) = 0$, then $w(x,t) := u_t(x,t)$ satisfies (6).*

PROOF. Let $\xi := (r+t)/2$, $\rho := (r-t)/2$, $v := r\,u(x,t)$. Then (1) becomes

$$v_{tt} - v_{rr} - r^{-2}Av + qv = 0, \quad A = -\Delta^*, \tag{7}$$

where Δ^* is the angular part of the Laplacian. In the variables ξ, ρ, this equation becomes

$$v_{\xi\rho} + (\xi+\rho)^{-2}\,Av - qv = 0. \tag{8}$$

The boundary conditions (2) become

$$v = 0 \qquad \text{at } \rho = 0 \quad \text{and} \quad \text{at } \xi = 0. \tag{9}$$

Since the operator A does not act on the variables ρ and ξ, it follows from (8) and (9) that

$$v_{\xi\rho} = 0 \qquad \text{at } \rho = 0 \quad \text{and} \quad \text{at } \xi = 0. \tag{10}$$

One has $ru_t = v_t = (v_\xi - v_\rho)/2$. Thus, $2v_t|_{t=r} = (v_\xi - v_\rho)|_{\rho=0} = v_\xi|_{\rho=0} - v_\rho|_{\rho=0}$. It follows from (9) that $v_\xi|_{\rho=0} = 0$. Therefore, $2v_t|_{t=r} = -v_\rho|_{\rho=0}$. It follows from (10) that $v_\rho|_{\rho=0} = c$, where c does not depend on ξ (c may depend on the angular variables). If one proves that $c = 0$, then the proof of Lemma 1 is complete. One has

$$-2ru_t\,|_{t=r} = v_\rho\big|_{\rho=0} = c. \tag{11}$$

By the assumption, u_t is a locally continuous function. Therefore, taking $r \to 0$ in (11) yields $c = 0$. Lemma 1 is proved.

The proof of Lemma 1 is close to an argument in [2]. This completes the proof of Theorem 1.

REMARK 1. Condition (3) is satisfied if

$$\lim_{R\to\infty} \int_{|s|=R} \int_{-R}^{R} \left[|u_r|^2 + |u_t|^2\right] ds\, dt = 0. \tag{3''}$$

REMARK 2. In [4], equation (7) with an abstract operator $A \geq 0$ was studied in a Hilbert space.

REMARK 3. One can give an example of nonuniqueness of the solution to a Goursat-type problem in the class of functions which do not decay at infinity (so that condition (3) does not hold). For instance (cf. [5, p. 238]), the function

$$u(x_1, x_2, t) := \int_{|x-y|\leq t} \frac{\rho^m \cos n\varphi\, \rho\, d\rho\, d\varphi}{(t^2 - |x-y|^2)^{1/2}}, \qquad u \not\equiv 0, \quad y_1 = \rho\cos\varphi, \quad y_2 = \rho\sin\varphi,$$

m and n are positive integers, $n > m+1$, n is even, m is odd, solves the wave equation $u_{tt} - u_{x_1x_1} - u_{x_2x_2} = 0$ in $\mathbf{R}_x^2 \times \mathbf{R}_t$, $r := (x_1^2 + x_2^2)^{1/2}$, $x_1 = r\cos\theta$, $x_2 = r\sin\theta$, satisfies the conditions $u = 0$ at $t = 0$ and $u = 0$ at $t = r$. To check the last claim, one calculates

$$\begin{aligned} u(x,r) &= \int_{r^2 - 2r\rho\cos(\theta-\varphi)+\rho\leq r^2} \frac{\rho^{m+1}\cos n\varphi\, d\rho\, d\varphi}{\left[2r\rho\cos(\theta-\varphi) - \rho^2\right]^{1/2}} = \int_{\rho\leq 2r\cos\alpha} \frac{\rho^{m+\frac{1}{2}}\, d\rho \cos n\varphi\, d\varphi}{\left[2r\cos(\theta-\varphi)-\rho\right]^{1/2}} \\ &= \int_{-\pi/2+\theta}^{\pi/2+\theta} d\varphi \cos n\varphi \int_0^{\gamma} \frac{\rho^{m+\frac{1}{2}}\, d\rho}{(\gamma-\rho)^{1/2}} = \int_{-\pi/2}^{\pi/2} d\alpha \cos[n(\theta+\alpha)]\,(2r\cos\alpha)^{m+1} \\ &= c\,(2r)^{m+1}\left[\cos(n\theta)\int_{-\pi/2}^{\pi/2}(\cos\alpha)^{m+1}\cos(n\alpha)\, d\alpha \right. \\ &\quad \left. -\sin(n\theta)\int_{-\pi/2}^{\pi/2}(\cos\alpha)^{m+1}\sin(n\alpha)\, d\alpha\right]. \end{aligned}$$

Here, $\alpha = \varphi - \theta$, $\gamma = 2r\cos\alpha$, $c = \int_0^1 \frac{t^{m+1}\, dt}{(1-t)^{1/2}}$. If $n > m+1$, n is even, m is odd, then

$$\int_{-\pi/2}^{\pi/2} (\cos\alpha)^{m+1}\cos(n\alpha)\, d\alpha = 0, \qquad \int_{-\pi/2}^{\pi/2} (\cos\alpha)^{m+1}\sin(n\alpha)\, d\alpha = 0$$

so that $u(x,r) = 0$. The function $u(x_1, x_2, t) \not\equiv 0$ since $u_t = r^m\cos(n\theta)$ at $t = 0$.

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