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Existence of Solutions for a Class of Resonant Elliptic Problems

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INTRODUCTION

In this paper we consider *resonant* elliptic problems of the form

$$-\Delta u = \lambda_1 u + g(u) \qquad \text{in } \Omega,\ u = 0 \text{ on } \partial\Omega, \tag{P}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, λ_1 is the first eigenvalue of the problem $-\Delta u = \lambda u$ in Ω, $u = 0$ on $\partial\Omega$, and the nonlinearity $g\colon \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying the growth condition

$$|g(s)| \leqslant a\,|s|^{\sigma} + b \qquad \forall s \in \mathbb{R}, \tag{$*$}_\sigma$$

where a, $b > 0$ and $\sigma \geqslant 0$ are constants. When g is bounded, (P) is a resonant problem at λ_1, in the sense that $\lim_{|s| \to \infty} f(s)/s = \lambda_1$ where $f(s) = \lambda_1 s + g(s)$. If, in addition, one has

$$\lim_{|s| \to \infty} g(s) = 0 \qquad \text{and} \qquad \lim_{|s| \to \infty} G(s) = \hat{\beta} \in \mathbb{R},$$

where $G(s) = \int_0^s g(t)\,dt$, then (P) is called (cf. [6]) a *strong resonant problem* at λ_1.

In [6] the authors consider some situations of strong resonance, including the case of higher eigenvalues. Here, it is our objective to

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study other situations in which one has *one-sided strong resonance*, more precisely, we assume that the nonlinearity g satisfies

$$\lim_{s \to +\infty} g(s) = 0 \qquad \text{and} \qquad \lim_{s \to +\infty} G(s) = 0. \qquad (g_1)$$

Denoting by $g(\pm\infty)$ and $G(\pm\infty)$ the corresponding limits $\lim_{s \to \pm\infty} g(s)$ and $\lim_{s \to \pm\infty} G(s)$, we are therefore assuming that $g(+\infty) = 0$, $\beta = G(+\infty) = 0$ and we consider various cases depending on the value of $G(-\infty) = \alpha \in [-\infty, +\infty]$. In a previous paper [12] the case $\alpha = +\infty$ was considered and, therefore, we restrict our attention to the other situations.

In Theorems 1 to 3 below, we will be assuming that the nonlinearity g has *subcritical growth*, that is, $(*)_\sigma$ holds with $\sigma < (N+2)/(N-2)$ if $N \geqslant 3$ and $\sigma < \infty$ if $N = 1, 2$.

THEOREM 1. *Assume* (g_1) *and* $-\infty \leqslant \alpha \leqslant 0$. *In addition, assume*

$$G(s) \geqslant 0 \quad \text{if} \quad 0 < s < \delta \ (\text{or} \ -\delta < s < 0), \text{ for some } \delta > 0, \qquad (G_1)$$

if $-\infty < \alpha \leqslant 0$. *Then, problem* (P) *possesses a nonzero solution* $u \in H_0^1(\Omega)$.

When α is positive we need to impose further restrictions on the nonlinearity g.

THEOREM 2. *Assume* (g_1), $0 < \alpha < \infty$, *and*

$$g(-\infty) = 0, \qquad (g_2)$$

$$G(s) \leqslant \tfrac{1}{2}(\lambda_2 - \lambda_1)\, s^2 \qquad \forall s \in \mathbb{R}. \qquad (G_2)$$

Then, problem (P) *possesses a nonzero solution* $u \in H_0^1(\Omega)$.

Regarding multiplicity, we are able to show existence of two nonzero solutions when the nonlinearity g satisfies

$$g(-\infty) = G(-\infty) = 0, \qquad (\hat{g}_2)$$

namely, we have the following

THEOREM 3 (Multiplicity). *Under conditions* (g_1), $(\hat{g}_2)$, (G_1), *and* (G_2), *problem* (P) *has at least two nonzero solutions.*

These results extend and complement some of the results in [6, 15, 21, 27, 30]. We observe that the solutions $u \in H_0^1$ obtained in Theorems 1 to 3 are weak solutions in H_0^1, in the sense that

$$\int_\Omega \nabla u \cdot \nabla \theta \, dx - \int_\Omega \lambda_1 u \theta \, dx - \int_\Omega g(u) \theta \, dx = 0 \qquad \forall \theta \in H_0^1.$$

In fact, since we are assuming that g has subcritical growth, the functional $I: H_0^1 \to \mathbb{R}$ given by

$$I(u) = \int_\Omega \frac{1}{2}(|\nabla u|^2 - \lambda_1 u^2)\,dx - \int_\Omega G(u)\,dx$$
$$= \frac{1}{2}(\|u\|^2 - \lambda_1 |u|_2^2) - N(u),$$

is of class C^1 and the solutions we obtain are critical points of I.

On the other hand, if $g: \mathbb{R} \to \mathbb{R}$ is only assumed to satisfy the *supercritical* growth condition $(*)_\sigma$ with $\sigma = 2^*$, namely

$$|g(s)| \leqslant a\,|s|^{2^*} + b \qquad \forall s \in \mathbb{R} \text{ (and some } a, b > 0), \qquad (g_3)$$

where $2^* = 2N/(N-2)$ $(N \geqslant 3)$ is the limiting exponent for the Sobolev embedding $H_0^1 \subset L^p$, then the functional $I: H_0^1 \to [-\infty, +\infty]$ is not necessarily differentiable and, in this case, we look for weak solutions $u \in H_0^1$ in the sense of distributions, that is, functions $u \in H_0^1$ such that

$$\int_\Omega \nabla u \cdot \nabla\theta\,dx - \int_\Omega \lambda_1 u\theta\,dx - \int_\Omega g(u)\theta\,dx = 0 \qquad \forall \theta \in C_0^\infty.$$

THEOREM 4. *Assume* (g_3) *and* $-\infty \leqslant \beta \leqslant 0$, $-\infty \leqslant \alpha \leqslant 0$. *If* (G_1) *holds, then problem* (P) *has a nonzero solution* $u \in H_0^1$ *in the sense of distributions, which minimizes the functional* I.

It should be noticed that solely under the hypotheses (g_3), $\beta \in [-\infty, 0]$, $\alpha \in [-\infty, 0]$ and without a *local sign condition* such as (G_1) problem (P) could have $u = 0$ as the unique minimum of the functional I.

Theorem 4 partially complements the main result in [20], where condition (g_3) was considered (a similar supercritical condition was also considered in [4]). Under the assumption (g_3) (and in the x-dependent case), existence of a solution in the sense of distributions is shown in [20], provided that $G(x, s) = \int_0^s g(x, t)\,dt$ satisfies a *quadratic growth condition from above* and $B_\infty(x) = \limsup_{|s| \to \infty} 2G(x, s)/s^2$ is such that

$$i(B_\infty) = \inf\left\{\int_\Omega [|\nabla v|^2 - B_\infty(x)\,v^2]\,dx \;|\; v \in H_0^1,\ |v|_2 = 1\right\} > \lambda_1.$$

Clearly, in the situation of Theorem 4 we could have $B_\infty \equiv 0$, hence $i(B_\infty) = \lambda_1$. In Section 2 we will state and prove another related result of this type where $i(B_\infty) = \lambda_1$ is allowed.

We remark that there is a rich literature dealing with resonant problems, starting with a very nice result due to Landesman and Lazer [23]. Besides the already cited papers, we refer the interested reader to, e.g., [1–5, 7–14, 16–19, 24–26, 28, 29, 31] and references therein.

1. Proofs of Theorems 1, 2, and 3

We start recalling that a C^1 functional $I: E \to \mathbb{R}$ (E a Banach space) satisfies the local Palais–Smale condition $(\mathrm{PS})_c$ at the level $c \in \mathbb{R}$ if, whenever a sequence (u_n) in E is such that

$$I(u_n) \to c, \qquad I'(u_n) \to 0,$$

then (u_n) has a convergent subsequence. We need the following preliminary results, which are inspired from [12, Lemma 7; 5, Theorem 3.4]. Their proofs are given in Section 3.

Lemma 1. *Assume that $g: \mathbb{R} \to \mathbb{R}$ is bounded and there exist the limits $\beta = G(+\infty) \in [-\infty, +\infty]$, $\alpha = G(-\infty) \in [-\infty, +\infty]$. In addition, assume that $g(+\infty) = 0$ (resp. $g(-\infty) = 0$) in case $\beta \in \mathbb{R}$ (resp. $\alpha \in \mathbb{R}$). Then*

$$\{c \in \mathbb{R} \mid I \text{ satisfies } (\mathrm{PS})_c\} = \mathbb{R} \backslash \{-\alpha\,|\Omega|, -\beta\,|\Omega|\}.$$

Lemma 2. *Assume that $g: \mathbb{R} \to \mathbb{R}$ has subcritical growth and satisfies (g_1). Then I satisfies $(\mathrm{PS})_c$ for every $c \neq 0$ such that $c < -\alpha\,|\Omega|$.*

Proof of Theorem 1. *Case* $\alpha = -\infty$. It follows from Lemma 1 that I satisfies $(\mathrm{PS})_c$ for all $c \neq 0$. Now, consider the orthogonal complement W of $\langle \phi_1 \rangle$ and, for each $t \in \mathbb{R}$ let $m_t = \inf_{W_t} I$, where $W_t = \{t\phi_1 + w \mid w \in W\}$, and notice that $m_t > -\infty$ is attained in view of the coercivity of I on W. Also, since in this case the functional I is bounded from below, we have that $-\infty < m = \inf_{H_0^1} I \leqslant I(0) = 0$ and $m \leqslant m_t$ for every $t \in \mathbb{R}$. Fix some $T > 0$.

(i) If $m < 0$ then, since I satisfies $(\mathrm{PS})_m$, it follows that $m < 0 = I(0)$ is a critical value of I.

(ii) If $m = 0 \leqslant m_T$ then either we have $m_T = 0$ and, therefore, $I(u_T) = 0 = m$ for some $u_T = T\phi_1 + w \in W_T$, or else we have $m_T > 0$. In this latter case, noticing that $\lim_{t \to +\infty} I(t\phi_1) = 0$ in view of (g_1), we can apply the Saddle Point Theorem of Rabinowitz [28, 29] to conclude that I has a critical value $c \geqslant m_T > 0 = I(0)$.

Thus, in either one of the possibilities (i) or (ii), I has a critical point $u \neq 0$.

Case $-\infty < \alpha \leqslant 0$. In view of Lemma 2, I satisfies $(\mathrm{PS})_c$ for all $c \neq 0$, $c < -\alpha\,|\Omega|$. Again, since the functional I is bounded from below, we have

$-\infty < m \leqslant 0 = I(0)$. If $m<0$ it follows that I satisfies $(\mathrm{PS})_m$ and $m<0=I(0)$ is a critical value of I. On the other hand, if $m=0$ then, by (G_1) we have

$$I(t\phi_1) = -\int_\Omega G(t\phi_1)\,dx \leqslant 0$$

for $t>0$ (resp. $t<0$) small and, hence, there exists $u \neq 0$ such that $I(u)=0$. ∎

Proof of Theorem 2. Since Lemma 1 gives that I satisfies $(\mathrm{PS})_c$ if $c \neq 0$, $-\alpha\,|\Omega|$ and since we still have in this case that $-\infty < m \leqslant 0$, we can not guarantee that m is attained. Instead, we consider the infimum in the half-space $H_+ = \{t\phi_1 + w \mid t>0,\ w \in W\}$,

$$-\infty < m_+ = \inf_{H_+} I \leqslant 0,$$

and proceed to show that m_+ is attained at some $u_+ \in H_+$ in case $m_+ < 0$. First of all, we notice that $\partial H_+ = W$ and that $m_0 = \inf_W I = 0$ in view of $I(0)=0$ and hypothesis (G_2). Now, if $m_+ = 0$, we look at $m_T = \inf_{W_T} I$ for a fixed $T>0$, hence $m_T \geqslant m_+ = 0$, and proceed as in the first case of Theorem 1, considering the possibilities $m_T = 0$ and $m_T > 0$. On the other hand, if $m_+ < 0$, we pick a minimizing sequence $u_n = t_n\phi_1 + w_n$ $(t_n > 0)$, that is,

$$I(u_n) \to m_+ < 0, \tag{1}$$

and proceed to show that (u_n) is bounded.

In fact, the sequence (w_n) is bounded since

$$I(u_n) = \tfrac{1}{2}(\|w_n\|^2 - \lambda_1\,|w_n|_2^2) - N(u_n) = q(w_n) - N(u_n),$$

where $q \geqslant 0$ is coercive on W and N is bounded on H_0^1. But then, we must also have $t_n > 0$ bounded since, otherwise, Lebesgue's dominated convergence theorem and (g_1) applied to

$$N(u_n) = \int_\Omega G(t_n\phi_1 + w_n)\,dx$$

(recall that G is bounded) would imply that $N(u_n) \to 0$, hence

$$\lim_{n\to\infty} I(u_n) \geqslant 0,$$

contradicting (1). Thus (u_n) must be bounded and, for a subsequence (still denoted by (u_n)) and some $\hat{u} \in \bar{H}_+$, we obtain that

$$u_n \rightharpoonup \hat{u}, \qquad u_n \to \hat{u} \text{ a.e. and in } L^2.$$

Now, it follows by Lebesgue's dominated convergence theorem that $N(u_n) \to N(\hat{u})$ and, hence, by weak lower semicontinuity of q, that

$$I(\hat{u}) = q(\hat{u}) - N(\hat{u}) \leqslant \liminf q(u_n) - \lim N(u_n) = \liminf I(u_n) = m_+.$$

Therefore, we obtain that $I(\hat{u}) = m_+$ and, since we are assuming $m_+ < 0$, it necessarily follows that $\hat{u} \notin \partial H_+ = W$ and $\hat{u} \in H_+$ is a local minimum of I on H_+. In particular, $\hat{u}$ is a nonzero solution of (P). The proof of Theorem 2 is complete. ∎

Remark. Theorem 2 can be proved without condition (g_2) as long as we assume the local sign condition $G(s) \geqslant 0$ if $0 < s < \delta$.

Proof of Theorem 3. We start recalling that, in view of Lemma 1, I satisfies $(\mathrm{PS})_c$ for every $c \neq 0$. Fix $T_- < 0 < T_+$ and, which the notation of Theorem 1, consider the infima m_{T_-} and m_{T_+}. Also, define

$$m_\pm = \inf_{H_\pm} I,$$

where $H_\pm = \{t\phi_1 + w \mid t > 0 \ (t < 0),\ w \in W\}$. We consider various cases depending on the values of m_{T_-} and m_{T_+}.

Case (a). $m_{T_-} \leqslant 0$, $m_{T_+} \leqslant 0$. In this case we have $m_- \leqslant 0$, $m_+ \leqslant 0$, and, arguing as in Theorem 2, we obtain two nonzero solutions $u_- \in H_-$ and $u_+ \in H_+$.

Case (b). $m_{T_-} > 0$, $m_{T_+} \leqslant 0$. As above, there exists a nonzero solution $u_+ \in H_+$ with $I(u_+) = m_+ \leqslant 0$. On the other hand, since $I(0) = 0$, $\lim_{t \to -\infty} I(t\phi_1) = 0$ and $m_{T_-} > 0$, the Saddle Point Theorem gives another critical value $c \geqslant m_{T_-} > 0 = I(0)$.

Case (c) $m_{T_-} \leqslant 0$, $m_{T_+} > 0$. This case is similar to Case (b).

Case (d) $m_{T_-} > 0$, $m_{T_+} > 0$. We first observe that, as in Case (b) (or Case (c)), the functional I has a critical value $c > 0$ by the Saddle Point Theorem. On the other hand, if we define

$$\hat{m} = \inf_U I, \qquad \text{where} \quad U = \{t\phi_1 + w \mid T_- < t < T_+,\ w \in W\},$$

then $-\infty < \hat{m} \leqslant 0 = I(0)$ and, arguing as in Case (a), we conclude that $\hat{m} \leqslant 0$ is a critical value of I. In fact, if $\hat{m} = 0$ then, again, as in the proof of Theorem 1, condition (G_1) implies the existence of $0 \neq u \in U$ such that $I(u) = 0$. ∎

Remarks. (1) It should be noticed that, even without the local sign condition on $G(s)$, Theorem 3 yields multiplicity of solutions (one of which may be the zero solution).

(2) It should be also noticed from the argument of Theorem 1 (case $\alpha = -\infty$) that conditions (g_1), $(\hat{g}_2)$ alone are sufficient to guarantee existence of one nonzero solution in Theorem 3.

2. Proof of Theorem 4 and Some Related Results

In view of (g_3) and as $\alpha, \beta < +\infty$, our functional $I: H_0^1 \to (-\infty, +\infty]$ is well-defined and bounded from below. Therefore, we have $-\infty < m = \inf_{H_0^1} I \leq 0 = I(0)$. If $m = 0$, the conclusion follows from the *local sign condition* (G_1) (cf. proof of Theorem 1). Thus, without loss of generality, we may suppose that $m < 0$.

Letting $u_n = t_n \phi_1 + w_n$ be a minimizing sequence, that is,

$$I(u_n) \to m < 0, \tag{2}$$

we will show that (u_n) is bounded. In fact, as in Theorem 2, we conclude that the sequence (w_n) is bounded since

$$I(u_n) = \tfrac{1}{2}(\|w_n\|^2 - \lambda_1 |w_n|_2^2) - N(u_n) = q(w_n) - N(u_n),$$

where $q \geq 0$ is coercive on W and $-N$ is bounded from below on H_0^1 (recall that $-G(s)$ is bounded from below). On the other hand, we must also have $|t_n|$ bounded since, otherwise, Fatou's Lemma applied to $-N(u_n)$ would yield

$$\liminf I(u_n) \geq \min\{-\alpha\,|\Omega|,\ -\beta\,|\Omega|\},$$

hence $m \geq 0$, which contradicts (2). Thus (u_n) must be bounded and, for a subsequence (still denoted by (u_n)) and some $\hat{u} \in H_0^1$, we obtain that

$$u_n \rightharpoonup \hat{u}_n, \qquad u_n \to \hat{u} \text{ a.e. and in } L^2.$$

In particular, Fatou's lemma gives us $-N(\hat{u}) \leq \liminf[-N(u_n)]$ which, together with the weak lower semicontinuity of q, yields

$$I(\hat{u}) = q(\hat{u}) - N(\hat{u}) \leq \liminf q(u_n) + \liminf[-N(u_n)] \leq \liminf I(u_n) = m.$$

Thus, we obtain $I(\hat{u}) = m < 0$ and $u \neq 0$ is a minimizer for the functional I. Finally, the fact that $\hat{u}$ is a solution of (P) in the sense of distributions will follow using the hypothesis (g_3) and an argument as in [20] (cf. also [4, 22]) based on Fatou's Lemma, which we omit here. The proof of Theorem 4 is complete. ∎

Theorems 1 to 4 could be naturally extended to allow an x-dependence on the nonlinearity g. In fact, we now prove a further related result for such a resonant problem. More precisely, we will consider problems of the form

$$-\Delta u = \lambda_1 u + g(x, u) \qquad \text{in } \Omega,\ u = 0 \text{ on } \partial\Omega, \tag{$\hat{P}$}$$

where $g: \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the *supercritical growth condition* (g_3) (with $b > 0$ replaced by $b(x) \in L^1(\Omega)$) and the primitive $G(x, s) = \int_0^s g(x, t)\, dt$ satisfies the following *subquadratic growth conditions from above*:

$$G(x, s) \leqslant \tfrac{1}{2} A\, |s|^\alpha + B(x) \text{ a.e. } x \in \Omega,\ \forall s \in \mathbb{R},$$
$$\text{for some } A > 0,\ B(x) \in L^1(\Omega), \text{ and } 1 < \alpha < 2; \tag{g_4}$$

$$G(x, s) \leqslant -\tfrac{1}{2} \delta\, |s|^\beta + B_0(x) \text{ a.e. } x \in \Omega_0,\ \forall s \in \mathbb{R},$$
$$\text{for some } \delta > 0,\ B_0(x) \in L^1(\Omega),\ 1 < \alpha < \beta < 2, \text{ and } \Omega_0 \subset \Omega$$
$$\text{of positive measure.} \tag{g_5}$$

THEOREM 5. *Under conditions* (g_3)–(g_5), *problem* $(\hat{P})$ *has a solution* $u \in H_0^1$ *in the sense of distributions, which minimizes the functional* I.

As already mentioned in the Introduction, in [20] it is assumed that $g(x, s)$ satisfies (g_3) and then shown existence of a solution of $(\hat{P})$ in the sense of distributions provided that $G(x, s)$ is *quadratic from above* (that is, satisfies (g_4) with $\alpha = 2$) and $B_\infty(x) = \limsup_{|s| \to \infty} 2G(x, s)/s^2$ is such that

$$i(B_\infty) = \inf \left\{ \int_\Omega [|\nabla v|^2 - B_\infty(x)\, v^2]\, dx \mid v \in H_0^1,\ |v|_2 = 1 \right\} > \lambda_1.$$

We notice that the above condition $i(B_\infty) > \lambda_1$ implies that one must have $B_\infty(x) < 0$ on some set of positive measure. Thus, Theorem 4 complements the aforementioned result since conditions (g_4), (g_5) clearly imply that $B_\infty(x) \leqslant 0$ and, in fact, one could have situations for which (g_4), (g_5) hold and where $B_\infty \equiv 0$, so that $i(B_\infty) = \lambda_1$ and the result of [20] could not be used.

Proof of Theorem 5. We claim that our functional

$$I(u) = \frac{1}{2} (\|u\|^2 - \lambda_1 |u|_2^2) - \int_\Omega G(x, u)\, dx$$

is *coercive*, that is, $I(u) \to +\infty$ as $\|u\| \to \infty$. Indeed, suppose by contradiction that

$$I(u_n) = \frac{1}{2}\left(\|u_n\|^2 - \lambda_1 |u_n|_2^2\right) - \int_\Omega G(x, u_n)\, dx \leqslant C, \tag{3}$$

for some constant C and some sequence (u_n) with $\|u_n\| \to \infty$. Letting $v_n = u_n/|u_n|_2$ and dividing (3) by $|u_n|_2^2$, we obtain in view of (g_4) and of the continuous embedding $H_0^1 \subset L^\alpha$ that

$$\begin{aligned} \frac{1}{2}\left(\|v_n\|^2 - \lambda_1\right) &\leqslant \frac{A}{2}\,\frac{|v_n|_\alpha^\alpha}{|u_n|_2^{2-\alpha}} + \frac{\int_\Omega B}{|u_n|_2^2} + \frac{C}{|u_n|_2^2} \\ &\leqslant M\,\frac{\|v_n\|^\alpha}{|u_n|_2^{2-\alpha}} + \frac{N}{|u_n|_2^2}. \end{aligned} \tag{4}$$

Now, (3) implies that $|u_n|_2 \to \infty$ since, otherwise, we would obtain

$$\|u_n\|^2 \leqslant \lambda_1 |u_n|_2^2 + A\,|u_n|_2^2 + 2\int_\Omega B + 2C \leqslant D,$$

as $\alpha < 2$. Therefore, estimate (4) yields $\|v_n\|^2 - \lambda_1 \leqslant M_0 \|v_n\|^\alpha + N_0$ for all n large, hence

$$\|v_n\| \leqslant \text{constant},$$

again using $\alpha < 2$. Passing to a subsequence if necessary, we obtain

$$v_n \rightharpoonup v, \qquad v_n \to v \text{ a.e. and in } L^2,$$

for some $v \in H_0^1$ with $|v|_2 = 1$ (since $|v_n|_2 = 1$). But then, (4) gives

$$\tfrac{1}{2}\left(\|v\|^2 - \lambda_1\right) \leqslant \tfrac{1}{2}\liminf\left(\|v_n\|^2 - \lambda_1\right) \leqslant 0,$$

so that necessarily $v = \phi_1$ is a λ_1-eigenfunction with $|v|_2 = 1$. Now, writing

$$u_n = t_n \phi_1 + w_n,$$

with w_n orthogonal to ϕ_1 and recalling that $v_n \to v$ in L^2, we obtain that

$$\frac{t_n}{|u_n|_2} \to 1 \qquad \text{and} \qquad \frac{w_n}{t_n} \to 0 \text{ in } L^2. \tag{5}$$

On the other hand, using (g_4) and (g_5) to estimate the two integrals in

$$I(u_n) = \frac{1}{2}\left(\|w_n\|^2 - \lambda_1 |w_n|_2^2\right) - \int_{\Omega_0} G(x, u_n)\, dx - \int_{\Omega\setminus\Omega_0} G(x, u_n)\, dx,$$

we obtain

$$I(u_n) \geq \frac{\lambda}{2} |w_n|_2^2 + \frac{\delta}{2} |u_n|_{\beta, \Omega_0}^\beta - \frac{A}{2} |u_n|_\alpha^\alpha - \gamma,$$

where $\lambda = \lambda_2 - \lambda_1 > 0$, $\gamma \in \mathbb{R}$, and $|\cdot|_{\beta, \Omega_0}$ denotes the L^β-norm in Ω_0. We can rewrite the above expression as

$$I(u_n) \geq \frac{\lambda}{2} |w_n|_2^2 + \frac{\delta |t_n|^\beta}{2} |\phi_1 + \hat{w}_n|_{\beta, \Omega_0}^\beta - \frac{A |t_n|^\alpha}{2} |\phi_1 + \hat{w}_n|_\alpha^\alpha - \gamma, \tag{6}$$

where $\hat{w}_n = w_n/t_n \to 0$ in $L^\beta(\Omega_0)$ and in $L^\alpha(\Omega)$ in view of (5) and the fact that $1 < \alpha$, $\beta < 2$. Therefore, since $\alpha < \beta$ and $|u_n|_2^2 = t_n^2 + |w_n|_2^2 \to \infty$, (6) implies that

$$I(u_n) \to +\infty,$$

which is a contradiction to (3). Thus, the functional I is coercive.

Now, hypothesis (g_4) implies that I is weakly lower semicontinuous (cf. [20], where α can be taken equal to 2) and, therefore, I is bounded from below and there exists $\hat{u} \in H_0^1$ such that

$$I(\hat{u}) = \inf_{H_0^1} I.$$

Finally, using hypothesis (g_3) and again a Fatou's lemma argument as in [4, 20], it follows that the minimizer $\hat{u}$ is a solution of $(\hat{P})$ in the sense of distributions. The proof of Theorem 5 is complete. ∎

Remarks. (1) In view of condition (g_3) (or, more generally, a condition of the type $\sup_{|s| \leq r} |G(x, s)| \in L^1(\Omega)$), it is clear that conditions (g_4) and (g_5) are implied, respectively, by the *uniform* conditions

$$\limsup_{|s| \to \infty} 2G(x, s)/|s|^\alpha \leq A < +\infty, \text{ uniformly for a.e. } x \in \Omega, \tag{$\hat{g}_4$}$$

$$\limsup_{|s| \to \infty} 2G(x, s)/|s|^\beta \leq -\delta < 0, \text{ uniformly for a.e. } x \in \Omega_0. \tag{$\hat{g}_5$}$$

However, since $B(x)$ and $B_0(x)$ are only assumed to be in $L^1(\Omega)$, rather than in $L^\infty(\Omega)$, conditions (g_4), (g_5) do not necessarily imply $(\hat{g}_4)$, $(\hat{g}_5)$.

(2) Some comments on Theorem 5 are now in order. Aside from the fact that the *supercritical* condition (g_3) suffices to prove that minimizers are solutions in the sense of distributions (cf. [4, 20]), both Theorem 5 and

the main result of [20] are based on the fact that the functional I is shown to be *coercive* (so that the basic minimization result of the calculus of variations may be used). In [20], the coercivity is a consequence of hypotheses (g_4) (with $\alpha=2$) and $i(B_\infty)>\lambda_1$. On the other hand, in Theorem 5 the coercivity follows from conditions (g_4) and (g_5), which could hold true in situations where $i(B_\infty)=\lambda_1$. These observations suggest that the question of coercivity of the functional I should be further explored and, hopefully, one should be able to unify and better understand such results through more general conditions on the primitive $G(x, s)$.

3. Proofs of Lemmas 1 and 2

We omit the proof of Lemma 1 since it is similar to that of [12, Lemma 7].

Proof of Lemma 2. Considering $u_n \subset H_0^1(\Omega)$ satisfying

(i) $I(u_n) \to c \neq 0$,

(ii) $I'(u_n) \to 0$,

(iii) $\|u_n\| \to \infty$,

we will show that $c \geqslant -\alpha\, |\Omega|$. As before, we write $u_n = t_n\phi_1 + w_n$ so that

$$I(u_n) = q(w_n) - N(u_n),$$

where $q \geqslant 0$ is coercive on W and $-N$ is bounded from below on H_0^1. So, it follows that

$$\|w_n\| \leqslant M \qquad \forall n \in \mathbb{N} \tag{7}$$

and, without loss of generality, we may assume that

$$\begin{aligned} w_n &\rightharpoonup w && \text{weakly in } H_0^1 \\ w_n &\to w && \text{strongly in } L^p \\ w_n(x) &\to w(x) && \text{a.e. in } \Omega \\ |w_n(x)| &\leqslant h_p(x) && \text{a.e. in } \Omega, \text{ where } h_p \in L^p, \end{aligned} \tag{8}$$

and $1 \leqslant p < 2N/(N-2)$ if $N \geqslant 3$. Now, (iii) and (7) imply that $|t_n| \to \infty$.

Claim. If $t_n \to +\infty$ as $n \to \infty$ then $\|w_n\| \to 0$.

Indeed, since

$$\langle I'(u_n), w_n \rangle = \|w_n\|^2 - \lambda_1\, |w_n|_2^2 - \int g(u_n)\, w_n \to 0 \tag{9}$$

in view of (ii) and (7), it suffices to show that the integral term goes to zero as $n \to \infty$. Let $s_0 > 0$ be such that $|g(s)| \leqslant \varepsilon \ \forall s \geqslant s_0$ and consider the sets

$$A_n = \{x \in \Omega \mid t_n \phi_1(x) + w_n(x) \geqslant s_0\},$$
$$B_n = \{x \in \Omega \mid t_n \phi_1(x) + w_n(x) < s_0\},$$

so that $\Omega = A_n \cup B_n$. We clearly have

$$\left| \int_{A_n} g(t_n \phi_1 + w_n)\, w_n \right| \leqslant \varepsilon \int_{A_n} |w_n| \leqslant \varepsilon\, |h_1|_1, \tag{10}$$

where h_1 is given by (8). On the other hand, using (8) and $(*)_\sigma$ we obtain

$$|g(t_n \phi_1(x) + w_n(x))\, w_n(x)|\, \chi_{B_n}(x) \leqslant (a \mid t_n \phi_1(x) + w_n(x)|^\sigma + b)\, |w_n(x)|$$
$$\leqslant (a_1 \mid w_n(x)|^\sigma + a_1 s_0^\sigma + b)\, |w_n(x)|$$

using the fact that $|w_n(x) + t_n \phi_1(x)| \leqslant |w_n(x)| + s_0$ if $x \in B_n$. Thus, considering $h_{\sigma+1}$ given by (8), we obtain the estimate

$$|g(t_n \phi_1(x) + w_n(x))\, w_n(x)|\, \chi_{B_n}(x) \leqslant b_1 [(h_{\sigma+1}(x))^{\sigma+1} + 1],$$

where the function on the right hand side belongs to $L^1(\Omega)$ in view of (8), as $\sigma + 1 < 2N/(N-2)$. Since $\chi_{B_n}(x) \to 0$ for a.e. $x \in \Omega$, we get by Lebesgue's Theorem that

$$\int_{B_n} g(t_n \phi_1 + w_n)\, w_n \to 0. \tag{11}$$

Hence, (10), and (11) imply that $\int_\Omega g(u_n)\, w_n \to 0$ so that

$$\|w_n\| \to 0 \tag{12}$$

as desired and the Claim is proved.

Next, using (g_1), (8), (12), and arguments similar to those above, we may conclude that

$$I(u_n) \to 0 \qquad \text{if} \quad t_n \to +\infty,$$

which is a contradiction to $c \neq 0$. Thus, we must have $t_n \to -\infty$. Finally, using (8) and the fact that $-G(s)$ is bounded from below, we can apply Fatou's Lemma to obtain

$$\liminf I(u_n) \geqslant \liminf[-N(u_n)] \geqslant -\alpha\, |\Omega|$$

since $t_n \to -\infty$. The proof of Lemma 2 is complete. ∎

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