ACADEMIC PRESS

Available at WWW.MATHEMATICSWEB.ORG POWERED BY SCIENCE @ DIRECT•

Journal of Number Theory 102 (2003) 11–22

JOURNAL OF Number Theory

http://www.elsevier.com/locate/jnt

Solution-free sets for linear equations

Tomasz Łuczak[a,b,*] and Tomasz Schoen[a]

[a] *Department of Discrete Mathematics, Adam Mickiewicz University, 61-614 Poznań, Poland*

[b] *Department of Mathematics and Computer Science, Emory University, 400 Dowman Drive, MSC E416, Atlanta, GA 30322, USA*

Received 25 October 2000; revised 16 December 2002

Communicated by D. Goss

Abstract

We study the maximum size and the structure of sets of natural numbers which contain no solution of one or more linear equations. Thus, for every natural i and $k \geqslant 2$, we find the minimum $\alpha = \alpha(i,k)$ such that if the upper density of a strongly k-sum-free set $A \subseteq \mathbb{N}$ is at least α, then A is contained in a maximal strongly k-sum-free set which is a union of at most i arithmetic progressions. We also determine the maximum density of sets of natural numbers without solutions to the equation $x = y + az$, where a is a fixed integer.

MSC: 11B05; 11D04; 11B75

Keywords: Sum-free sets; Critical densities

1. Introduction

For a given subset A of an abelian group G and an integer $k \geqslant 2$, let

$$S_k(A) = \bigcup_{i=2}^{k} iA,$$

where $iA = A + \cdots + A$ (i times), and $S_1(A) = A$. We say that A is *strongly k-sum-free* if $A \cap S_k(A) = \emptyset$. In particular, the fact that A is strongly 2-sum-free means that $A \cap (A + A) = \emptyset$; in this case we say that A is *sum-free*.

*Corresponding author. Department of Mathematics and Computer Science, Emory University, 400 Dowman Drive, MSC E416, Atlanta, GA 30322, USA. Fax: +1-404-727-5611.

E-mail addresses: tomasz@mathcs.emory.edu (T. Łuczak), schoen@amu.edu.pl (T. Schoen).

0022-314X/03/$ - see front matter
doi:10.1016/S0022-314X(03)00064-7

The structure and the size of large k-sum-free subsets of abelian groups has been extensively studied for $k=2$; much less is known about the case $k \geqslant 3$. In the following section we find an upper bound for the size of strongly k-sum-free subsets of a finite abelian group G and characterize the structure of strongly k-sum-free subsets which are, in a way, "extremal" ones.

Then we use this result to study the structure of sets of natural numbers which contain no solution to one or more linear equations. We first consider the structure of strongly k-sum-free subsets A of $\mathbb{N}$, showing that if the upper density $\bar{\mathrm{d}}(A)$ of A is larger than $i/((2k-1)i-k-1)$ for some natural $i \in \mathbb{N}$, then A is a union of at most $i-1$ arithmetic progression of the same difference. We also give a characterization of sets for which this estimate is sharp. In the last section of the note we find a sharp estimate for the upper density of sets $A \subseteq \mathbb{N}$ which contain no solution to the equation $x = y + az$ for a given integer $a \neq 0$.

2. Strongly sum-free subsets of abelian groups

Most of results which deal with the structure of large sum-free subsets of abelian groups are based on the following well-known theorem proved by Kneser in [4].

Theorem 1. *Let $A, B \subseteq G$, where G is a finite abelian group. Then*

$$|A+B| \geqslant |A| + |B| - |\Gamma(A+B)|, \tag{1}$$

where $\Gamma(X)$ is the stabilizer of a set $X \subseteq G$.

A direct application of Theorem 1 yields

$$|S_k(A)| \geqslant k|A| - (k-1)|\Gamma(S_k(A))|,$$

but if A is strongly k-sum-free we can prove a considerable better estimate for $|S_k(A)|$.

Lemma 2. *Let $A \subseteq G$ be a strongly k-sum-free set, where G is a finite abelian group. Then*

$$|S_\ell(A)| \geqslant 2(\ell-1)|A| - (\ell-1)|\Gamma(S_\ell(A))|, \tag{2}$$

for each ℓ such that $2 \leqslant \ell \leqslant k$.

We deduce Lemma 2 from a slightly more general result. For a natural numbers d, k, and $A \subseteq G$, we define $S_{k,d}(A)$ setting $S_{1,d}(A) = A$ and

$$S_{k,d} = \bigcup_{i=1}^{\lceil (k-1)/d \rceil} (id+1)A.$$

Hence, in particular, $S_{k,1}(A) = S_k(A)$. We shall show the following generalization of Lemma 2.

Lemma 3. *Let $A \subseteq G$ be a strongly k-sum-free set of an abelian group G, and let $\ell = du + 1 \leqslant k$ for some natural numbers d and u. Then*

$$|S_{ud+1,d}(A)| \geqslant (d+1)u|A| - (d-1)u\gamma_d(A) - u|\Gamma(S_{ud+1,d}(A))|, \tag{3}$$

where $\gamma_1(A) = 0$ and for $d \geqslant 2$ we put $\gamma_d(A) = |\Gamma(dA)|$.

We start with the following observation.

Fact 4. *Let A be a subset of abelian group and $k \geqslant 2$. Then for all $\ell = du + 1 \leqslant k - d$, we have*

$$\Gamma(S_{\ell,d}(A)) \subseteq \Gamma(S_{\ell+d,d}(A)) \tag{4}$$

and

$$\Gamma(\ell A) \subseteq \Gamma((\ell + d)A). \tag{5}$$

Furthermore, if $d = 1$ and A is a strongly k-sum-free set which is maximal, (i.e., which is not properly contained in any other strongly k-sum-free set), then in (4) the equality holds.

Proof. In order to verify (4) it is enough to show that

$$\Gamma(S_{\ell,d}(A)) + (\ell + d)A \subseteq S_{\ell+d,d}(A).$$

Let $g \in \Gamma(S_{\ell,d}(A))$. Then, from the definition of $\Gamma(S_{\ell,d}(A))$, we have $g + S_{\ell,d}(A) = S_{\ell,d}(A)$, so

$$\begin{aligned} g + (\ell + d)A = g + \ell A + dA &\subseteq g + S_{\ell,d}(A) + dA \\ &\subseteq S_{\ell,d}(A) + dA \subseteq S_{\ell+d,d}(A). \end{aligned}$$

Hence (4) holds. The same argument with $S_{i,d}(A)$ replaced by iA for $i = \ell, \ell + d$, gives (5).

Now assume that $d = 1$, A is a maximal strongly k-sum-free set, and

$$\Gamma(A) \neq \Gamma(S_{k,1}(A)) = \Gamma(S_k(A)).$$

Then, for some $a \in A$ and $h \in \Gamma(S_k(A))$, we have $a + h \notin A$. We shall show that then the set $A \cup \{a + h\} \supsetneq A$ is strongly k-sum-free, and so A is not maximal.

Indeed, note that

$$S_k(A + h) \subseteq S_k(A) + \{h, 2h, \ldots, kh\} \subseteq S_k(A) + \Gamma(S_k(A)) = S_k(A),$$

so that

$$S_k(A \cup \{a+h\}) = S_k(A).$$

Moreover,

$$A \cap (S_k(A) - h) \subseteq A \cap (S_k(A) - \Gamma(S_k(A))) = A \cap S_k(A) = \emptyset.$$

Thus, $(A+h) \cap S_k(A) = \emptyset$, i.e., $(A \cup \{a+h\}) \cap S_k(A) = \emptyset$, and so the set $A \cup \{a+h\}$ is strongly k-sum-free. $\square$

Proof of Lemma 3. We use induction on u. For $u = 1$ we have $S_{d+1,d}(A) = (d+1)A$. From Theorem 1 it follows that

$$|(d+1)A| = |dA + A| \geqslant |dA| + |A| - |\Gamma((d+1)A)|.$$

Furthermore, using once again Kneser's theorems ($d-1$ times) and (5) (with $d = 1$) we infer that

$$|dA| \geqslant d|A| - (d-1)\gamma_d(A). \tag{6}$$

Hence

$$|S_{d+1,d}(A)| \geqslant (d+1)|A| - (d-1)\gamma_d(A) - |\Gamma(S_{d+1,d}(A))|,$$

which verifies the assertion for $u = 1$.

Now suppose that $u \geqslant 2$. Observe that

$$\begin{aligned} S_{ud+1,d}(A) &= \bigcup_{i=1}^{u} (id+1)A = \left(\bigcup_{i=0}^{u-1} (id+1)A \right) + A \\ &= (A \cup S_{(u-1)d+1}(A)) + A. \end{aligned}$$

Thus, Kneser's theorem, (6), and the fact that $A \cap S_\ell(A) = \emptyset$, give

$$\begin{aligned} |S_{ud+1,d}(A)| &\geqslant |A \cup S_{(u-1)d+1,d}(A)| + |dA| - |\Gamma(S_{ud+1,d}(A))| \\ &\geqslant (d+1)|A| + |S_{(u-1)d+1,d}(A)| \\ &\quad - (d-1)\gamma_d(A) - |\Gamma(S_{ud+1,d}(A))| . \end{aligned}$$

Now (3) follows from the induction hypothesis and (4). $\square$

Our next result characterizes a special class of "large" sets for which in (2) the equality holds.

Lemma 5. *Let A, $|A| = r$, be a maximal strongly k-sum-free subset of a finite abelian group G of order $m = (2k-1)r - (k-1)$, and let $|\Gamma(A)| = 1$. Then there is an* *isomorphism* $\phi : G \to \mathbb{Z}_m$ *such that*

$$\phi(A) = \{r, r+1, \ldots, 2r-1\}.$$

The proof of Lemma 5 relies on the well-known result of Kemperman [3], which describes the sets A, B for which in (1) the equality holds. We say that a subset C of an abelian group G is *quasi-periodic* if there exists a subgroup F of G, with $2 \leqslant |F| < |G|$, such that C can be partition into two sets C' and C'', where C' is a non-empty union of F-cosets and a residual set C'' is contained in a F-coset. Kemperman's theorem can be stated as follows.

Theorem 6. *Let A and B be subsets of an abelian group G such that $|A|, |B| \geqslant 2$, $|\Gamma(A+B)| = 1$ and $|A+B| = |A| + |B| - 1$. Then either $A+B$ is an arithmetic progression, or $A+B$ is quasi-periodic.*

Proof of Lemma 5. Let A fulfill the assumption of Lemma 5. Then, from Fact 4, we get $|\Gamma(S_k(A))| = |\Gamma(A)| = 1$. Note also that Lemma 2 implies that $G = A \cup S_k(A)$.

If $A = \{a\}$, then $G = A \cup S_k(A)$ is a cyclic group generated by a and the assertion easily follows. Thus, we assume that $|A| = r \geqslant 2$. We show first that $S_k(A)$ is not quasi-periodic.

Indeed, suppose to the contrary that there is a proper subgroup F of G such that all but one F-cosets are contained in either A or $S_k(A)$, and the residual F-coset, say, $h+F$, contains elements of both A and $S_k(A)$.

First we exclude the possibility $A \subsetneq h+F$. Note that from the fact that A is strongly k-sum-free and $A \cup S_k(A) = G$, it follows that $h+F$ is a generator of the quotient group G/F and $kA \subseteq F$. Let $d = |G|/|F| \geqslant 2$ be the rank of the element $h+F$ in G/F. Then $k = du$ for some $u \geqslant 2$. Note also that

$$A \cup S_{(u-1)d+1,d}(A) = h+F \quad \text{and} \quad A \cap S_{(u-1)d+1,d}(A) = \emptyset. \tag{7}$$

We shall use Lemma 3 to show that

$$|A| + |S_{(u-1)d+1,d}(A)| > |F|,$$

to get a contradiction.

Observe that (7) and the fact that the sets A and $S_{(u-1)d+1,d}(A)$ are non-empty imply that $|\Gamma(S_{(u-1)d+1,d}(A))| = |\Gamma(A)| = 1$. Thus, by (4),

$$|\Gamma(S_{d+1,d}(A))| = |\Gamma((d+1)A)| = 1,$$

and so from (5) we infer that for $d \geqslant 2$ we have $|\Gamma(dA)| = 1$. Hence, Lemma 3 gives

$$\begin{aligned} |A \cup S_{(u-1)d+1,d}(A)| &\geqslant r + (d+1)(u-1)r - (d-1)(u-1) - (u-1) \\ &= dru - dr + ru - du + d. \end{aligned}$$

Furthermore, since $r \geqslant 2$,

$$|F| = \frac{|G|}{d} = \frac{(2du-1)r-(du-1)}{d} < 2ur - u.$$

Thus,

$$\begin{aligned} |A \cup S_{(u-1)d+1,d}(A)| - |F| &> dru - dr - ru - du + d + u \\ &= u(d-1)(r-1) - d(r-1) \\ &\geqslant 2(d-1)(r-1) - d(r-1) \\ &= (d-2)(r-1) \\ &\geqslant 0, \end{aligned}$$

and so A cannot be properly contained in any coset $h+F$.

Therefore, there exist $g, h \in F$ such that $g + F \subseteq A$, $(h+F) \cap A \neq \emptyset$ and $(h+F) \cap S_k(A) \neq \emptyset$. Without loss of generality, we may also assume that $h \in S_k(A)$, so that for some ℓ, $2 \leqslant \ell \leqslant k$, there are elements $a_1, \ldots, a_\ell \in A$, such that $h = a_1 + \cdots + a_\ell$. Note that if $a_1 + F = h + F$, then $a_2 + \cdots + a_\ell \in F$, which is impossible because in such a case we would have both

$$g + a_2 + \cdots + a_\ell + F = g + F \subseteq A$$

and

$$g + a_2 + \cdots + a_\ell + F \subseteq a_2 + \cdots + a_\ell + A \subseteq \ell A \subseteq S_k(A),$$

contradicting the assumption that $A \cap S_\ell(A) = \emptyset$. Since clearly $(a_1 + F) \cap A \neq \emptyset$, and $h + F$ is the unique coset sharing points with both A and $S_k(A)$, it follows that $a_1 + F \subseteq A$. The same argument shows that $a_2 + F \subseteq A$. Hence

$$h + F = (a_1 + F) + (a_2 + F) + (a_3 + \cdots + a_\ell) \subseteq A + A + (\ell - 2)A \subseteq S_k(A),$$

contradicting the fact that $(h+F) \cap A \neq \emptyset$. Consequently, $S_k(A)$ is not quasi-periodic.

Observe now that from the proof of Lemma 3 it follows that

$$|S_k(A)| = |(A \cup S_{k-1}(A)) + A| = |A \cup S_{k-1}(A)| + |A| - 1.$$

Hence, Kemperman's theorem implies that $S_k(A) = (A \cup S_{k-1}(A)) + A$ is an arithmetic progression.

Let $S_k(A) = \{a, a+b, \ldots, a+(m-1)b\}$, and let H denote the subgroup generated in G by b. Since $G = A \cup S_k(A)$ and $S_k(A) \subseteq a + H$, G is a cyclic group generated by b (note that either $H = G$, or $|S_k(A)| = |H| = |G|/2$ which implies $k = 2$, $G = \mathbb{Z}_2$, $A = \{1\}$). Thus, $a = nb$ for some $n \in \mathbb{N}$ and $S_k(A) =$

$\{nb, (n+1)b, \ldots, (n+m-1)b\}$. Define an isomorphism $\phi : G \to \mathbb{Z}_m$ by setting $\phi(b) = 1$. Then

$$\phi(A) = \mathbb{Z}_m \backslash \phi(S_k(A)) = \{t, t+1, \ldots, t+r-1\}$$

for some t, $0 \leqslant t \leqslant m-1$. Now let $B = \{t, t+1, \ldots, t+r-1\}$. Then $S_k(B) = \{2t, 2t+1, \ldots, k(t+r-1)\}$ and, since $B \cup S_k(B) = \mathbb{Z}_m$ and $B \cap S_k(B) = \emptyset$, we must have $t + r = 2t \pmod m$, i.e., $t = r$. $\square$

For $k = 2$ Lemma 5 can be restated in the following particularly appealing form; for this case it was proved by Rhemtulla and Street [7].

Lemma 7. *Let S be a sum-free subset of a finite abelian group G with more than $|G|/3$ elements. Then there exist a sum-free set $A \subseteq G$, a natural number r, and a homomorphism $\phi : G \to \mathbb{Z}_{3r-1}$, such that $S \subseteq A$, $|A|/|G| = r/(3r-1)$ and*

$$\phi(A) = \{r, r+1, \ldots, 2r-1\}.$$

Proof. Let A be the maximal sum-free containing S. We may assume that $|\Gamma(A)| = 1$; otherwise we consider the set of all the cosets of $\Gamma(A)$. Since A is maximal, Fact 4 implies that $|\Gamma(A+A)| = 1$, and so from Theorem 1 we get

$$|G| \geqslant |A| + |A+A| \geqslant 3|A| - 1.$$

Thus, since $|A|/|G| > 1/3$, we have $|G| = 3|A| - 1$. Consequently, A fulfills the assumption of Lemma 5 and the assertion follows. $\square$

3. Dense strongly sum-free sets of natural numbers

The size and the structure of strongly k-sum-free sets were considered by the authors in [6], where we showed the following result.

Theorem 8. *For every $k \geqslant 2$ and $\varepsilon > 0$ there exists a natural number $r = r(k, \varepsilon)$ such that each strongly k-sum-free set $A \subseteq \mathbb{N}$ of the upper density $\bar{\mathrm{d}}(A) > 1/(2k-1) + \varepsilon$ is contained in a strongly k-sum-free set which is a union of at most r arithmetic progressions.*

Furthermore, if $\bar{\mathrm{d}}(A) > 1/(k+1)$, then A is contained in a strongly k-sum-free arithmetic progression.

The main result of this section improves the above result, giving the best possible estimates for the critical densities of strongly k-sum-free sets contained in strongly k-sum-free sets which are unions of at most i arithmetic progressions.

Theorem 9. *Let $A \subseteq \mathbb{N}$ be a strongly k-sum-free set such that for some natural number $i \geqslant 2$*

$$\bar{\mathrm{d}}(A) > \frac{i}{(2k-1)i-k+1}. \tag{8}$$

Then A is contained in a strongly k-sum-free set, which is union of at most $i-1$ arithmetic progressions with the same difference. Moreover, if A is a maximal strongly k-sum-free set such that

$$\bar{\mathrm{d}}(A) = \frac{i}{(2k-1)i-k+1}$$

then there exist $r \in \{1, \dots, (2k-1)i-k+1\}$, satisfying $(r, (2k-1)i-k+1) = 1$, such that

$$A = \{n \in \mathbb{N}\colon\ n \equiv rs (\mathrm{mod}\ ((2k-1)i-k+1))\ \textit{for some}\ s \in \{i, \dots, 2i-1\}\}.$$

Proof. Assume that A is a strongly k-sum-free set satisfying (8). Since

$$\frac{i}{(2k-1)i-k+1} > \frac{1}{2k-1}$$

for any $i \in \mathbb{N}$, Theorem 8 implies that A is contained in a maximal strongly k-sum-free set B, which consists of arithmetic progressions with the same difference D. Thus, there are distinct integers $b_1, \dots, b_j \in \{0, 1, \dots, D-1\}$ such that

$$B = \{n \in \mathbb{N}\colon\ n \equiv b_1, \dots, b_j \ (\mathrm{mod}\ D)\}.$$

Put $B' = \{b_1, \dots, b_j\}$. Then B' is a maximal strongly k-sum-free set in $\mathbb{Z}_D$, hence by Fact 4 we have $\Gamma(B') = \Gamma(S_k(B'))$. Moreover, choosing minimal possible period D of the set B, we get $1 = |\Gamma(B')| = |\Gamma(S_k(B'))|$.

Thus, Lemma 2 applied to the set B' gives

$$|S_k(B')| \geqslant 2(k-1)|B'| - (k-1),$$

and since $B' \cap S_k(B') = \emptyset$, one has

$$|B'| + |S_k(B')| \leqslant D,$$

so that

$$D \geqslant (2k-1)j - k + 1.$$

Hence

$$\frac{i}{(2k-1)i-k+1} < \bar{\mathrm{d}}(A) \leqslant \bar{\mathrm{d}}(B) = \frac{j}{D} \leqslant \frac{j}{(2k-1)j-k+1}, \tag{9}$$

which implies $j < i$ and the first part of the assertion follows.

Now assume that A fulfills the assumptions of the second part of Theorem 9. Then in $|B'| = j = i$, $|\Gamma(B')| = 1$, and $D = (2k-1)i - k + 1$. Using Lemma 5 we infer that there is an isomorphism ϕ of $\mathbb{Z}_{(2k-1)i-k+1}$ which maps B' onto the set $\{i, i+1, \dots, 2i-1\}$. Now, to complete the proof, it is enough to observe that each isomorphism ψ of $\mathbb{Z}_{(2k-1)i-k+1}$ can be written as $\psi(s) = rs$, where $r \in \{1, \dots, (2k-1)i-k+1\}$ and $(r, (2k-1)i-k+1) = 1$. $\square$

Let us explicitly state the important special case of Theorem 9, which improves the second part of Theorem 8.

Corollary 10. *Let $A \subseteq \mathbb{N}$ be a strongly k-sum-free set of the upper density larger than $2/(3k-1)$. Then A is contained in a strongly k-sum-free arithmetic progression.*

For $k = 2$ in the proof of Theorem 9 instead of Lemma 5 one can use a more precise Lemma 7 and obtain the following complete characterization of sum-free subsets of $\mathbb{N}$ of the upper density larger than $1/3$. This result settles in the affirmative a conjecture of Calkin [1] (see also [2]).

Theorem 11. *If set $A \subseteq \mathbb{N}$ is sum-free and $\bar{\mathrm{d}}(A) > 1/3$, then there exist natural numbers k and $j \in \{1, \dots, 3k-1\}$, $(j, 3k-1) = 1$ such that*

$$A \subseteq \{n \in \mathbb{N}\colon\ n \equiv ji \pmod{3k-1} \text{ for some } i \in \{k, \dots, 2k-1\}\}.$$

4. The equation $x = y + az$

Let $a \neq 0$ be an integer and let $\Omega(a)$ denote the family of all subsets of $\mathbb{N}$ containing no solutions to the equation $x = y + az$. In this part of the note we investigate the maximum upper density of sets from $\Omega(a)$, i.e., we study the behaviour of the function

$$\mu(a) = \max_{A \in \Omega(a)} \bar{\mathrm{d}}(A).$$

For $a \neq 0$ we define $\lambda(a) = 1$ if neither $a+1$, nor $a-1$, has a positive divisor congruent to 2 modulo 3, and

$$\lambda(a) = \min\{n \in \mathbb{N}\colon\ n \text{ divides } a+1 \text{ or } a-1 \text{ and } n \equiv 2 \pmod 3\}$$

if such a divisor exists. The main result of this section states that the value of $\lambda(a)$ determines the behaviour of $\mu(a)$.

Theorem 12. *Let $a \neq 0$ be a given integer. Then*

$$\mu(a) = \tfrac{1}{3},$$

for all a with $\lambda(a) = 1$, and

$$\mu(a) = \frac{\lambda(a)+1}{3\lambda(a)},$$

whenever $\lambda(a) > 1$.

Moreover, if $\lambda(a) > 1$ and $\bar{\mathrm{d}}(A) > 1/3$ for some $A \in \Omega(a)$, then there exists $s, r \in \mathbb{N}$ such that $a \equiv \pm 1 \pmod{3r-1}$, $(s, 3r-1) = 1$ and

$$A \subseteq \{n \in \mathbb{N}\colon\ n \equiv is \pmod{3r-1} \text{ for some } i \in \{r, \ldots, 2r-1\}\}.$$

In the proof of Theorem 12 we use the following, somewhat technical, result.

Fact 13. *Let $A = \{r, \ldots, 2r-1\} \subseteq \mathbb{Z}_{3r-1}$ and $u \in \mathbb{Z}_{3r-1}$. If $uA \subseteq A$, then either $u = 1$ or $u = 3r-2$.*

Proof. Observe first that $u \neq 0$. Note also that if $i \in A$ satisfies $ui = \min uA$, then

$$i = r \quad \text{or } i = 2r-1. \tag{10}$$

Indeed, if $r < i < 2r-1$, then $u(i-1), ui, u(i+1) \in uA \subseteq A$. Hence, we must have $\pm u \in \{1, 2, \ldots, r-1\}$. On the other hand, since $ui - u, ui + u \geqslant \min uA = ui$ and $r \leqslant ui \leqslant 2r-1$, we must have $u \in \{r, \ldots, 2r-1\}$. Contradiction. Hence (10) holds.

Now suppose, that $\gcd(u, 3r-1) = d > 2$. Set $s = (3r-1)/d$. Then $1 \leqslant s \leqslant r-1$ and $r + s \in A$. Moreover, $ur = u(r \pm s)$ and $u(2r-1) \equiv u(2r-1 \pm s)$, which contradicts (10).

Observe also that $\gcd(u, 3r-1) = 2$ gives $(3r-1)/2 \in A$, so that $u(3r-1)/2 = 0$, contradicting the fact that $uA \subseteq A$.

Thus, we must have $\gcd(u, 3r-1) = 1$, so that $uA = A$. Hence (10) implies that either $ur = r$, or $u(2r-1) = r$. The former case gives $u = 1$, the latter one $u = 3r-2$. $\square$

Proof of Theorem 12. For a given $a \neq 0$ put $\lambda = \lambda(a)$. To get a lower bound for μ consider the following sets

$$S = \{n \in \mathbb{N}\colon\ n \equiv 1 \pmod 3\}$$

and

$$S' = \{n \in \mathbb{N}\colon\ n \equiv (\lambda+1)/3, \ldots, 2(\lambda+1)/3 - 1 \pmod{\lambda}\}.$$

If $\lambda = 1$ then S contains no solutions to the equation $x = y + az$. In order to see it, observe that any such solution yields $a \equiv 0 \pmod 3$, while $\lambda = 1$ implies $a \equiv 2 \pmod 3$.

If $\lambda > 1$ then we have $a \equiv \pm 1 \pmod{\lambda}$. Therefore, the set S' is free of solutions to $x = y + az$.

Now let us assume that $\mu(a) > 1/3$. We show that then $\lambda(a) > 1$ and, in fact, characterize all sets A from $\Omega(a)$ of the upper density larger than $1/3$.

Thus, let $A \in \Omega(a)$ have the upper density larger than $1/3$. We show first that A is sum-free. Indeed, if there are numbers $b_1, b_2, b_3 \in A$ such that

$$b_1 = b_2 + b_3,$$

then, since A is free of solutions to the equation $x = y + az$, the sets

$$A, A + ab_1, A + ab_2,$$

are pairwise disjoint. Hence

$$\bar{\mathrm{d}}(A) \leqslant 1/3,$$

contradicting the choice of A.

Thus, the set A is sum-free. Since $\bar{\mathrm{d}}(A) > 1/3$ from Theorem 11 it follows that for some $j, r \in \mathbb{N}$, $(j, 3r-1) = 1$ we have

$$A \subseteq \{n \in \mathbb{N}\colon\ n \equiv ji \pmod{(3r-1)} \text{ for some } i \in \{r, \dots, 2r-1\}\}.$$

Without lost of the generality we may assume that $j = 1$, so that

$$A \subseteq \{n \in \mathbb{N}\colon\ n \equiv r, \dots, 2r-1 \pmod{(3r-1)}\}.$$

Let $A' \subseteq \{r, \dots, 2r-1\}$ be defined as

$$A' = \{t \in \mathbb{Z}_{3r-1}\colon\ n \equiv t \pmod{(3r-1)} \text{ for some } n \in A\}.$$

Since $\bar{\mathrm{d}}(A) > 1/3$ it follows that $A' = \{r, \dots, 2r-1\}$ and the set A' contains no solutions to the equation $x = y + az$ in the group $\mathbb{Z}_{3r-1}$. Furthermore, since

$$(A' - A') \cup A' = \mathbb{Z}_{3r-1} \quad \text{and} \quad (A' - A') \cap aA' = \emptyset,$$

we have $aA' \subseteq A'$. Therefore, by Lemma 13, one has

$$a \equiv \pm 1 \pmod{(3r-1)}, \tag{11}$$

so that $\lambda(a) > 1$. Consequently, each maximal subset A from $\Omega(a)$ of the upper density larger than $1/3$ has density $r/(3r-1)$ for some r for which (11) holds. Since the smallest possible r which fulfills (11) is equal to $(\lambda(a)+1)/3$, the assertion follows. $\square$

We remark that the fact that the densest sets in $\Omega(a)$ are very regular is strongly related to a special form of the equation $x = y + az$. Consider for instance the family $\hat{\Omega}(a)$ of sets of natural numbers which contains no solutions to the equation $x + y = az$, and let $\hat{\mu}(a)$ denote the maximum upper density of the sets from $\hat{\Omega}(a)$. It was shown by Lucht [5] (see also Schoen [8]), that if a is large enough the sets $A \in \hat{\Omega}(a)$ for which $\bar{d}(A) = \mu(A)$ are not periodic; in fact lower density of each such set is strictly smaller than $\mu(A)$.

References

[1] N.J. Calkin, Sum-free sets and measure spaces, Ph.D. Dissertation, University of Waterloo, 1988.
[2] P.J. Cameron, On the structure of a random sum-free set, Probab. Theory Related Fields 76 (1987) 523–531.
[3] J.H.B. Kemperman, On small sumsets in abelian group, Acta Math. 103 (1960) 63–88.
[4] M. Kneser, Ein Satz über abelschen Gruppen mit Anwendungen auf die Geometrie der Zahlen, Math. Zeit. 61 (1955) 429–434.
[5] L. Lucht, Dichteschranken für die Lösbarkeit gewisser linearer Gleichungen, J. Reine Angew. Math. 285 (1976) 209–217.
[6] T. Łuczak, T. Schoen, On the infinite sum-free sets of natural numbers, J. Number Theory 66 (1997) 211–224.
[7] A.H. Rhemtulla, A.P. Street, Maximum sum-free sets in elementary Abelian p-groups, Canad. Math. Bull. 14 (1971) 73–80.
[8] T. Schoen, On sets of natural numbers without solutions to a noninvariant linear equation, Acta Arith. 193 (2000) 149–155.