Probabilistic Aspects of Machine Decomposition Theory*

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ABSTRACT

Results obtained concern the likelihood that randomly chosen machines admit nontrivial decompositions of their state behavior.

INTRODUCTION

In this paper some results of probabilistic nature are obtained concerning the function \( \beta_{np} : \text{(finite state) machines} \rightarrow \text{integers} \), where \( \beta_{np} \) is defined as follows: given an \( n \)-state, \( p \)-input machine \( \delta \), \( \beta_{np}(\delta) \) is the number of nontrivial SP partitions of \( \delta [1], [2] \).

A justification for this investigation is that \( \beta_{np}(\delta) \) can be regarded as a rough measure of the decomposability of \( \delta \), since according to Hartmanis and Stearns \([1], [2]\), the decompositions of a machine are associated with its nontrivial SP partitions.

The material is divided into seven sections. In the first we obtain some simple results concerning partitions and recall the basic facts about machine decomposition.

In Section 2 we formally introduce \( \beta_{np} \) and derive a formula for its expectation, which can be regarded as a measure of the average decomposability of \( n \)-states, \( p \)-input machines. In Sections 3 and 4 the dependency of \( \beta_{np} \) on the parameters \( n \) and \( p \) is investigated for \( n \rightarrow \infty \). This amounts to study the decomposability of large machines as a function of the sizes of memory and input. It turns out that for \( n \rightarrow \infty \) almost all \( n \)-state, \( p \)-input machines have an unbounded number of decompositions if the input \( p \) increases as a function \( p(n) \) of the memory \( n \), in such a way that \( \lim_{n \rightarrow \infty} p(n)/\ln n = 0 \). Conversely almost all \( n \)-state, \( p \)-input machines are for \( n \rightarrow \infty \) indecomposable if \( p \) increases as a function of \( n \) in such a way that \( \lim_{n \rightarrow \infty} \ln n/p(n) = 0 \).

Section 5 is devoted to clock decompositions \([3]\), i.e., decompositions which have an autonomous machine or clock as first component. It turns out that the results already obtained about decompositions of large machines also hold for clock decompositions; i.e., for \( n \rightarrow \infty \) almost all \( n \)-state, \( p(n) \)-input machines have an unbounded

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number of clock decompositions or do not have any clock decomposition if \( p(n)/\ln n \to 0 \) or \( \ln n/p(n) \to 0 \), respectively. This result is quite surprising since clock decompositions seem to be rather specialized and therefore much less probable than general decompositions.

Finally in Section 6 and 7 some other results obtained in [4] and open problems are indicated.

1. Preliminaries

We denote by \(|S|\) the cardinality of a set \( S \) and by \([n]\) the set \( \{1, 2, \ldots, n\} \) where \( n \) is any positive integer.

A partition \( \pi \) on a set \( S \) is a collection of disjoint subsets of \( S \) whose set union is \( S \). The subsets are called blocks of \( \pi \).

A partition on \([n]\) is nontrivial if it has more than 1 and less than \( n \) blocks. When we write out a partition we list its blocks in order of nondecreasing cardinality. For example, if partition \( \pi \) on \([7]\) has blocks \( \{1, 5\}, \{3, 6, 7\} \) and \( \{2, 4\} \), then we write

\[
\pi = \{(1, 5), (2, 4), (3, 6, 7)\},
\]

or

\[
\pi = \{(2, 4), (1, 5), (3, 6, 7)\}.
\]

A subdivision of a positive integer \( n \) is a sequence of integers \( \psi = n_1 n_2 \cdots n_k \), with \( 0 < n_1 \leq n_2 \leq \cdots \leq n_k \), such that \( \sum_{i=1}^{k} n_i = n \). The subdivision \( \psi = n_1 n_2 \cdots n_k \) will also be written as \( \nu_1 a_1 \nu_2 a_2 \cdots \nu_r a_r \) where \( \nu_1, \nu_2, \ldots, \nu_r \) are the distinct numbers appearing in \( \psi \), with \( \nu_i < \nu_{i+1} \) and \( a_i \) is the number of repetitions of the number \( \nu_i \) in \( \psi \). For example, the subdivision \( 2335 \) of \( 13 \) is also written \( 23^{3}5^{1} \) or, more simply, \( 2335 \).

Subdivisions are well known in combinatorial analysis under the name of partitions of an integer ([5]; chapter 6); however, in order to avoid confusion between the concepts of partitions on a set and partition of the number of elements of the set itself, we will not use the traditional term.

A subdivision \( \psi \) of \( n \) is nontrivial if \( \psi \neq 1^n \) and \( \psi \neq n \). We shall denote by \( \Psi_n \) the set of all nontrivial subdivisions of \( n \).

The subdivision of a partition \( \pi = \{B_1, B_2, \ldots, B_k\} \) on \([n]\) is the subdivision \( \psi = |B_1| |B_2| \cdots |B_k| \) of the integer \( n \). For example, the subdivision of partition \( \pi = \{(2), (6, 8), \{1, 3\}, \{4, 5, 7, 9\}\} \) on \([9]\) is \( 1224 = 12^24 \).

**Lemma 1.1.** The number of partitions on \([n]\) having a given subdivision

\[
\psi = n_1 n_2 \cdots n_k = \nu_1 a_1 \nu_2 a_2 \cdots \nu_r a_r
\]
is given by:

\[ P(\psi) = \frac{n!}{\prod_{i=1}^{k} (n_i)! \prod_{i=1}^{k} (a_i!)} = \frac{n!}{\prod_{i=1}^{k} (v_i)! (a_i!)^p} \]  

(1.1)

**Proof.** The number of ways of assigning the elements of \([n]\) into \(k\) disjoint subsets \(S_{11}, S_{12}, \ldots, S_{1a_1}, S_{21}, S_{22}, \ldots, S_{2a_2}, \ldots, S_{r_1}, S_{r_2}, \ldots, S_{r_a}\), with \(|S_{ij}| = v_i\), is given by the multinomial coefficient

\[ \frac{n!}{\prod_{i=1}^{k} (n_i)!} \]

if we assume that all the subsets are distinguishable.

But all the assignments obtainable one from the other by permuting the labels of \(S_{11}, S_{12}, \ldots, S_{1a_1}, S_{21}, S_{22}, \ldots, S_{2a_2}, \ldots, S_{r_1}, S_{r_2}, \ldots, S_{r_a}\) correspond to the same partition with subdivision \(\psi\). Clearly, the number of such permutations is \(a_i!\). Hence the total number of partitions with subdivision \(\psi\) is obtained by dividing \(n!/\prod_{i=1}^{k} (n_i)!\) by the factor \(\prod_{i=1}^{k} (a_i!)\).

A machine with \(n\) states and \(p\) inputs, or \((n, p)\)-machine, is a mapping \(\delta: [n] \times [p] \rightarrow \{0, 1\}\); for \(B \subseteq [n]\) and \(x \in [p]\) we denote by \(\delta(B, x)\) the set \(\bigcup_{i \in B} \delta(i, x)\); also we denote by \([n]^{[n] \times [p]}\) the set of all \((n, p)\)-machines. Clearly \([n]^{[n] \times [p]} = n^{np}\).

An \((n, p)\) machine is decomposable if there exist two machines \(\delta_1 \in [n_1]^{[n_1] \times [p]}\) and \(\delta_2 \in [n_2]^{[n_2] \times [n_1] p}\), where \(n_1 < n\) and \(n_2 < n\), and there exists a \(1 \rightarrow 1\) mapping \(\alpha\) from a subset of \([n_1] \times [n_2]\) onto \([n]\) such that whenever \(\alpha((i, j))\) is defined and \(x \in [p]\), then also \(\alpha(\delta_1(i, x), \delta_2(j, (i-1)p + x))\) is defined and equals \(\delta_2(\alpha(i, j), x)\). These decompositions correspond to the nontrivial serial decompositions of the state behavior of \(I\).

A partition \(\pi = \{B_1, B_2, \ldots, B_k\}\) on \([n]\) has substitution property (SP) for an \((n, p)\)-machine \(\delta\) if for each \(B_i (i = 1 \cdots k)\) and \(x \in [p]\) there exists a \(B_{h(i, x)}\) such that

\[ \delta(B_i, x) \subseteq B_{h(i, x)} \]

In \([I]\) the following is proved.

**Theorem 1.1.** An \((n, p)\)-machine \(\delta\) is decomposable if there exists a nontrivial partition \(\pi\) on \([n]\) which has SP for \(\delta\).

2. **Average Number of SP Partitions**

Theorem 1.1 and other results of \([I]\) indicate that the decomposition of a machine can be analyzed in terms of its nontrivial SP partitions. In particular, the number of nontrivial SP partitions of a machine \(\delta\) can be regarded as a measure of the decomposability of \(\delta\).
DEFINITION 2.1. For any $\delta \in [n]^{[n] \times [p]}$, let $\beta_{np}(\delta)$ be the number of nontrivial SP partitions of $\delta$.

$\beta_{np}$ will be considered as a random variable defined over $[n]^{[n] \times [p]}$, which is considered as a sample space of equiprobable elements. In the following, we shall obtain a formula for the expectation of $\beta_{np}$.

DEFINITION 2.2. Given a subdivision $\psi = n_1n_2 \cdots n_k$ of the integer $n$, let $\pi$ be a fixed partition on $[n]$ with subdivision $\psi$. We define $Q(\psi, p)$ as the number of $(n, p)$-machines for which $\pi$ has SP. We shall also denote $Q(\psi, 1)$ as $Q(\psi)$.

Note that for reasons of symmetry $Q(\psi, p)$ does not depend upon the choice of $\pi$.

LEMMA 2.1.

$$Q(\psi, p) = n = Q(\psi)$$

Proof. An $(n, p)$-machine is described by the $np$ integers $\delta(i, x)$, for $i \in [n], x \in [p]$. In how many ways is it possible to assign the $n_i$ values of the restriction of $\delta$ to $B_i \times \{x\}$, to obtain an $(n, p)$-machine for which $\pi$ has SP? From Definition 2.1 there exists an index $h(j, x)$ such that $\delta(B_j, x) \subseteq B_{h(j, x)}$. If $h(j, x) = 1$, this gives $n_1^{n_i}$ possibilities, if $h(j, x) = 2, n_2^{n_i}$ possibilities,..., if $h(j, x) = k, n_k^{n_i}$ possibilities. Consequently, the $n_i$ values of the restriction of $\delta$ to $B_i \times \{x\}$ can be chosen in

$$\sum_{i=1}^{k} n_i^{n_j}$$

ways.

An $(n, p)$-machine can be obtained by independently assigning the restrictions of $\delta$ to the $kp$ sets $B_i \times \{x\}$, for $j \in [k]$ and $x \in [p]$. Hence the number of $(n, p)$-machines for which $\pi$ has SP is given by (2.1).

THEOREM 2.1. The average number of nontrivial SP partitions of an $(n, p)$-machine is given by:

$$E(n, p) = \sum_{\psi \in \mathcal{P}_n} \Psi(\psi) \left( \frac{Q(\psi)}{n^p} \right)^p$$

Proof. From the definition of expectation we have:

$$E(n, p) = \sum_{\delta} n^{-np} \beta_{np}(\delta)$$

where the sum is extended to all the machines $\delta$ in $[n]^{[n] \times [p]}$. 

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We rewrite this expression in the equivalent form:

\[ E(n, p) = n^{-np} \sum_{\pi} M_{np}(\pi) \]

where \( M_{np}(\pi) \) is the number of machines in \([n]^{[n]} \times [p]^{[p]}\) for which partition \( \pi \) has SP and the sum is extended to all the nontrivial partitions on \([n]\). From Lemma 2.1 \( M_{np}(\pi) = Q(\psi)^p \), \( \psi \) being the subdivision of the integer \( n \) associated with \( \pi \). Taking into account the fact that from Lemma 1.1 there are \( P(\psi) \) partitions having the same subdivision \( \psi \), we immediately obtain (2.2).

A severe limitation of (2.2) is that its computational complexity is proportional to the number of elements of \( \Psi_n \). Since this number increases quite rapidly with \( n \), (2.2) is not practically applicable for large \( n \).

It is, however, possible to study the statistical decomposability of large machines using suitable bounding techniques. This investigation leads to very simple results as illustrated in the next section.

3. ASYMPTOTIC DECOMPOSABILITY (FIRST PART)

Let \( \tau_j \) denote the partition \( \{\{j\}, \{[n] \setminus \{j\}\}\} \) on \([n]\).

**Definition 3.1.** A partition \( \pi \) on \([n]\) is elementary for an \((n, p)\)-machine \( \delta \) if:

a) \( \pi = \tau_j \) for some \( j \in [n] \).

b) \( \delta([n], x) \subseteq [n] \setminus \{j\} \) for all \( x \in [p] \).

Clearly a partition which is elementary for a machine \( \delta \) is also nontrivial and has SP for \( \delta \).

**Lemma 3.1.** The average number of elementary partitions of an \((n, p)\)-machine is given by:

\[ L(n, p) = n \left(1 - \frac{1}{n}\right)^{np} \]

**Proof.** As in Theorem 2.1 we obtain

\[ L(n, p) = n^{-np} \sum_{j=1}^{n} N_{np}(\tau_j) \]

(3.1)

where \( N_{np}(\tau_j) \) is the number of \((n, p)\)-machines for which the partition \( \tau_j \) is elementary. From Definition 3.1 this is equal to the number of machines such that all the \( np \) values of \( \delta(i, x) \), for \( i \in [n] \) and \( x \in [p] \), are in \([n] \setminus \{j\}\), which is a set of \( n - 1 \) elements. Hence

\[ N_{np}(\tau_j) = (n - 1)^{np} \]

from which (3.1) immediately follows.
LEMMA 3.2. The variance of the number of elementary partitions of an \((n, p)\)-machine is given by:

\[
V(n, p) = L(n, p) + (n^2 - n) \left(1 - \frac{2}{n}\right)^{np} - L(n, p)^2
\]

Proof. We introduce a matrix \(e_{ij}\) \((i = 1 \cdots n, j = 1 \cdots n)\), where:

\[
e_{ij} = 1 \quad \text{if partition } \tau_j \text{ is elementary for the } (n, p)\text{-machine } \delta_i
\]

\[
e_{ij} = 0 \quad \text{otherwise}
\]

Then:

\[
V(n, p) = \sum_{i=1}^{n} n^{np} \left(\sum_{j=1}^{n} e_{ij}\right)^2 - L(n, p)^2
\]

\[
= n^{-np} \sum_{j=1}^{n} \sum_{l=1}^{n} \sum_{i=1}^{n} e_{ij} e_{il} - L(n, p)^2
\]

But

\[
\sum_{i=1}^{n^{np}} e_{ij} e_{il}
\]

can be interpreted as the number of \((n, p)\)-machines for which both the partitions \(\tau_j\) and \(\tau_l\) are elementary. This in turn is equal to the number of \((n, p)\)-machines \(\delta\) such that \(\delta([n], x) \subset [n] - \{i, j\}\) for all \(x \in [p]\).

Hence clearly

\[
\sum_{i=1}^{n^{np}} e_{ij} e_{il} = (n - 1)^{np} \quad \text{if } j = l
\]

\[
\sum_{i=1}^{n^{np}} e_{ij} e_{il} = (n - 2)^{np} \quad \text{if } j \neq l
\]

Consequently

\[
V(n, p) = n^{-np} \left(\sum_{j=1}^{n} \sum_{l=1}^{n} e_{ij} e_{il} + \sum_{j=1}^{n} \sum_{l=1 \neq j}^{n} e_{ij} e_{il}\right) - L(n, p)^2
\]

\[
= n^{-np} \{n(n - 1)^{np} + (n^2 - n)(n - 2)^{np}\} - L(n, p)^2
\]

\[
= L(n, p) + (n^2 - n) \left(1 - \frac{2}{n}\right)^{np} - L(n, p)^2
\]
**Lemma 3.3.**

\[ V(n, p) < L(n, p) \]

**Proof.**

\[
V(n, p) < L(n, p) + n^2 \left( 1 - \frac{2}{n} \right)^n - L(n, p)^2
\]

\[
= L(n, p) + n^2 \left\{ (1 - \frac{2}{n})^n - (1 - \frac{1}{n})^2 \right\}
\]

\[
< L(n, p)
\]

since

\[ 1 - \frac{2}{n} < \left( 1 - \frac{1}{n} \right)^2. \]

**Lemma 3.4.** Let \( pr(X) \) denote the probability of the event \( X \), and let \( \epsilon_{np(\delta)} \) be the number of elementary partitions of \( \delta \). Then:

\[ pr[\epsilon_{np} < \frac{1}{2}L(n, p)] < \frac{4}{L(n, p)} \]

**Proof.** We write for simplicity \( L, V, \epsilon \) in place of \( L(n, p), V(n, p), \epsilon_{np} \). We have:

\[ pr \left[ \epsilon < \frac{L}{2} \right] \leq pr \left[ \left| \epsilon - L \right| > \frac{L}{2} \right] \leq pr \left[ \left| \epsilon - L \right| > \frac{L}{2} V \right] \]

since from Lemma 3.3. \( L > V \).

But from Chebyshev's inequality ([6]; chapter 5, page 226):

\[ pr \left[ \left| \epsilon - L \right| > \frac{L}{2} V \right] < \frac{1}{\left( \frac{L}{2} \right)^2} = \frac{4}{L} \]

**Theorem 3.1.** Let \( p \) be a function from the naturals to the naturals such that

\[ \lim_{n \to \infty} \frac{p(n)}{\ln n} = 0 \]

Then

\[ \lim_{n \to \infty} E(n, p(n)) = \infty \]

and for any integer \( K \)

\[ \lim_{n \to \infty} pr[\beta_{np(n)} \geq K] = 1 \]
Proof. We have from Lemma 3.1:
\[
\lim_{n \to \infty} E(n, p(n)) \geq \lim_{n \to \infty} L(n, p(n)) = \lim_{n \to \infty} n \left(1 - \frac{1}{n}\right)^{\frac{1}{\ln n}} \\
\geq \lim_{n \to \infty} \ln \left[ e^{-1} \left(1 - \frac{1}{n}\right)^{\frac{1}{\ln n}} \right] \geq \lim_{n \to \infty} \left(1 - \frac{p(n)}{\ln n}\right) \cdot e^{\ln [1 - \frac{p(n)}{\ln n}]} = +\infty
\]
where we have used the relation
\[
\left(1 - \frac{1}{n}\right)^n = e^{-1} \left(1 - \frac{1}{n}\right) = e^{-\frac{1}{n} - \frac{1}{2n^2} - o\left(\frac{1}{n^3}\right)} > e^{-1 - \frac{1}{n}}
\]
Given an integer \(K\) from Lemma (3.4) we have:
\[
\lim_{n \to \infty} \Pr[\beta_{n,p(n)} \geq K] \geq \lim_{n \to \infty} \Pr[\epsilon_{n,p(n)} \geq K] \geq \lim_{n \to \infty} \Pr \left[ \epsilon_{n,p(n)} \geq \frac{L(n, p(n))}{2} \right] \\
= \lim_{n \to \infty} \left\{ 1 - \Pr \left[ \epsilon_{n,p(n)} < \frac{L(n, p(n))}{2} \right] \right\} \geq 1 - \lim_{n \to \infty} \frac{4}{L(n, p(n))} = 1
\]

4. ASYMPTOTIC DECOMPOSABILITY (SECOND PART)

In this section we shall prove that if \(p\) is a function such that
\[
\lim_{n \to \infty} \frac{\ln n}{p(n)} = 0
\]
then
\[
\lim_{n \to \infty} E(n, p(n)) = 0
\]
and consequently
\[
\lim_{n \to \infty} \Pr(\beta_{n,p(n)} = 0) = 1
\]

DEFINITION 4.1. Let \(\psi = n_1 n_2 \cdots n_k \in \mathcal{W}_n\). We associate with \(\psi\) a set \(g(\psi) \subset \mathcal{W}_{n+1}\), called the set of subdivisions generated by \(\psi\), as follows:
\[
g(\psi) = \{1n_1 n_2 \cdots n_k\} \quad \text{if} \quad n_1 = n_2
\]
\[
g(\psi) = \{1n_1 n_2 \cdots n_k, (n_1 + 1)n_2 \cdots n_k\} \quad \text{if} \quad n_1 < n_2 \quad \text{and} \quad \psi \neq 1(n - 1)
\]
\[
g(\psi) = \{1^2(n - 1), 2(n - 1), 1n\} \quad \text{if} \quad \psi = 1(n - 1).
\]

For example,
\[
g(2^24) = \{12^24\}, \quad g(1^35) = \{1^45\}, \quad g(24) = \{124, 34\},
\]
\[
g(123) = \{1^223, 2^3\}, \quad g(14) = \{1^44, 24, 15\}.
\]
LEMMA 4.1. Let $\psi' = n_1n_2 \cdots n_k \in \Psi_{n+1}$. There exists one and only one subdivision $\psi \in \Psi_n$ such that $\psi' \in g(\psi)$. $\psi$ is given by $n_1n_2 \cdots n_k$ if $n_1 = 1$ and $\psi' \neq 1n$, by $(n_1 - 1)n_2 \cdots n_k$ if $n_1 > 1$, by $1(n - 1)$ if $\psi' = 1n$.

LEMMA 4.2.

$$\Psi_{n+1} = \bigcup_{\psi \in \Psi_n} g(\psi)$$

$$g(\psi_1) \cap g(\psi_2) = \emptyset \quad \text{for} \quad \psi_1 \neq \psi_2.$$

Let $h$ be a constant greater than $1/\ln [4/(e + 1)]$, $\psi$ a subdivision of an integer and $\Psi$ a subset of subdivisions. We define

$$E(\psi) = P(\psi) \left[ \frac{Q(\psi)}{n^n} \right]^h \ln n$$

$$E(\Psi) = \sum_{\psi \in \Psi} E(\psi)$$

where $n$ is the integer of which $\psi$ is a subdivision. Clearly by Theorem 2.1

$$E(n, h \ln n) = \sum_{\psi \in \Psi_n} E(\psi)$$

$$E(n + 1, h \ln(n + 1)) = \sum_{\psi \in \Psi_n} E(g(\psi)).$$

LEMMA 4.3.

$$\frac{E(n + 1, h \ln(n + 1))}{E(n, h \ln n)} \leq \max_{\psi \in \Psi_n} \frac{E(g(\psi))}{E(\psi)}$$

In the following lemmas an upper bound will be obtained for the quantity

$$\max_{\psi \in \Psi_n} \frac{E(g(\psi))}{E(\psi)}.$$

LEMMA 4.4. Let $\psi = n_1n_2 \cdots n_k \in \Psi_n$, $\psi_1 = 1n_2 \cdots n_k \in \Psi_{n+1}$, and

$$\rho_1(n, \psi) = \frac{E(\psi_1)}{E(\psi)}.$$

Then for large $n$:

$$\rho_1(n, \psi) < 1 - \frac{\ln n}{n} \quad (4.1)$$

Proof. We rewrite $\psi = 1^{m}n_{m+1} \cdots n_k$, where $n_{m+1} > 1$ and by convention $1^n \cdots n_k$ denotes $n_1 \cdots n_k$ for $m = 0$. We have

$$\rho_1 = \frac{P(\psi_1)}{P(\psi)} \left[ \frac{Q(\psi_1)}{(n + 1)^{n+1}} \right]^{h \ln(n+1)} \left[ \frac{n^n}{Q(\psi)} \right]^{h \ln n}.$$
By Lemma 1.1

\[
\frac{P(\psi_1)}{P(\psi)} = \frac{n + 1}{m + 1}
\]

and

\[
\left[ \frac{Q(\psi)}{(n + 1)^{n+1}} \right]^{\frac{h \ln(n + 1)}{n}} < \left[ \frac{Q(\psi)}{(n + 1)^{n+1}} \right]^{h \ln n}
\]

since

\[
Q(\psi) < (n + 1)^{n+1}.
\]

Hence

\[
\rho_1 < \frac{n + 1}{m + 1} \left[ \frac{Q(\psi)}{Q(\psi)} \right]^{\frac{n}{(n + 1)^{n+1}}}^{h \ln n}
\]

\[
= \frac{n + 1}{m + 1} \left[ \frac{(n + 1)^{m+1}}{n^m} \prod_{j=m+1}^{k} \left( \frac{1}{m + n_{m,j}^{n_i} + \cdots + n_{k}^{n_i}} \right) \right]^{\frac{n}{(n + 1)^{n+1}}}^{h \ln n}
\]

\[
< \frac{n + 1}{m + 1} \left[ \left( \frac{n + 1}{n} \right)^{m-n} \left( \frac{n + 1}{n} \right)^{k-m-n} \right]^{\frac{n}{(n + 1)^{n+1}}}^{h \ln n}
\]

\[
= \frac{n + 1}{m + 1} \left[ \frac{n + 1}{n} \right]^{(k-n)h \ln n}
\]

where use has been made of the inequality

\[
\frac{1 + m + n_{m,j}^{n_i} + \cdots + n_{k}^{n_i}}{m + n_{m,j}^{n_i} + \cdots + n_{k}^{n_i}} < \frac{n + 1}{n}.
\]

But

\[
k \leq m + \frac{n - m}{2}
\]

since \(n_i = 1\) for \(i = 1 \cdots m\) and \(n_i \geq 2\) for \(i = m + 1 \cdots k\). Then

\[
\rho_1 < \frac{n + 1}{m + 1} \left[ \frac{n + 1}{n} \right]^{\frac{m-n}{2h \ln n}} = (n + 1) \left[ \frac{n + 1}{n} \right]^{\frac{m}{2h \ln n}} f(m)
\]

where

\[
f(m) = \frac{1}{m + 1} \left[ \frac{n + 1}{n} \right]^{\frac{m}{2h \ln n}}.
\]

\(f(m)\) is a concave function of \(m\) for \(0 \leq m \leq n - 2\).
In fact:
\[
\frac{df}{dm} = f(m) \left[ -\frac{1}{m+1} + \frac{h \ln n}{2} \ln \frac{n+1}{n} \right]
\]
\[
\frac{d^2f}{dm^2} = f(m) \left[ -\frac{1}{m+1} + \frac{h \ln n}{2} \ln \frac{n+1}{n} \right]^2 + f(m) \frac{1}{(m+1)^2} > 0
\]

Hence
\[
f(m) \leq \max\{f(0), f(n-2)\}
\]

But
\[
f(0) = 1
\]
\[
f(n-2) = \frac{1}{n-1} \left( 1 + \frac{1}{n} \right)^{\frac{n-2}{2} h \ln n} > \frac{1}{n-1} \left( 1 + \frac{1}{n} \right)^{2n \ln n}
\]
\[
> \frac{e \ln n}{n-1} > f(0)
\]

where we have used the condition \( h > 1/\ln [4/(e + 1)] > 4(1 - 2/n)^{-1} \) (for large \( n \)).

Therefore
\[
\rho_1 < (n+1) \left[ \frac{n+1}{n} \right]^{\frac{n-2}{2} h \ln n} f(n-2)
\]
\[
= \frac{n+1}{n-1} \left[ \frac{n}{n+1} \right]^{h \ln n} = 1 + \frac{2}{n} - \frac{h \ln n}{n} + o\left( \left( \frac{\ln n}{n} \right)^2 \right)
\]
\[
< 1 - \frac{\ln n}{n} \quad \text{(for large } n)\].

**Lemma 4.5.** Let \( \psi = n_1 n_2 \cdots n_k \in \mathcal{P}_n \), with \( n_1 < n_2, \psi_2 = (n_1 + 1)n_2 \cdots n_k \in \mathcal{P}_{n+1} \) and \( \rho_2(n, \psi) = E(\psi_2)/E(\psi) \). Then for large \( n \):
\[
\rho_2(n, \psi) < \frac{\left( \frac{e}{e+1} \right)^n}{n}.
\]  

**Proof.** We have by Theorem 2.1
\[
\rho_2 = \frac{P(\psi_2)}{P(\psi)} \left[ \frac{Q(\psi_2)}{(n+1)^{n+1}} \right]^{h \ln (n+1)} \left[ \frac{n^n}{Q(\psi)} \right]^{h \ln n}
\]

Clearly by Lemma 1.1
\[
\frac{P(\psi_2)}{P(\psi)} = \frac{n+1}{n_1+1} \leq \frac{n+1}{n_1+1}
\]
where \( a \) is the number of occurrences of the number \( n_1 + 1 \) in \( \psi_2 \). As in Lemma 4.4

\[
\left[ \frac{Q(\psi_2)}{(n + 1)^{n+1}} \right]^{h \ln(n+1)} < \left[ \frac{Q(\psi_2)}{(n + 1)^{n+1}} \right]^{h \ln n}
\]

so that

\[
\rho_2 < \frac{n + 1}{n_1 + 1} \left[ \frac{Q(\psi_2)}{Q(\psi)} \right]^{n_1} \left( \frac{n}{n + 1} \right)^{n+1} \ln n
\]

\[
< \frac{n + 1}{n_1 + 1} \left[ \frac{Q(\psi_2)}{Q(\psi)} \right]^{n_1} \left( \frac{1}{(n + 1) e^{1/n}} \right)^{h \ln n}
\]

(4.3)

since for large \( n \)

\[
\frac{n^n}{(n + 1)^{n+1}} = \frac{1}{(n + 1) \left( \frac{1}{n} \right)^n} < \frac{1}{(n + 1) e^{1/n}}
\]

We now rewrite \( Q(\psi_2)/Q(\psi) \) as \( \prod_{j=1}^k R_j \) where

\[
R_1 = \frac{(n_1 + 1) n_1^{n_1+1} + n_2^{n_2+1} + \cdots + n_k^{n_k+1}}{n_1^{n_1} + n_2^{n_2} + \cdots + n_k^{n_k}}
\]

\[
R_j = \frac{(n_1 + 1) n_j^{n_j} + n_2^{n_j} + \cdots + n_k^{n_j}}{n_1^{n_1} + n_2^{n_2} + \cdots + n_k^{n_k}} \quad (j = 2 \cdots k)
\]

and we found bounds for \( R_1 \) and \( \prod_{j=2}^k R_j \) which will be useful in the following.

Since

\[
(n_1 + 1)^{n_1+1} = (n_1 + 1) \left( 1 + \frac{1}{n_1} \right) n_1^{n_1} < (n_1 + 1) e n_1^{n_1}
\]

we have:

\[
R_1 \leq \frac{(n_1 + 1) e n_1^{n_1} + n_2^{n_1+1} + \cdots + n_k^{n_1+1}}{n_1^{n_1} + n_2^{n_1} + \cdots + n_k^{n_1} \leq \max\{(n_1 + 1) e, n_k\}} \quad (4.4)
\]

Also for \( j > 1 \)

\[
R_j \leq \frac{(n_1 + 1)^{n_j} + n_2^{n_j} + \cdots + n_k^{n_j}}{n_2^{n_j} + \cdots + n_k^{n_j}} \leq \frac{k(n_1 + 1)^{n_j}}{(k - 1)(n_1 + 1)^{n_j}} = \frac{k}{k - 1}
\]
where use has been made of the inequality:

\[
\frac{a + b}{b} \leq \frac{a + b_1}{b_1} \quad \text{for} \quad b \geq b_1
\]

Then:

\[
\prod_{j=2}^{k} R_j \leq \left( \frac{k}{k-1} \right)^{k-1} = \left( 1 + \frac{1}{k-1} \right)^{k-1} < e
\]  \hspace{1cm} (4.5)

To bound \( \rho_2 \) we consider now various cases:

a) Let \((n_1 + 1)e \leq n_k \leq (n + 1)e - 1\). Then from (4.3), (4.4), and (4.5):

\[
\rho_2 < \frac{n + 1}{n_1 + 1} \left[ \frac{n_k e^{(1 - \frac{1}{n})h \ln n}}{(n + 1) e^{1 - 1/n}} \right]^{h \ln n} < ne^{(1 - \frac{1}{n})h \ln n} = n^{1 - (1 - \frac{1}{n})h} \]  \hspace{1cm} (4.6)

b) Let \((n_1 + 1)e \leq n_k, (n + 1)e - 1 < n_k\). Then:

\[
R_j < \frac{(n_1 + 1)^{n_1} + n_2^{n_2} + \cdots + n_k^{n_k}}{n_2^{n_1} + \cdots + n_k^{n_k}} \leq \frac{(n_1 + 1)^{n_1} + n_k^{n_k}}{n_k^{n_k}}
\]

\[
\prod_{j=2}^{k} R_j < \left[ 1 + \left( \frac{n_1 + 1}{n_k} \right)^2 \right]^{k-1} < e^{(k-1) \left( \frac{n_k + 1}{n_k} \right)^2}
\]

\[
\leq e^{ \frac{(n+1-n_k)(n+1)}{n_k^2} } < e^{1 - e^{-1}}
\]

since \((k - 1)(n_1 + 1) \leq n + 1 - n_k\).

Using (4.4) and (4.5):

\[
\rho_2 < \frac{n + 1}{n_1 + 1} \left[ \frac{n_k e^{1 - \frac{1}{n}}}{(n + 1) e^{1 - 1/n}} \right]^{h \ln n} < ne^{(-e^{-1} + \frac{1}{n})h \ln n}
\]

\[
= n^{1 - h(e^{-1} - \frac{1}{n})} \]  \hspace{1cm} (4.7)

c) Let \((n_1 + 1)e > n_k, k > 2\). Then from (4.3), (4.4) and (4.5):

\[
\rho_2 < \frac{n + 1}{n_1 + 1} \left[ \frac{(n_1 + 1) e \cdot e^{h \ln n}}{(n + 1) e^{1 - 1/n}} \right]^{h \ln n}
\]

\[
= e^{(1 + \frac{1}{n})h \ln n} \left( \frac{n_1 + 1}{n + 1} \right)^{h \ln n} \leq e^{h \ln n + 1} \frac{1}{3^h \ln n - 1} \cdot < 3e^{ \left( \frac{e}{3} \right)^h \ln n} = 3e^{h(1 - \ln 3)}
\]  \hspace{1cm} (4.8)
d) Let \((n_1 + 1)e > n_k, k = 2\). As in the derivation of (4.4) we have:

\[
R_1 < \frac{(n_1 + 1)e n_1^{n_1} + (n - n_1)^{n_1}}{n_1^{n_1} + (n - n_1)^{n_1}} = (n - n_1) \frac{(n_1 + 1)e n_1^{n_1} + (n - n_1)^{n_1}}{n_1^{n_1} + (n - n_1)^{n_1}}
\]

\[
< (n - n_1) \frac{1}{2} \left[ \frac{(n_1 + 1)e}{n - n_1} + 1 \right]
\]

where we have used the inequality:

\[
\frac{xa + b}{a + b} < \frac{x + 1}{2} \quad \text{for } x > 1, \quad a < b
\]

Consequently:

\[
R_1 < \frac{1}{2} [(n_1 + 1)e + n - n_1] = \frac{1}{2} [(n_1 + 1)(e - 1) + (n + 1)]
\]

\[
\leq \frac{1}{2} \left[ \frac{n + 1}{2} (e - 1) + (n + 1) \right] = \frac{e + 1}{4} (n + 1)
\]

(4.9)

Then from (4.3), (4.5) and (4.9):

\[
\rho_2 < \frac{n + 1}{n_1 + 1} \left[ \frac{(e + 1)(n + 1)e^{\frac{1}{2} \ln n}}{4e^{1/n}(n + 1)} \right]^{h \ln n}
\]

\[
< \frac{(n_1 + 1) + e(n + 1)}{n_1 + 1} \left[ \frac{e + 1}{4} \right]^{h \ln n} \frac{h \ln n}{e n}
\]

\[
< (e + 1) e^{-\frac{1}{h} \ln n} = e(e + 1)n^{-1}
\]

(4.10)

where we have used the inequalities \(n_k < e(n_1 + 1)\) and \(h > (\ln(4/e + 1))^{-1}\).

By evaluating the constants in the exponents of (4.6), (4.7) and (4.8), we find that the bound on \(\rho_2\) given by (4.10) dominates for large \(n\) the bounds given by (4.6), (4.7) and (4.8).

**Lemma 4.6.** Let \(\psi = 1(n - 1) \in \Psi_n\). Then for large \(n\):

\[
\frac{E(g(\psi))}{E(\psi)} < \frac{n}{n + 1}.
\]

(4.11)
Proof.

\[
\frac{E(g(\psi))}{E(\psi)} = \frac{E(1^2(n - 1)) + E(2(n - 1)) + E(1n)}{E(1(n - 1))} < \frac{n + 1}{2} \left[ \frac{(n + 1)^2 2 + (n - 1)^{n-1}}{n + 1} \cdot \frac{n}{(n + 1)^{n+1}} \right]^h \ln n
\]

\[
+ \frac{n + 1}{2} \left[ \frac{4 + (n - 1)^2}{n} \frac{2^{n-1} + (n - 1)^{n-1}}{1 + (n - 1)^{n-1}} \frac{n^n}{(n + 1)^{n+1}} \right]^h \ln n
\]

\[
+ \frac{n + 1}{n} \left[ \frac{(n + 1)(1 + n^n)}{(n + 1)^{n+1}} \right]^h \ln(n^{(n+1)}) \left[ \frac{n^n}{(n + 1)^{n+1}} \right]^h \ln n
\]

\[
< (n + 1) \left\{ e^{-1} \left[ 1 + \frac{1}{n} \right] \right\}^h \ln n
\]

\[
+ \frac{n + 1}{n} \left[ 1 + 0 \left( \frac{\ln(n + 1)}{n^n} \right) \right] \frac{\left( 1 + \frac{1}{n-1} \right)^{(n-1)} \ln n}{(1 + \frac{1}{n})^n} \ln(n^{(n+1)})
\]

\[
< n^{-(h-2)} + \frac{n + 1}{n} \left[ 1 + 0 \left( \frac{\ln(n + 1)}{n^n} \right) \right] \left( 1 + \frac{1}{n} \right)^n \ln n
\]

\[
< n^{-(h-2)} + \frac{n + 1}{n} \left( 1 + \frac{1}{n} \right)^{\frac{h}{2n}} = n^{-(h-2)} + \left( \frac{n}{n + 1} \right)^{\frac{h}{2} - 1} < \frac{n}{n + 1}
\]

(for large \( n \)).

**Theorem 4.1.** Let \( p \) be a function from the naturals to the naturals such that:

\[
\lim_{n \to \infty} \frac{\ln n}{p(n)} = 0
\]

Then:

\[
\lim_{n \to \infty} E(n, p(n)) = 0
\]
which immediately implies

\[
\lim_{n \to \infty} \Pr[\beta_{n,p(n)} = 0] = 1.
\]

Proof. Let \( \psi = n_1 n_2 \cdots n_k \), with \( n_1 < n_2 \) and \( \psi \neq 1(n - 1) \). Then for large \( n \) from Lemmas 4.4 and 4.5:

\[
\frac{E(g(\psi))}{E(\psi)} = \rho_1(n, \psi) + \rho_2(n, \psi)
\]

\[
< 1 - \frac{\ln n}{n} + \frac{e(e + 1)}{n}
\]

(4.12)

The same bound holds a fortiori for \( \psi = n_1 n_2 \cdots n_k \), with \( n_1 = n_2 \) since in this case

\[
\frac{E(g(\psi))}{E(\psi)} = \rho_1(n, \psi)
\]

Finally, for \( \psi = (1(n - 1)) \), Lemma 4.6 shows that for large \( n \):

\[
\frac{E(g(\psi))}{E(\psi)} < \frac{n}{n + 1}
\]

(4.13)

The bound for \( E(g(\psi)) / E(\psi) \) given by (4.13) clearly dominates the bound given by (4.12), and consequently holds also for the ratio \( E(n + 1, h \ln(n + 1)) / E(n, h \ln n) \) from Lemma 4.3.

Let now \( n_0 \) be a fixed integer such that (4.12) and (4.13) hold with \( n = n_0 \). We have for \( n > n_0 \):

\[
\frac{E(n, h \ln n)}{E(n_0, h \ln n_0)} = \prod_{m=n_0}^{n-1} \frac{E(m + 1, h \ln(m + 1))}{E(m, h \ln m)} < \prod_{m=n_0}^{n-1} \frac{m}{m + 1} = \frac{n_0}{n}
\]

Let \( p \) be a function such that

\[
\lim_{n \to \infty} \frac{\ln n}{p(n)} = 0
\]

Then for sufficiently large \( n \):

\[
E(n, p(n)) < E(n, h \ln n).
\]

Hence:

\[
\lim_{n \to \infty} E(n, p(n)) \leq \lim_{n \to \infty} E(n, h \ln n) \leq \lim_{n \to \infty} E(n_0, h \ln n_0) \frac{n_0}{n} = 0
\]
Since for all \( n \), \( E(n, p(n)) > 0 \) we obtain
\[
\lim_{n \to \infty} E(n, p(n)) = 0.
\]

5. Clock Decomposability

**Definition 5.1.** An \((n, p)\)-machine \( \delta \) is a clock if
\[
\forall i \in [n], \quad \forall x \in [p] \quad \delta(i, x) = \delta(i, 1).
\]
A clock is essentially equivalent to a 1-input machine. We have defined it as an input independent \((n, p)\)-machine to be consistent in the following with our previous definition of machine decomposition.

**Definition 5.2.** A decomposition of a machine \( \delta \) into two machines \( \delta_1 \) and \( \delta_2 \) is a clock decomposition if \( \delta_1 \) is a clock.

**Definition 5.3.** A partition \( \pi = \{B_1, B_2, \ldots, B_k\} \) on \([n]\) is a clock partition for an \((n, p)\)-machine \( \delta \) if it has SP for \( \delta \) and \( \forall j \in [k] \) there exists a \( B_{h(j)} \) such that \( \forall x \in [p] \)
\[
\delta(B_{h(j)}, x) \subseteq B_{h(j)}.
\]

In [1, section 4.5] the following result is proved:

**Theorem 5.1.** An \((n, p)\)-machine \( \delta \) admits a clock decomposition iff there exists a nontrivial partition \( \pi \) on \([n]\) which is a clock partition for \( \delta \).

Let now \( \chi_{np} : [n][n][p] \to \text{naturals} \) be the function defined as follows: for any \( \delta \in [n][n][p] \), \( \chi_{np}(\delta) \) is the number of nontrivial clock partitions of \( \delta \) itself.

**Theorem 5.2.** Both Theorems 3.1 and 4.1 still hold if we substitute "Expectation of \( \chi_{np} \)" in place of \( E(n, p) \) and \( \chi_{np} \) in place of \( \beta_{np} \) in the statements of the theorems.

**Proof.** Immediate from the proofs of Theorems 3.1 and 4.1 and from the relation
\[
\epsilon_{np} \leq \chi_{np} \leq \beta_{np}.
\]

6. Other Results

In this section we quote without proof some results of probabilistic nature concerning other structural properties of machines. For a detailed proof see [4].

In the following \( p(n) \) denotes a function from the naturals to the naturals.
THEOREM 6.1. Let \( \alpha_{np}(\delta) \) denote the number of nontrivial partition pairs ([1], Section 3) of an \((n, p)\)-machine \( \delta \).

For \( p(n) \) satisfying the condition

\[
\lim_{n \to \infty} \frac{p(n)}{\frac{3}{2}n \ln n} < 1
\]

we have

\[
\lim_{n \to \infty} \text{Expectation}[\alpha_{np(n)}] = \infty
\]

and for any fixed integer \( K \):

\[
\lim_{n \to \infty} \text{Prob}[\alpha_{np(n)} \geq K] = 1
\]

Conversely for \( p(n) \) satisfying the condition

\[
\lim_{n \to \infty} \frac{\frac{3}{2}n \ln n}{p(n)} < 1
\]

we have

\[
\lim_{n \to \infty} \text{Prob}[\alpha_{np(n)} = 0] = 1
\]

THEOREM 6.2. Let \( \nu_{np}(\delta) \) denote the number of nontrivial uniform partitions [8], of an \((n, p)\)-machine \( \delta \).

Then for any fixed integer \( q > 0 \):

\[
\lim_{n \to \infty} \text{Expectation}[\nu_{nq}] = 0
\]

and consequently:

\[
\lim_{n \to \infty} \text{Prob}[\nu_{nq} = 0] = 1
\]

THEOREM 6.3. Let \( \gamma_{np}(\delta) \) denote the number of nontrivial SP partitions of a group \((n, p)\)-machine \( \delta \) (a machine such that for any \( x \in [p] \) the restriction of \( \delta \) to \([n] \times \{x\} \) is a permutation). Then

\[
\lim_{n \to \infty} \text{Expectation}[\gamma_{n1}] = \infty
\]

and

\[
\lim_{n \to \infty} \text{Prob}[\gamma_{n1} \geq 1] = 1
\]
Conversely for any fixed integer \( q > 1 \)

\[
\lim_{n \to \infty} \text{Expectation}[\gamma_{\eta q}] = 0
\]

and consequently

\[
\lim_{n \to \infty} \text{Prob}[\gamma_{\eta q} = 0] = 1
\]

7. OPEN PROBLEMS

The following open problems are proposed:

a) The proof of Theorem 3.1 shows that the theorem itself can be proved assuming only that the function \( p(n) \) satisfies the condition.

\[
\lim_{n \to \infty} \frac{p(n)}{\ln n} < 1
\]

Similarly the proof of Theorem 4.1 shows the theorem itself can be proved assuming only that the function \( p(n) \) satisfies the condition

\[
\lim_{n \to \infty} \frac{\ln n}{p(n)} < \ln \frac{4}{e + 1}
\]

It is conjectured that Theorem 4.1 holds if we assume only that \( p(n) \) satisfies the condition.

\[
\lim_{n \to \infty} \frac{\ln n}{p(n)} < 1
\]

b) Let \( \eta \) be a real number such that \( 0 < \eta \leq 1 \).

An \((n, p)\)-machine \( \delta \) is \( \eta \)-decomposable if it is decomposable into two machines \( \delta_1 \in [n_1]^{[n_1 \times \eta]} \) and \( \delta_2 \in [n_2]^{[n_2 \times (n_2 - \eta)]} \) where \( n_1 < \eta n \) and \( n_2 < \eta n \). Clearly all decompositions are 1-decompositions. An open problem is to find for each \( \eta \) a function \( f_\eta(n) \) such that if \( p(n) \) is such that

\[
\lim_{n \to \infty} \frac{p(n)}{f_\eta(n)} = 0
\]

then for \( n \to \infty \) almost all \((n, p(n))\)-machines are \( \eta \)-decomposable while if

\[
\lim_{n \to \infty} \frac{f_\eta(n)}{p(n)} = 0
\]
then for $n \to \infty$ almost all $(n, p(n))$-machines are not $\eta$-decomposable. Theorems 3.1 and 4.1 shows that $f_1(n) = \ln n$, and it would be interesting to obtain $f_\eta(n)$ for $\eta < 1$.

c) Finally one could consider the case of machines with output. The quantity to be probabilistically analyzed would then be the "number of nontrivial output consistent $SP$ partitions of a $n$-state, $p$-input, $q$-output machines" (see [1], Section 2.5). In particular this would lead to a probabilistic analysis of the reducibility properties of machines.

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