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The Neumann problem and Helmholtz decomposition in convex domains

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Abstract

We show that the Neumann problem for Laplace's equation in a convex domain Ω with boundary data in $L^p(\partial\Omega)$ is uniquely solvable for $1 < p < \infty$. As a consequence, we obtain the Helmholtz decomposition of vector fields in $L^p(\Omega, \mathbb{R}^d)$.
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Keywords: Neumann problem; Convex domains; Helmholtz decomposition

1. Introduction

The main purpose of this paper is to prove the following.

Theorem 1.1. *Let Ω be a bounded convex domain in $\mathbb{R}^d$, $d \geqslant 2$. Let $1 < p < \infty$. Then the L^p Neumann problem for $\Delta u = 0$ in Ω is uniquely solvable. That is, given any $f \in L^p(\partial\Omega)$ with mean value zero, there exists a harmonic function u in Ω, unique up to constants, such that $(\nabla u)^* \in L^p(\partial\Omega)$ and $\frac{\partial u}{\partial n} = f$ n.t. on $\partial\Omega$. Moreover, the solution satisfies the estimate $\|(\nabla u)^*\|_p \leqslant C\|f\|_p$, where C depends only on d, p and the Lipschitz character of Ω.*

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Here and thereafter $(\nabla u)^*$ denotes the nontangential maximal function of ∇u and n the unit outward normal to $\partial\Omega$. By $\frac{\partial u}{\partial n} = f$ n.t. on $\partial\Omega$ we mean that for a.e. $P \in \partial\Omega$, $\langle \nabla u(x), n(P)\rangle$ converges to $f(P)$ as $x \to P$ nontangentially. We remark that for a harmonic function u and $p \geqslant 2$, $(\nabla u)^* \in L^p(\partial\Omega)$ implies that u is in the Sobolev space $L^p_{1+\frac{1}{p}}(\Omega)$ [8]. The solutions in Theorem 1.1 satisfy the estimate $\|\nabla u\|_{L^p_{1/p}(\Omega)} \leqslant C\|f\|_p$ for $2 \leqslant p < \infty$.

It is known that the L^p Neumann problem for $\Delta u = 0$ in a bounded C^1 domain is uniquely solvable for any $p \in (1,\infty)$ [2]. However, if Ω is a general Lipschitz domain, the sharp range of p's, for which the L^p Neumann problem in Ω is solvable, is $1 < p < 2 + \varepsilon$, where $\varepsilon > 0$ depends on Ω (see [7,15,1]; also see [9] for references on related work on boundary value problems in Lipschitz domains). In [10], for any given Lipschitz domain Ω and $p > 2$, Kim and Shen established a necessary and sufficient condition for the solvability of the L^p Neumann problem in Ω. More precisely, it is shown in [10] that the L^p Neumann problem for $\Delta u = 0$ in Ω is solvable if and only if there exist positive constants C_0 and r_0 such that for any $0 < r < r_0$ and $Q \in \partial\Omega$, the following weak reverse Hölder inequality on $\partial\Omega$,

$$\left\{\frac{1}{r^{d-1}}\int_{B(Q,r)\cap\partial\Omega}\left|(\nabla v)^*\right|^p d\sigma\right\}^{1/p} \leqslant C_0\left\{\frac{1}{r^{d-1}}\int_{B(Q,2r)\cap\partial\Omega}\left|(\nabla v)^*\right|^2 d\sigma\right\}^{1/2}, \tag{1.1}$$

holds for any harmonic function v in Ω satisfying $(\nabla v)^* \in L^2(\partial\Omega)$ and $\frac{\partial v}{\partial n} = 0$ on $B(Q,3r)\cap\partial\Omega$ (see [10, Theorem 1.1]). Using this condition, Kim and Shen [10] obtained the solvability of the L^p Neumann problem for $\Delta u = 0$ in bounded convex domains in $\mathbb{R}^d$ for $1 < p < \infty$ if $d = 2$; for $1 < p < 4$ if $d = 3$; and for $1 < p < 3 + \varepsilon$ if $d \geqslant 4$ (see [10, Theorem 1.2]). Theorem 1.1 extends the results in [10] in the case $d \geqslant 3$ and completely solves the L^p Neumann problem for Laplace's equation in convex domains.

Our approach to Theorem 1.1 follows the proof of Theorem 1.2 in [10]. To establish the weak reverse Hölder inequality (1.1), we use the square function estimates for harmonic functions in Lipschitz domains and the local $W^{2,2}$ estimate in convex domains. This reduces the problem to the estimate of

$$\sup_{x\in B(P,r)}\left|\nabla^2 v(x)\right|^{p-2}\left[\delta(x)\right]^{p-1-t}, \tag{1.2}$$

where $t \in (0,1)$, $\delta(x) = \operatorname{dist}(x,\partial\Omega)$, and v is a harmonic function in Ω such that $\frac{\partial v}{\partial n} = 0$ on $B(P,3r)\cap\partial\Omega$ and $(\nabla v)^* \in L^2(\partial\Omega)$. In [10] the authors used the interior estimates and the local $W^{2,2}$ to obtain that for any $x \in B(P,r)\cap\Omega$,

$$\left|\nabla^2 v(x)\right| \leqslant \frac{C}{r}\left[\frac{r}{\delta(x)}\right]^{\frac{d}{2}}\left\{\frac{1}{r^d}\int_{B(P,3r)\cap\Omega}\left|\nabla v(y)\right|^2 dy\right\}^{1/2}. \tag{1.3}$$

By a reflection argument the classical De Giorgi–Nash estimate implies that for any $x \in B(P,r)\cap\Omega$,

$$\left|\nabla^2 v(x)\right| \leqslant \frac{C}{r}\left[\frac{r}{\delta(x)}\right]^{2-\alpha}\left\{\frac{1}{r^d}\int_{B(P,3r)\cap\Omega}\left|\nabla v(y)\right|^2 dy\right\}^{1/2}, \tag{1.4}$$

where $\alpha > 0$ depends on Ω. Substituting (1.3) for $d = 2, 3$ and (1.4) for $d \geqslant 4$ into (1.2) and choosing t sufficiently close to 0, we see that the exponent of $\delta(x)$ would be positive for any $p > 2$ if $d = 2$; for $p < 4$ if $d = 3$; and for $p < 3 + \varepsilon$ if $d \geqslant 4$. This leads to the restriction of p for $d \geqslant 3$ in [10, Theorem 1.2]. In this paper we will show that if $d \geqslant 3$, for any $x \in B(P, r) \cap \Omega$,

$$\left|\nabla^2 v(x)\right| \leqslant \frac{C_\eta}{r}\left[\frac{r}{\delta(x)}\right]^{1+\eta}\left\{\frac{1}{r^d}\int\limits_{B(P,3r)\cap\Omega}\left|\nabla v(y)\right|^2 dy\right\}^{1/2} \tag{1.5}$$

for any $\eta > 0$. Substituting (1.5) into (1.2), we see that the exponent of $\delta(x)$ is $-\eta(p-2)+1-t$, which would be positive for any $p > 2$ if $\eta > 0$ is sufficiently small.

To show (1.5), we will prove that if Ω is a convex domain with smooth boundary, then for any $q > 2$,

$$\left\{\frac{1}{r^d}\int\limits_{B(Q,r)\cap\Omega}|\nabla v|^q\,dx\right\}^{1/q} \leqslant C\left\{\frac{1}{r^d}\int\limits_{B(Q,2r)\cap\Omega}|\nabla v|^2\,dx\right\}^{1/2}, \tag{1.6}$$

whenever v is harmonic in Ω and $v \in C^2(\overline{\Omega})$, $\frac{\partial v}{\partial n} = 0$ in $B(Q, 3r) \cap \partial\Omega$. The constant C in (1.6) depends only on d, q and the Lipschitz character of Ω. Our proof of (1.6) is inspired by a recent paper of V. Maz'ya [12] (also see [11]), in which he established the L^∞ gradient estimate for solutions of the Neumann–Laplace problem in convex domains. More precisely, it is proved in [12] that if $q > d$ and $f \in L^q(\Omega)$ with mean value zero, then $\|\nabla u\|_{L^\infty(\Omega)} \leqslant C\|f\|_{L^q(\Omega)}$, where $-\Delta u = f$ in Ω and $\frac{\partial u}{\partial n} = 0$ on $\partial\Omega$. Although the proof of (1.6) does not rely on this estimate, the formulation of our main technical lemma, Lemma 2.2, as well as its proof, is motivated by [12].

With Theorem 1.1 at our disposal, following the potential approach developed by Fabes, Mendez, Mitrea [3] in Lipschitz domains, we may study the solvability of the Poisson equation with Neumann boundary conditions in convex domains. In particular, consider the boundary value problem

$$\begin{cases} \Delta u = f \in L^p_{-1,0}(\Omega), \\ \dfrac{\partial u}{\partial n} = g \in B^p_{-1/p}(\partial\Omega), \\ u \in W^{1,p}(\Omega). \end{cases} \tag{1.7}$$

Here $L^p_{-1,0}(\Omega)$ is the dual of $L^q_1(\Omega) = W^{1,q}(\Omega)$ and $B^p_{-1/p}(\partial\Omega)$ the dual of the Besov space $B^q_{1/p}(\partial\Omega)$ on $\partial\Omega$, where $q = \frac{p}{p-1}$. We will call $u \in W^{1,p}(\Omega)$ a solution to (1.7) with data (f, g), if

$$\int\limits_\Omega \nabla u \cdot \nabla\phi\,dx = -\langle f, \phi\rangle_{L^p_{-1,0}(\Omega)\times L^q_1(\Omega)} + \langle g, Tr(\phi)\rangle_{B^p_{-1/p}(\partial\Omega)\times B^q_{1/p}(\partial\Omega)} \tag{1.8}$$

for any $\phi \in W^{1,q}(\Omega)$, where $Tr(\phi)$ denotes the trace of ϕ on $\partial\Omega$.

Theorem 1.2. *Let Ω be a bounded convex domain in $\mathbb{R}^d$, $d \geqslant 2$. Let $1 < p < \infty$. Then for any $f \in L^p_{-1,0}(\Omega)$ and $g \in B^p_{-1/p}(\partial\Omega)$ satisfying the compatibility condition $\langle f, 1\rangle = \langle g, 1\rangle$,* *the Poisson problem* (1.7) *has a unique* (*up to constants*) *solution* u. *Moreover, the solution* u *satisfies the estimate*

$$\|\nabla u\|_{L^p(\Omega)} \leqslant C\big\{\|f\|_{L^p_{-1,0}(\Omega)} + \|g\|_{B^p_{-1/p}(\partial\Omega)}\big\}, \tag{1.9}$$

where C *depends only on* d, p *and the Lipschitz character of* Ω.

We remark that for bounded Lipschitz or C^1 domains, the inhomogeneous Neumann problem, $\Delta u = f \in L^p_{1/p-s-1,0}(\Omega)$ in Ω, $\frac{\partial u}{\partial n} = g \in B^p_{-s}(\partial\Omega)$ and $u \in L^p_{1-s+1/p}(\Omega)$ with $s \in (0,1)$ and $p \in (1,\infty)$, was studied in [3], where the authors obtained the solvability for the sharp ranges of p and s. Analogous results for the inhomogeneous Dirichlet problem in Lipschitz or C^1 domains may be found in [8]. In particular, it follows from [3] that the boundary value problem (1.7) is solvable for $p \in ((3/2)-\varepsilon, 3+\varepsilon)$ if Ω is Lipschitz; and for $p \in (1,\infty)$ if Ω is C^1.

Let $L^p_\sigma(\Omega)$ denote the subspace of functions $\mathbf{v}$ in $L^p(\Omega,\mathbb{R}^d)$ such that $\int_\Omega \mathbf{v}\cdot\nabla\phi\,dx = 0$ for any $\phi \in C^1(\mathbb{R}^d)$. As a corollary of Theorem 1.2, we establish the Helmholtz decomposition of L^p vector fields on convex domains for $1 < p < \infty$.

Theorem 1.3. *Let* Ω *be a bounded convex domain in* $\mathbb{R}^d$, $d \geqslant 2$ *and* $1 < p < \infty$. *Then*

$$L^p\big(\Omega,\mathbb{R}^d\big) = \operatorname{grad} W^{1,p}(\Omega) \oplus L^p_\sigma(\Omega). \tag{1.10}$$

That is, given any $\mathbf{u} \in L^p(\Omega,\mathbb{R}^d)$, *there exist* $\phi \in W^{1,p}(\Omega)$, *unique up to a constant, and a unique* $\mathbf{v} \in L^p_\sigma(\Omega)$ *such that* $\mathbf{u} = \nabla\phi + \mathbf{v}$. *Moreover,*

$$\max\big\{\|\nabla\phi\|_{L^p(\Omega)}, \|\mathbf{v}\|_{L^p(\Omega)}\big\} \leqslant C_p\|\mathbf{u}\|_{L^p(\Omega)}, \tag{1.11}$$

where C_p *depends only on* d, p *and the Lipschitz character of* Ω.

A useful tool in the study of the Navier–Stokes equations, the Helmholtz decomposition (1.10) is well known for smooth domains (see e.g. [4]). It was proved in [3] that (1.10)–(1.11) hold for $p \in ((3/2)-\varepsilon, 3+\varepsilon)$ if Ω is Lipschitz; and for $p \in (1,\infty)$ if Ω is C^1. The range $(3/2)-\varepsilon < p < 3+\varepsilon$ is known to be sharp for Lipschitz domains in $\mathbb{R}^d$, $d \geqslant 3$ (see [3]).

2. Estimates on smooth convex domains

The purpose of this section is to establish the following.

Theorem 2.1. *Let* Ω *be a bounded convex domain in* $\mathbb{R}^d$, $d \geqslant 3$ *with* C^2 *boundary. Let* $u \in C^3(\overline{\Omega})$. *Suppose that* $\Delta u = 0$ *in* Ω *and* $\frac{\partial u}{\partial n} = 0$ *on* $B(Q,3r)\cap\partial\Omega$ *for some* $Q \in \partial\Omega$ *and* $0 < r < r_0$. *Then for any* $q > 2$,

$$\left\{\frac{1}{r^d}\int\limits_{B(Q,r)\cap\Omega} |\nabla u|^q\,dx\right\}^{1/q} \leqslant C\left\{\frac{1}{r^d}\int\limits_{B(Q,2r)\cap\Omega} |\nabla u|^2\,dx\right\}^{1/2}, \tag{2.1}$$

where C *depends only on* d, q *and the Lipschitz character of* Ω.

The summation convention will be used in this section.

The proof of Theorem 2.1 relies on the following lemma. As we mentioned in Introduction, the formulation of Lemma 2.2 as well as its proof is inspired by a paper of Maz'ya [12].

Lemma 2.2. *Let Ω be a bounded convex domain with C^2 boundary. Suppose that $\mathbf{v} = (v_1, \dots, v_d) \in C^2(\overline{\Omega}, \mathbb{R}^d)$ and $\mathbf{v} \cdot n = 0$ on $\partial\Omega$. Let $g = |\mathbf{v}|^2$. Then for a.e. $t \in (0, \infty)$,*

$$\int_{\{g=t\}} |\nabla g|\, d\sigma \leqslant 2\sqrt{t} \int_{\{g=t\}} \left\{ \left(\sum_{i,j} \left| \frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i} \right|^2 \right)^{1/2} + \left|\operatorname{div}(\mathbf{v})\right| \right\} d\sigma + 2 \int_{\{g>t\}} \left\{ \left|\operatorname{div}(\mathbf{v})\right|^2 - \frac{\partial v_i}{\partial x_j} \cdot \frac{\partial v_j}{\partial x_i} \right\} dx, \tag{2.2}$$

where $\sigma = H^{d-1}$ denotes the $(d-1)$-dimensional Hausdorff measure and $\{g = t\} = \{x \in \Omega\colon g(x) = t\}$, $\{g > t\} = \{x \in \Omega\colon g(x) > t\}$.

Proof. Let Ψ be a nonnegative Lipschitz function on $[0, \infty)$. It follows from integration by parts that

$$\begin{aligned}
\int_\Omega \Psi\big(|\mathbf{v}|^2\big) \frac{\partial v_i}{\partial x_j} \cdot \frac{\partial v_j}{\partial x_i}\, dx &= -2 \int_\Omega \Psi'\big(|\mathbf{v}|^2\big) v_k \cdot \frac{\partial v_k}{\partial x_j} \cdot v_i \cdot \frac{\partial v_j}{\partial x_i}\, dx \\
&\quad - \int_\Omega \Psi\big(|\mathbf{v}|^2\big) \cdot v_i \cdot \frac{\partial}{\partial x_i} \big\{\operatorname{div}(\mathbf{v})\big\}\, dx \\
&\quad + \int_{\partial\Omega} \Psi\big(|\mathbf{v}|^2\big) v_i n_j \frac{\partial v_j}{\partial x_i}\, d\sigma \\
&= -2 \int_\Omega \Psi'\big(|\mathbf{v}|^2\big) v_k \cdot \frac{\partial v_k}{\partial x_j} \cdot v_i \cdot \frac{\partial v_j}{\partial x_i}\, dx \\
&\quad + 2 \int_\Omega \Psi'\big(|\mathbf{v}|^2\big) v_k \cdot \frac{\partial v_k}{\partial x_i} \cdot v_i \cdot \operatorname{div}(\mathbf{v})\, dx \\
&\quad + \int_\Omega \Psi\big(|\mathbf{v}|^2\big) \big\{\operatorname{div}(\mathbf{v})\big\}^2\, dx \\
&\quad + \int_{\partial\Omega} \Psi\big(|\mathbf{v}|^2\big) \left\{ v_i n_j \frac{\partial v_j}{\partial x_i} - v_i n_i \operatorname{div}(\mathbf{v}) \right\} d\sigma.
\end{aligned} \tag{2.3}$$

This gives

$$\int_\Omega \Psi\big(|\mathbf{v}|^2\big) \left\{ \big\{\operatorname{div}(\mathbf{v})\big\}^2 - \frac{\partial v_i}{\partial x_j} \cdot \frac{\partial v_j}{\partial x_i} \right\} dx$$

$$= \int_{\partial\Omega} \Psi\big(|\mathbf{v}|^2\big)\left\{v_i n_i \operatorname{div}(\mathbf{v}) - v_i n_j \frac{\partial v_j}{\partial x_i}\right\} d\sigma$$

$$+ 2\int_{\Omega} \Psi'\big(|\mathbf{v}|^2\big)\left\{v_k \cdot \frac{\partial v_k}{\partial x_j} \cdot v_i \cdot \frac{\partial v_j}{\partial x_i} - v_k \cdot \frac{\partial v_k}{\partial x_i} \cdot v_i \cdot \operatorname{div}(\mathbf{v})\right\} dx. \tag{2.4}$$

Using the assumptions that $\mathbf{v}\cdot n = 0$ on $\partial\Omega$ and Ω is a convex domain with C^2 boundary, we observe that

$$v_i n_i \operatorname{div}(\mathbf{v}) - v_i n_j \frac{\partial v_j}{\partial x_i} = -\beta(\mathbf{v}_T; \mathbf{v}_T) \geqslant 0 \quad \text{on } \partial\Omega,$$

where $\mathbf{v}_T = \mathbf{v} - (\mathbf{v}\cdot n)n$ is the tangential component of $\mathbf{v}$ on $\partial\Omega$ and $\beta(\cdot,\cdot)$ the second fundamental quadratic form of $\partial\Omega$ (see [6, pp. 133–134]). Hence,

$$2\int_{\Omega} \Psi'\big(|\mathbf{v}|^2\big) \cdot v_k \cdot \frac{\partial v_k}{\partial x_j} \cdot v_i \cdot \frac{\partial v_j}{\partial x_i}\, dx \leqslant 2\int_{\Omega} \Psi'\big(|\mathbf{v}|^2\big) \cdot v_k \cdot \frac{\partial v_k}{\partial x_i} \cdot v_i \cdot \operatorname{div}(\mathbf{v})\, dx$$

$$+ \int_{\Omega} \Psi\big(|\mathbf{v}|^2\big)\left\{\big\{\operatorname{div}(\mathbf{v})\big\}^2 - \frac{\partial v_i}{\partial x_j} \cdot \frac{\partial v_j}{\partial x_i}\right\} dx. \tag{2.5}$$

Let $g = |\mathbf{v}|^2$. Then $|\nabla g|^2 = 4 v_k \cdot \frac{\partial v_k}{\partial x_j} \cdot v_i \cdot \frac{\partial v_i}{\partial x_j}$. It follows from (2.5) that

$$\frac{1}{2}\int_{\Omega} \Psi'(g)|\nabla g|^2\, dx \leqslant 2\int_{\Omega} \Psi'(g) v_k \cdot \frac{\partial v_k}{\partial x_j} \cdot v_i \left\{\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right\} dx$$

$$+ 2\int_{\Omega} \Psi'(g) \cdot v_k \cdot \frac{\partial v_k}{\partial x_i} \cdot v_i \cdot \operatorname{div}(\mathbf{v})\, dx$$

$$+ \int_{\Omega} \Psi(g)\left\{\big\{\operatorname{div}(\mathbf{v})\big\}^2 - \frac{\partial v_i}{\partial x_j} \cdot \frac{\partial v_j}{\partial x_i}\right\} dx. \tag{2.6}$$

We now fix $0 < t < \tau < \infty$. Let Ψ be continuous so that $\Psi(s) = 1$ for $s \geqslant \tau$, $\Psi(s) = 0$ for $s \leqslant t$, and Ψ is linear on $[t, \tau]$. In view of (2.6), we obtain

$$\frac{1}{2(\tau - t)} \int_{t<g<\tau} |\nabla g|^2\, dx \leqslant \frac{1}{\tau - t} \int_{t<g<\tau} |\nabla g||\mathbf{v}|\left\{\sum_{i,j}\left|\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right|^2\right\}^{1/2} dx$$

$$+ \frac{1}{\tau - t} \int_{t<g<\tau} |\nabla g||\mathbf{v}|\big|\operatorname{div}(\mathbf{v})\big|\, dx$$

$$+ \int_{g>t} \Psi(g)\left\{\big\{\operatorname{div}(\mathbf{v})\big\}^2 - \frac{\partial v_i}{\partial x_j} \cdot \frac{\partial v_j}{\partial x_i}\right\} dx. \tag{2.7}$$

By the co-area formula, we may rewrite (2.7) as

$$\frac{1}{2(\tau-t)}\int_t^\tau\int_{g=s}|\nabla g|\,d\sigma\,ds \leqslant \frac{1}{\tau-t}\int_t^\tau\int_{g=s}|\mathbf{v}|\left\{\sum_{i,j}\left|\frac{\partial v_i}{\partial x_j}-\frac{\partial v_j}{\partial x_i}\right|^2\right\}^{1/2}d\sigma\,ds$$
$$+\frac{1}{\tau-t}\int_t^\tau\int_{g=s}|\mathbf{v}|\left|\operatorname{div}(\mathbf{v})\right|d\sigma\,ds$$
$$+\int_{g>t}\Psi(g)\left\{\left\{\operatorname{div}(\mathbf{v})\right\}^2-\frac{\partial v_i}{\partial x_j}\cdot\frac{\partial v_j}{\partial x_i}\right\}dx. \tag{2.8}$$

Letting $\tau\to t^+$ in (2.8), we obtain the desired estimate by the Lebesgue's differentiation theorem. $\square$

Next we apply Lemma 2.2 to harmonic functions in Ω with normal derivatives vanishing on part of the boundary.

Lemma 2.3. *Let Ω be a bounded convex domain with C^2 boundary and $Q\in\partial\Omega$. Let $u\in C^3(\overline{\Omega})$. Suppose that $\Delta u=0$ in Ω and $\frac{\partial u}{\partial n}=0$ on $B(Q,2r)\cap\partial\Omega$ for some $r>0$. Then for a.e. $t\in(0,\infty)$,*

$$\int_{g=t}|\nabla g|\,d\sigma\leqslant 6\sqrt{t}\int_{g=t}|\nabla u||\nabla\varphi|\,d\sigma+2\int_{g>t}|\nabla u|^2|\nabla\varphi|^2\,dx, \tag{2.9}$$

where $g=|(\nabla u)\varphi|^2$ and $\varphi\in C_0^\infty(B(Q,2r))$.

Proof. Let $\mathbf{v}=(\nabla u)\varphi$. Then $\mathbf{v}\cdot n=0$ on $\partial\Omega$ and

$$\frac{\partial v_i}{\partial x_j}=\varphi\frac{\partial^2 u}{\partial x_i\partial x_j}+\frac{\partial u}{\partial x_i}\frac{\partial\varphi}{\partial x_j}.$$

It follows that $\operatorname{div}(\mathbf{v})=(\Delta u)\varphi+\nabla u\cdot\nabla\varphi=\nabla u\cdot\nabla\varphi$ and

$$\frac{\partial v_i}{\partial x_j}-\frac{\partial v_j}{\partial x_i}=\frac{\partial u}{\partial x_i}\frac{\partial\varphi}{\partial x_j}-\frac{\partial u}{\partial x_j}\frac{\partial\varphi}{\partial x_i}.$$

Hence,

$$\left(\sum_{i,j}\left|\frac{\partial v_i}{\partial x_j}-\frac{\partial v_j}{\partial x_i}\right|^2\right)^{1/2}+\left|\operatorname{div}(\mathbf{v})\right|=\left\{2|\nabla u|^2|\nabla\varphi|^2-2(\nabla u\cdot\nabla\varphi)^2\right\}^{1/2}+|\nabla u\cdot\nabla\varphi|$$
$$\leqslant 3|\nabla u||\nabla\varphi|. \tag{2.10}$$

Next note that

$$\begin{aligned}\left|\operatorname{div}(\mathbf{v})\right|^2-\frac{\partial v_j}{\partial x_i}\frac{\partial v_i}{\partial x_j}&=-\varphi^2\left|\nabla^2u\right|^2-2\varphi\cdot\frac{\partial^2 u}{\partial x_i\partial x_j}\cdot\frac{\partial u}{\partial x_i}\cdot\frac{\partial\varphi}{\partial x_j}\\&=-\sum_{i,j}\left(\varphi\frac{\partial^2 u}{\partial x_i\partial x_j}+\frac{\partial u}{\partial x_i}\frac{\partial\varphi}{\partial x_j}\right)^2+|\nabla u|^2|\nabla\varphi|^2\\&\leqslant|\nabla u|^2|\nabla\varphi|^2.\end{aligned}\tag{2.11}$$

In view of (2.10)–(2.11), estimate (2.9) in Lemma 2.3 now follows readily from Lemma 2.2. □

Lemma 2.4. *Let Ω be a bounded convex domain in $\mathbb{R}^d$, $d\geqslant 3$. Let f, g be two nonnegative functions on $\overline{\Omega}$. Suppose that $f\in C(\overline{\Omega})$, $g\in C^1(\overline{\Omega})$ and*

$$\int\limits_{g=t}|\nabla g|\,d\sigma\leqslant C_0\left\{\sqrt{t}\int\limits_{g=t}f\,d\sigma+\int\limits_{g>t}f^2\,dx\right\}\tag{2.12}$$

for a.e. $t\in(0,\infty)$. Then there exists C depending only on d, q, C_0 and the Lipschitz character of Ω such that

$$\left\{\int\limits_{\Omega}g^q\,dx\right\}^{1/q}\leqslant C\left\{\int\limits_{\Omega}f^{2p}\,dx\right\}^{1/p}+C|\Omega|^{\frac{1}{q}-1}\int\limits_{\Omega}g\,dx,\tag{2.13}$$

where $p>1$ and $\frac{1}{q}=\frac{1}{p}-\frac{2}{d}$.

Proof. By considering $g_\delta=g+\delta$ and then letting $\delta\to 0^+$, we may assume that g is bounded from below by a positive constant. Using the co-area formula and (2.12), we obtain

$$\begin{aligned}\int\limits_{\Omega}g^\alpha|\nabla g|^2\,dx&=\int\limits_0^\infty t^\alpha\int\limits_{g=t}|\nabla g|\,d\sigma\,dt\\&\leqslant C_0\int\limits_0^\infty t^\alpha\left\{t^{\frac{1}{2}}\int\limits_{g=t}f\,d\sigma+\int\limits_{g>t}f^2\,dx\right\}dt\\&\leqslant C\int\limits_{\Omega}g^{\alpha+\frac{1}{2}}|\nabla g|f\,dx+C\int\limits_{\Omega}g^{\alpha+1}f^2\,dx,\end{aligned}\tag{2.14}$$

where $\alpha>-1$. By the Cauchy inequality with an $\varepsilon>0$,

$$\int\limits_{\Omega}g^{\alpha+\frac{1}{2}}|\nabla g|f\,dx\leqslant\varepsilon\int\limits_{\Omega}g^\alpha|\nabla g|^2\,dx+C_\varepsilon\int\limits_{\Omega}g^{\alpha+1}f^2\,dx.$$

This, together with (2.14), implies that

$$\int_\Omega g^\alpha |\nabla g|^2\,dx \leqslant C\int_\Omega g^{\alpha+1} f^2\,dx. \tag{2.15}$$

Since Ω is convex, there exists a constant C, depending only on d and $[\operatorname{diam}(\Omega)]^d/|\Omega|$, such that

$$\left\{\int_\Omega |w-w_\Omega|^{\frac{2d}{d-2}}\,dx\right\}^{\frac{d-2}{d}} \leqslant C\int_\Omega |\nabla w|^2\,dx, \tag{2.16}$$

where $w\in C^1(\overline{\Omega})$ and w_Ω denotes the average of w over Ω. Let $\beta>(1/2)$ and $w=g^\beta$ in (2.16). We obtain

$$\left\{\int_\Omega g^{\frac{2d\beta}{d-2}}\,dx\right\}^{\frac{d-2}{d}} \leqslant C\int_\Omega g^{2\beta-2}|\nabla g|^2\,dx + C|\Omega|^{-1-\frac{2}{d}}\left\{\int_\Omega g^\beta\,dx\right\}^2. \tag{2.17}$$

Let $\alpha=2\beta-2$. It follows from (2.15) and (2.17) that

$$\left\{\int_\Omega g^{\frac{2d\beta}{d-2}}\,dx\right\}^{\frac{d-2}{d}} \leqslant C\int_\Omega g^{2\beta-1} f^2\,dx + C|\Omega|^{-1-\frac{2}{d}}\left\{\int_\Omega g^\beta\,dx\right\}^2, \tag{2.18}$$

for any $\beta>(1/2)$.

We now choose $p_0>1$ so that $(2\beta-1)p_0=\frac{2d\beta}{d-2}$. By Hölder's inequality,

$$\begin{aligned}\int_\Omega g^{2\beta-1} f^2\,dx &\leqslant \left\{\int_\Omega g^{(2\beta-1)p_0}\,dx\right\}^{1/p_0}\left\{\int_\Omega f^{2p_0'}\,dx\right\}^{1/p_0'}\\ &\leqslant \varepsilon\left\{\int_\Omega g^{(2\beta-1)p_0}\,dx\right\}^{\frac{p_1}{p_0}} + C_\varepsilon\left\{\int_\Omega f^{2p_0'}\,dx\right\}^{\frac{p_1'}{p_0'}},\end{aligned} \tag{2.19}$$

where $p_1=\frac{2\beta}{2\beta-1}$. Note that $\frac{p_1}{p_0}=\frac{d-2}{d}$. Also $2p_0'=\frac{4d\beta}{d-2+4\beta}$ and $\frac{p_1'}{p_0'}=\frac{d-2+4\beta}{d}$. In view of (2.18)–(2.19), we obtain

$$\left\{\int_\Omega g^{\frac{2d\beta}{d-2}}\,dx\right\}^{\frac{d-2}{d}} \leqslant C\left\{\int_\Omega f^{\frac{4d\beta}{d-2+4\beta}}\,dx\right\}^{\frac{d-2+4\beta}{d}} + C|\Omega|^{-1-\frac{2}{d}}\left\{\int_\Omega g^\beta\,dx\right\}^2. \tag{2.20}$$

Finally we let $p=\frac{2d\beta}{d-2+4\beta}$ and $q=\frac{2d\beta}{d-2}$. It follows from (2.20) that

$$\left\{\int_\Omega g^q\,dx\right\}^{1/q} \leqslant C\left\{\int_\Omega f^{2p}\,dx\right\}^{1/p} + C|\Omega|^{-\frac{1}{2\beta}-\frac{1}{d\beta}}\left\{\int_\Omega g^\beta\,dx\right\}^{1/\beta}. \tag{2.21}$$

Note that $\frac{1}{q} = \frac{1}{p} - \frac{2}{d}$ and $2\beta = (1 - \frac{2}{d})q$. Also $-\frac{1}{2\beta} - \frac{1}{d\beta} = \frac{1}{q} - \frac{1}{\beta}$. Since $2\beta < q$, the desired estimate follows from (2.21) by Hölder's inequality. □

We are now in a position to give the proof of Theorem 2.1.

Proof of Theorem 2.1. Let $1 < \rho < \tau < 2$. Choose $\varphi \in C_0^\infty(B(Q, \tau r))$ such that $\varphi = 1$ in $B(Q, \rho r)$ and $|\nabla\varphi| \leqslant C[(\tau - \rho)r]^{-1}$. It follows from Lemmas 2.3 and 2.4 that

$$\left\{ \int_\Omega \left|(\nabla u)\varphi\right|^{2q} dx \right\}^{1/q} \leqslant C \left\{ \int_\Omega |\nabla u|^{2p} |\nabla\varphi|^{2p} dx \right\}^{1/p} + C|\Omega|^{\frac{1}{q}-1} \int_\Omega \left|(\nabla u)\varphi\right|^2 dx, \qquad (2.22)$$

where $p > 1$ and $\frac{1}{q} = \frac{1}{p} - \frac{2}{d}$. This yields that

$$\left\{ \frac{1}{r^d} \int_{\Omega\cap B(Q,\rho r)} |\nabla u|^{2q} dx \right\}^{1/(2q)} \leqslant C \left\{ \frac{1}{r^d} \int_{\Omega\cap B(Q,\tau r)} |\nabla u|^{2p} dx \right\}^{1/(2p)} \qquad (2.23)$$

for any $p > 1$ and $\frac{1}{q} = \frac{1}{p} - \frac{2}{d}$, where $1 < \rho < \tau < 2$. By a simple iteration argument, we obtain

$$\left\{ \frac{1}{r^d} \int_{\Omega\cap B(Q,r)} |\nabla u|^{q} dx \right\}^{1/q} \leqslant C \left\{ \frac{1}{r^d} \int_{\Omega\cap B(Q,3r/2)} |\nabla u|^{p} dx \right\}^{1/p}, \qquad (2.24)$$

for any $2 < p < q < \infty$. Estimate (2.1) follows readily from (2.24) and the reverse Hölder inequality,

$$\left\{ \frac{1}{r^d} \int_{\Omega\cap B(Q,3r/2)} |\nabla u|^{\bar p} dx \right\}^{1/\bar p} \leqslant C \left\{ \frac{1}{r^d} \int_{\Omega\cap B(Q,2r)} |\nabla u|^{2} dx \right\}^{1/2}, \qquad (2.25)$$

where $\bar p = 2 + \eta$ and $\eta > 0$ depends on the Lipschitz character of Ω. We mention that (2.25) with $\bar p$ close to 2 holds even for weak solutions of elliptic systems of divergence form with bounded measurable coefficients. See e.g. [5, Chapter V] for the interior case. The boundary case follows from the interior case by a reflection argument. □

Remark 2.5. Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$, $d \geqslant 2$. Suppose that $\Delta u = 0$ in Ω, $(\nabla u)^* \in L^2(\partial\Omega)$ and $\frac{\partial u}{\partial n} = 0$ on $B(Q, 3r) \cap \partial\Omega$. Then the estimate (2.1) holds for $2 < q < 3 + \varepsilon$ if $d \geqslant 3$; and for $2 < q < 4 + \varepsilon$ if $d = 2$. To show this, one uses the fact that the L^2 Neumann problem in Lipschitz domains is solvable as well as the observation that u is C^α in $B(Q, 2r) \cap \overline{\Omega}$ for some $\alpha > 0$ if $d \geqslant 3$; and for some $\alpha > (1/2)$ if $d = 2$. We refer the reader to [14, pp. 188–189], where the same estimate was proved for a Lipschitz domain Ω, under the Dirichlet condition $u = 0$ on $B(Q, 3r) \cap \partial\Omega$. The proof in [14] extends easily to the case of the Neumann boundary condition. We point out that if $d \geqslant 3$, the C^α $(\alpha > 0)$ estimate follows from the De Giorgi–Nash estimate by a reflection argument. For the case $d = 2$, one may use the solvability of the L^p Neumann problem in Lipschitz domains for some $p = \bar p > 2$ and the square function estimates to show that $\nabla u \in L^{\bar p}_{1/\bar p}(B(Q, 2r) \cap \Omega) \subset L^{2\bar p}(B(Q, 2r) \cap \Omega)$. By

Sobolev imbedding, this implies that u is C^α on $B(Q,2r)\cap\overline{\Omega}$ for some $\alpha>(1/2)$. If Ω is C^1, the estimate (2.1) holds for any $d\geqslant 2$ and $q>2$. This follows from the fact that the L^p Neumann problem in C^1 domains is solvable for any $p>2$. Since the results in this paper do not use the estimates mentioned above, we omit the details here.

3. Weak reverse Hölder inequality on the boundary

The goal of this section is to prove the following.

Theorem 3.1. *Under the same conditions on Ω and u as in Theorem* 2.1*, we have*

$$\left\{\frac{1}{r^{d-1}}\int_{B(Q,r)\cap\partial\Omega}\left|(\nabla u)^*\right|^p d\sigma\right\}^{1/p}\leqslant C\left\{\frac{1}{r^{d-1}}\int_{B(Q,2r)\cap\partial\Omega}\left|(\nabla u)^*\right|^2 d\sigma\right\}^{1/2}, \tag{3.1}$$

for any $p>2$, where C depends only on d, p and the Lipschitz character of Ω.

We begin with a local $W^{2,2}$ estimate.

Lemma 3.2. *Under the same conditions on Ω and u as in Theorem* 2.1*, we have*

$$\int_{B(Q,r)\cap\Omega}\left|\nabla^2 u\right|^2 dx\leqslant\frac{C}{r^2}\int_{B(Q,2r)\cap\Omega}|\nabla u|^2\,dx \tag{3.2}$$

where C depends only on d.

Proof. See e.g. [10, p. 1826]. □

Let $\delta(x)=\operatorname{dist}(x,\partial\Omega)$.

Lemma 3.3. *Let w be a harmonic function in a bounded Lipschitz domain Ω. Let $p>2$. Fix $x_0\in\Omega$ such that $\delta(x_0)\geqslant c_0\operatorname{diam}(\Omega)$. Then for any $t\in(0,1)$,*

$$\begin{aligned}\int_{\partial\Omega}\left|(\nabla w)^*\right|^p d\sigma\leqslant{}& C\left\{\operatorname{diam}(\Omega)\right\}^t\sup_{x\in\Omega}\left|\nabla^2 w(x)\right|^{p-2}\left[\delta(x)\right]^{p-1-t}\int_{\Omega}\left|\nabla^2 w\right|^2 dy\\&+C\left|\nabla w(x_0)\right|^p|\partial\Omega|,\end{aligned} \tag{3.3}$$

where C depends only on d, p, t, c_0 and the Lipschitz character of Ω.

Proof. See e.g. [10, p. 1827]. □

Proof of Theorem 3.1. Since Ω is a Lipschitz domain, by rotation and translation, we may assume that $Q=0$ and

$$B(Q,C_0r_0)\cap\Omega=\left\{\left(x',x_d\right):\ x_d>\psi\left(x'\right)\right\}\cap B(Q,C_0r_0)$$

where $\psi : \mathbb{R}^{d-1} \to \mathbb{R}$ such that $\psi(0) = 0$ and $\|\nabla\psi\|_\infty \leqslant M$, and $C_0 = 10\sqrt{d}(1+M)$. Let

$$S(r) = \{(x', \psi(x')):\ |x'| < r\}.$$

We will show that if $u \in C^2(\overline{\Omega})$ is harmonic in Ω and $\frac{\partial u}{\partial n} = 0$ on $S(8r)$, then

$$\left\{\frac{1}{r^{d-1}} \int\limits_{S(r)} |(\nabla u)^*|^p\, d\sigma\right\}^{1/p} \leqslant C\left\{\frac{1}{r^{d-1}} \int\limits_{S(4r)} |(\nabla u)^*|^2\, d\sigma\right\}^{1/2}, \tag{3.4}$$

where C depends only on d, p and M. Estimate (3.1) follows from (3.4) by a simple covering argument.

For $P \in \partial\Omega$, define

$$\begin{aligned} \mathcal{M}_1(\nabla u)(P) &= \sup\{|\nabla u(x)|:\ x \in \Omega,\ |x-P| < C_0\delta(x) \text{ and } |x-P| \leqslant c_0 r\},\\ \mathcal{M}_2(\nabla u)(P) &= \sup\{|\nabla u(x)|:\ x \in \Omega,\ |x-P| < C_0\delta(x) \text{ and } |x-P| > c_0 r\}. \end{aligned} \tag{3.5}$$

Note that $(\nabla u)^* = \max\{\mathcal{M}_1(\nabla u), \mathcal{M}_2(\nabla u)\}$. The desired estimate for $\mathcal{M}_2(\nabla u)$ follows readily from the interior estimates for harmonic functions. To handle $\mathcal{M}_1(\nabla u)$, we apply Lemma 3.3 to u on the Lipschitz domain $Z(2r)$, where

$$Z(\rho) = \{(x', x_d):\ |x'| < \rho \text{ and } \psi(x') < x_d < 20\sqrt{d}(1+M)\rho\}.$$

This Yields That

$$\begin{aligned} \frac{1}{r^{d-1}} \int\limits_{S(r)} |\mathcal{M}_1(\nabla u)|^p\, d\sigma &\leqslant \frac{1}{r^{d-1}} \int\limits_{\partial Z(2r)} |(\nabla u)^*_{Z(2r)}|^p\, d\sigma \\ &\leqslant C r^{t-d+1} \sup_{Z(2r)} |\nabla^2 u(x)|^{p-2}[\delta(x)]^{p-1-t} \int\limits_{Z(2r)} |\nabla^2 u(y)|^2\, dy \\ &\quad + C|\nabla v(x_0)|^p, \end{aligned} \tag{3.6}$$

where $\delta(x) = \operatorname{dist}(x, Z(2r))$ and $(\nabla u)^*_{Z(2r)}$ denotes the nontangential maximal function of ∇u with respect to the domain $Z(2r)$. Note that the last term in the right-hand side of (3.6) may be treated easily, using the interior estimates.

Let I denote the first term in the right-hand side of (3.6). By Lemma 3.2,

$$I \leqslant C r^{t-d-1} \sup_{Z(2r)} |\nabla^2 u(x)|^{p-2}[\delta(x)]^{p-1-t} \int\limits_{Z(2r)} |\nabla u(y)|^2\, dy. \tag{3.7}$$

Let $x \in Z(2r)$. It follows from the interior estimates that for any $q > 2$,

$$\begin{aligned}
\left|\nabla^2 u(x)\right| &\leqslant \frac{C}{\delta(x)}\left\{\frac{1}{[\delta(x)]^d}\int\limits_{B(x,\delta(x))}|\nabla u|^q\,dx\right\}^{1/q}\\
&\leqslant \frac{Cr^{\frac{d}{q}}}{[\delta(x)]^{1+\frac{d}{q}}}\left\{\frac{1}{r^d}\int\limits_{Z(2r)}|\nabla u|^q\,dx\right\}^{1/q}\\
&\leqslant \frac{Cr^{\frac{d}{q}}}{[\delta(x)]^{1+\frac{d}{q}}}\left\{\frac{1}{r^d}\int\limits_{Z(4r)}|\nabla u|^2\,dx\right\}^{1/2},
\end{aligned}\tag{3.8}$$

where we have used estimate (2.1) in the last step. This, together with (3.7), implies that

$$I \leqslant Cr^{t-1+\frac{d}{q}(p-2)}\sup_{x\in Z(2r)}\left[\delta(x)\right]^{p-1-t-(1+\frac{d}{q})(p-2)}\left\{\frac{1}{r^d}\int\limits_{Z(4r)}|\nabla u|^2\,dy\right\}^{p/2}. \tag{3.9}$$

Since $p-1-t-(1+\frac{d}{q})(p-2)=1-t-\frac{d}{q}(p-2)$, we may choose $q>2$ so large that the exponent of $\delta(x)$ in (3.9) is positive. Using $\delta(x)\leqslant Cr$, we then obtain

$$I \leqslant C\left\{\frac{1}{r^d}\int\limits_{Z(4r)}|\nabla u|^2\,dy\right\}^{p/2} \leqslant C\left\{\frac{1}{r^{d-1}}\int\limits_{S(4r)}\left|(\nabla u)^*\right|^2\,d\sigma\right\}^{p/2}. \tag{3.10}$$

Thus we have proved that

$$\frac{1}{r^{d-1}}\int\limits_{S(r)}\left|\mathcal{M}_1(\nabla u)\right|^p\,d\sigma \leqslant C\left\{\frac{1}{r^{d-1}}\int\limits_{S(4r)}\left|(\nabla u)^*\right|^2\,d\sigma\right\}^{p/2}.$$

This, together with the same estimate for $\mathcal{M}_2(\nabla u)$, gives (3.4). □

Remark 3.4. Let $p>2$. It follows from Theorem 3.1 and [10, Theorem 1.1] that if Ω is a bounded convex domain with C^2 boundary, the L^p Neumann problem for $\Delta u=0$ in Ω is uniquely solvable. Moreover, the solution satisfies the estimate $\|(\nabla u)^*\|_p \leqslant C\|\frac{\partial u}{\partial n}\|_p$, where C depends only on d, p and the Lipschitz character of Ω.

4. Proof of Theorem 1.1

Let Ω be a bounded convex domain in $\mathbb{R}^d$, $d\geqslant 2$. Let $p>2$. We need to show that the L^p Neumann problem for $\Delta u=0$ in Ω is uniquely solvable. Since the case $d=2$ is contained in [10], we will assume that $d\geqslant 3$.

The uniqueness of the L^p Neumann problem follows directly from the uniqueness of the L^2 Neumann problem. To establish the existence, it suffices to show that if $f\in C_0^\infty(\mathbb{R}^d)$ and $\int_{\partial\Omega} f\,d\sigma=0$, then the solution of the L^2 Neumann problem for $\Delta u=0$ in Ω with boundary data $f|_{\partial\Omega}$ satisfies $\|(\nabla u)^*\|_p\leqslant C\|f\|_p$.

To this end we approximate Ω from the outside by a sequence of convex domains $\{\Omega_j\}$ with smooth boundaries and uniform Lipschitz characters. Let u_j be a solution to the L^2 Neumann problem for Laplace's equation in Ω_j with data $f_j - \alpha_j$, where α_j is the mean value of f on $\partial\Omega_j$. It follows from Remark 3.4 that

$$\big\|(\nabla u_j)^*\big\|_{L^p(\partial\Omega_j)} \leqslant C\|f_j - \alpha_j\|_{L^p(\partial\Omega_j)}, \tag{4.1}$$

where C depends only on d, p and the Lipschitz character of Ω. By a limiting argument (see e.g. [7]), there exists a subsequence, still denoted by $\{u_j\}$, such that $\nabla u_j \to \nabla v$ uniformly on any compact subset of Ω, where v is a variational solution of the Neumann problem in Ω with data $f|_{\partial\Omega}$. Using this and (4.1), we may deduce that $\|(\nabla v)^*\|_{L^p(\partial\Omega)} \leqslant C\|f\|_{L^p(\partial\Omega)}$. Since $u - v$ is constant by the uniqueness of the variational solutions, we obtain $\|(\nabla u)^*\|_{L^p(\partial\Omega)} \leqslant C\|f\|_{L^p(\partial\Omega)}$. This completes the proof.

5. Proof of Theorem 1.2

To establish the existence, we first reduce the problem to the case where $f = 0$. The argument is standard. Let $f \in L^p_{-1,0}(\Omega)$ and $w = \Pi_\Omega(f) =: \mathcal{R}_\Omega \Pi(\widetilde{f})$. Here $\mathcal{R}_\Omega$ denotes the operator restricting distributions in $\mathbb{R}^d$ to Ω, the map $\Pi : \mathcal{E}'(\mathbb{R}^d) \to \mathcal{D}'(\mathbb{R}^d)$ is given by the convolution with the fundamental solution for Δ in $\mathbb{R}^d$ with pole at the origin, and $\widetilde{f}$ is defined by $\langle \widetilde{f}, \phi\rangle = \langle f, \mathcal{R}_\Omega(\phi)\rangle$ for $\phi \in C^\infty(\mathbb{R}^d)$. Let $q = \frac{p}{p-1}$. For any $\phi \in B^q_s(\partial\Omega)$ with $s = \frac{1}{p}$, define

$$\big\langle \Lambda(f), \phi\big\rangle = \int_\Omega \nabla w \cdot \nabla\psi \, dx + \langle f, \psi\rangle_{L^p_{-1,0}(\Omega)\times L^q_1(\Omega)}, \tag{5.1}$$

where ψ is a function in $L^q_1(\Omega)$ such that $Tr(\psi) = \phi$ and $\|\psi\|_{L^q_1(\Omega)} \leqslant C\|\phi\|_{B^q_{1/p}(\partial\Omega)}$. Here we have used the fact that the trace operator $Tr : L^q_1(\Omega) \to B^q_{1/p}(\partial\Omega)$ is bounded and onto and that

$$\|\phi\|_{B^q_{1/p}(\partial\Omega)} \approx \inf\big\{\|\psi\|_{L^q_1(\Omega)} \colon Tr(\psi) = \phi\big\}.$$

Since

$$\int_\Omega \nabla w \cdot \nabla\psi \, dx + \langle f, \psi\rangle = 0 \quad \text{for any } \psi \in C_0^\infty(\Omega)$$

and $C_0^\infty(\Omega)$ is dense in $\{u \in L^q_1(\Omega) \colon Tr(u) = 0\}$, it is easy to see that $\langle \Lambda(f), \phi\rangle$ is well defined. Furthermore,

$$\begin{aligned}
\big|\big\langle \Lambda(f), \phi\big\rangle\big| &\leqslant \big\{\|w\|_{L^p_1(\Omega)} + \|f\|_{L^p_{-1,0}(\Omega)}\big\}\|\psi\|_{L^q_1(\Omega)} \\
&\leqslant C\|f\|_{L^p_{-1,0}(\Omega)}\|\psi\|_{L^q_1(\Omega)} \\
&\leqslant C\|f\|_{L^p_{-1,0}(\Omega)}\|\phi\|_{B^q_{1/p}(\partial\Omega)},
\end{aligned} \tag{5.2}$$

where we have used the Calderón–Zygmund estimate $\|w\|_{L^p_1(\Omega)} \leqslant C\|f\|_{L^p_{-1,0}(\Omega)}$. It follows that w is a weak solution to (1.7) with data $(f, \Lambda(f))$ and $\|\Lambda(f)\|_{B^p_{-1/p}(\partial\Omega)} \leqslant C\|f\|_{L^p_{-1,0}(\Omega)}$. Thus, by subtracting w from u, we may always assume that $f = 0$.

Next, we note that if Ω is a bounded Lipschitz domain, the solvability of the Neumann problem,

$$\begin{cases} \Delta u = 0 \quad \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = g \in B^p_{-s}(\partial\Omega) \quad \text{on } \partial\Omega, \\ u \in L^p_{1-s+1/p}(\Omega), \end{cases} \tag{5.3}$$

was established in [3] for $(s, 1/p)$ in the (open) convex polygon $\mathcal{P}$ formed by the vertices,

$$(1-\varepsilon, 0), \quad (1,0), \quad \big(1, (1+\varepsilon)/2\big), \quad (\varepsilon, 1), \quad (0,1), \quad \big(0, (1-\varepsilon)/2\big),$$

where $\varepsilon > 0$ depends on Ω. By interpolation, this, together with Theorem 1.1, implies that the Neumann problem (5.3) in a convex domain is uniquely solvable if $(s, 1/p)$ is the (open) convex polygon $\mathcal{P}_1$ formed by the vertices

$$(1-\varepsilon, 0), \quad (1,0), \quad \big(1, (1+\varepsilon)/2\big), \quad (\varepsilon, 1), \quad (0,1), \quad (0,0).$$

In particular, the Neumann problem (5.3) is solvable if $s = 1/p$ and $2 \leqslant p < \infty$. As a result, we have proved Theorem 1.2 for $2 \leqslant p < \infty$. The case $1 < p < 2$ will be proved by a duality argument, given in the next section (see Remark 6.5).

6. Proof of Theorem 1.3

Note that

$$\begin{aligned} L^p_\sigma(\Omega) &= \left\{ \mathbf{u} \in L^p\big(\Omega, \mathbb{R}^d\big)\colon \int_\Omega \mathbf{u} \cdot \nabla\psi \, dx = 0 \text{ for any } \psi \in W^{1,q}(\Omega) \right\} \\ &= \big\{ \mathbf{u} \in L^p\big(\Omega, \mathbb{R}^d\big)\colon \operatorname{div}(\mathbf{u}) = 0 \text{ in } \Omega \text{ and } \mathbf{u} \cdot n = 0 \text{ on } \partial\Omega \big\}, \end{aligned} \tag{6.1}$$

where $q = \frac{p}{p-1}$. Here $\mathbf{u} \cdot n$ is regarded as an element in $B^p_{-1/p}(\partial\Omega)$. Let X be a normed vector space and S a subset of X. The set

$$S^\perp = \big\{ \ell \in X^*\colon \langle \ell, f \rangle = 0 \text{ for all } f \in S \big\}$$

is called the set of annihilators of S. If S is a closed subspace of a reflexive Banach space X, then $(S^\perp)^\perp = S$ (see e.g. [13]). With this notation, we may write $L^p_\sigma(\Omega) = S^\perp \subset X = L^p(\Omega, \mathbb{R}^d)$, where $S = \operatorname{grad} W^{1,q}(\Omega) \subset L^q(\Omega, \mathbb{R}^d)$. Thus the L^p-Helmholtz decomposition (1.10) may be written as

$$L^p\big(\Omega, \mathbb{R}^d\big) = \operatorname{grad} W^{1,p}(\Omega) \oplus \big\{ \operatorname{grad} W^{1,q}(\Omega) \big\}^\perp. \tag{6.2}$$

Lemma 6.1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$ and $1 < p < \infty$. Then $C^\infty_\sigma(\Omega) = \{\mathbf{v} \in C^\infty_0(\Omega, \mathbb{R}^d)\colon \operatorname{div}(\mathbf{v}) = 0\}$ is dense in $L^p_\sigma(\Omega)$.*

Proof. Let $X^p(\Omega)$ denote the closure of $C_\sigma^\infty(\Omega)$ in $L^p(\Omega,\mathbb{R}^d)$. Clearly, $\operatorname{grad} W^{1,q}(\Omega) \subset (X^p(\Omega))^\perp$. On the other hand, if $\mathbf{u} \in L^q(\Omega,\mathbb{R}^d)$ and $\int_\Omega \mathbf{u}\cdot\mathbf{v}\,dx = 0$ for any $\mathbf{v} \in C_\sigma^\infty(\Omega)$, then $\mathbf{u} = -\nabla\psi$ for some $\psi \in L^1_{loc}(\Omega)$ (see e.g. [4, pp. 696–697] for a proof). This implies that $\mathbf{u} \in \operatorname{grad} W^{1,q}(\Omega)$. Hence we obtain $\operatorname{grad} W^{1,q}(\Omega) = (X^p(\Omega))^\perp$. It follows that $X^p(\Omega) = (\operatorname{grad} W^{1,q}(\Omega))^\perp = L^p_\sigma(\Omega)$ and thus $C_\sigma^\infty(\Omega)$ is dense in $L^p_\sigma(\Omega)$. □

Lemma 6.2. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$ and $1 < p < \infty$. If the Helmholtz decomposition* (1.10) *with the estimate* (1.11) *holds for the exponent p and constant C_p, then it holds for the dual exponent $q = \frac{p}{p-1}$ and constant $C_q = C_p$.*

Proof. This follows from the fact that if X_0, X_1 are closed subspaces of X and $X = X_0 \oplus X_1$, then $X^* = X_0^\perp \oplus X_1^\perp$. Note that if $\mathbf{u} \in L^p_\sigma(\Omega)$, $\mathbf{v} \in L^q_\sigma(\Omega)$, $\phi \in W^{1,p}(\Omega)$ and $\psi \in W^{1,q}(\Omega)$, then

$$\int_\Omega \mathbf{u}\cdot(\mathbf{v}+\nabla\psi)\,dx = \int_\Omega (\mathbf{u}+\nabla\phi)\cdot\mathbf{v}\,dx,$$

$$\int_\Omega \nabla\phi\cdot(\mathbf{v}+\nabla\psi)\,dx = \int_\Omega (\mathbf{u}+\nabla\phi)\cdot\nabla\psi\,dx. \tag{6.3}$$

By a simple duality argument, this shows that the estimate (1.11) holds for the exponent q and constant $C_q = C_p$. □

Next we will show that the L^p-Helmholtz decomposition is equivalent to the solvability of (5.3) for $s = 1/p$.

Theorem 6.3. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$ and $1 < p < \infty$. Then the L^p-Helmholtz decomposition* (1.10) *with the estimate* (1.11) *holds if and only if the Neumann problem* (5.3) *is uniquely solvable for $s = 1/p$.*

Proof. Suppose that (5.3) is uniquely solvable for $s = 1/p$. The uniqueness of the solutions to (5.3) implies that $L^p_\sigma(\Omega) \cap \operatorname{grad} W^{1,p}(\Omega) = \{0\}$. Given any $\mathbf{u} = (u_1,\dots,u_d) \in L^p(\Omega,\mathbb{R}^d)$, let

$$\phi(x) = \frac{\partial}{\partial x_i}\int_\Omega \Gamma(x-y)u_i(y)\,dy, \tag{6.4}$$

where $\Gamma(x)$ denotes the fundamental solution for Δ in $\mathbb{R}^d$ with pole at the origin. By the Calderón–Zygmund estimate, $\phi \in W^{1,p}(\Omega)$ and $\|\nabla\phi\|_{L^p(\Omega)} \leqslant C\|\mathbf{u}\|_{L^p(\Omega)}$. Since $\operatorname{div}(\mathbf{u} - \nabla\phi) = 0$ in Ω, it follows that $\Lambda = (\mathbf{u}-\nabla\phi)\cdot n \in B^p_{-1/p}(\partial\Omega)$ and $\|\Lambda\|_{B^p_{-1/p}(\partial\Omega)} \leqslant C\|\mathbf{u} - \nabla\phi\|_{L^p(\Omega)}$, where Λ may be defined by

$$\langle \Lambda, \varphi\rangle = \int_\Omega (\mathbf{u}-\nabla\phi)\cdot\nabla\widetilde{\varphi}\,dx$$

for $\varphi \in B^q_{1/p}(\partial\Omega)$ and $\widetilde{\varphi} \in W^{1,q}(\Omega)$ such that $Tr(\widetilde{\varphi}) = \varphi$ on $\partial\Omega$. We now let $\mathbf{v} = \mathbf{u} - \nabla(\phi + \psi) \in L^p(\Omega, \mathbb{R}^d)$, where $\psi \in W^{1,p}(\Omega)$ is a solution to (5.3) with boundary data Λ. Observe that for any $\varphi \in C^\infty(\mathbb{R}^d)$,

$$\int_\Omega \mathbf{v} \cdot \nabla\varphi \, dx = \langle \Lambda, \varphi \rangle - \int_\Omega \nabla\psi \cdot \nabla\varphi = 0.$$

Thus $\mathbf{v} \in L^p_\sigma(\Omega)$. Also note that

$$\begin{aligned} \big\|\nabla(\phi + \psi)\big\|_{L^p(\Omega)} &\leqslant C\big\{\|\nabla\phi\|_{L^p(\Omega)} + \|\Lambda\|_{B^p_{-1/p}(\partial\Omega)}\big\} \\ &\leqslant C\big\{\|\nabla\phi\|_{L^p(\Omega)} + \|\mathbf{u} - \nabla\phi\|_{L^p(\Omega)}\big\} \\ &\leqslant C\|\mathbf{u}\|_{L^p(\Omega)}. \end{aligned}$$

Since $\mathbf{u} = \mathbf{v} + \nabla(\phi + \psi)$, we obtain the Helmholtz decomposition (6.2).

Next suppose that the L^p-Helmholtz decomposition (1.10) with estimate (1.11) holds. The uniqueness for the Neumann problem (5.3) follows from the fact that

$$L^p_\sigma(\Omega) \cap \operatorname{grad} W^{1,p}(\Omega) = \{0\}.$$

To show the existence, let ψ be a solution of the L^2 Neumann problem in Ω with boundary data $\frac{\partial\psi}{\partial n} = g$, where $g \in L^\infty(\partial\Omega)$ and $\int_{\partial\Omega} g \, d\sigma = 0$. Given $\mathbf{u} \in L^q(\Omega, \mathbb{R}^d) \cap L^2(\Omega, \mathbb{R}^d)$, write $\mathbf{u} = \mathbf{v} + \nabla\phi$, where $\mathbf{v} \in L^q_\sigma(\Omega) \cap L^2_\sigma(\Omega)$ and $\phi \in W^{1,q}(\Omega) \cap W^{1,2}(\Omega)$. This is possible since the Helmholtz decomposition holds for exponents q and 2. It follows that

$$\begin{aligned} \left|\int_\Omega \nabla\psi \cdot \mathbf{u} \, dx\right| = \left|\int_\Omega \nabla\psi \cdot \nabla\phi \, dx\right| &= \left|\int_{\partial\Omega} \frac{\partial\psi}{\partial n} \cdot (\phi - \alpha) \, d\sigma\right| \\ &\leqslant \left\|\frac{\partial\psi}{\partial n}\right\|_{B^p_{-1/p}(\partial\Omega)} \|\phi - \alpha\|_{B^q_{1/p}(\partial\Omega)} \\ &\leqslant \left\|\frac{\partial\psi}{\partial n}\right\|_{B^p_{-1/p}(\partial\Omega)} \|\phi - \alpha\|_{W^{1,q}(\Omega)}, \end{aligned}$$

for any $\alpha \in \mathbb{R}$. By Poincaré inequality, this yields that

$$\left|\int_\Omega \nabla\psi \cdot \mathbf{u} \, dx\right| \leqslant C\left\|\frac{\partial\psi}{\partial n}\right\|_{B^p_{-1/p}(\partial\Omega)} \|\nabla\phi\|_{L^q(\Omega)}.$$

Using $\|\nabla\phi\|_{L^q(\Omega)} \leqslant C_q\|\mathbf{u}\|_{L^q(\Omega)}$, we then obtain

$$\|\nabla\psi\|_{L^p(\Omega)} \leqslant C\left\|\frac{\partial\psi}{\partial n}\right\|_{B^p_{-1/p}(\partial\Omega)} \tag{6.5}$$

by duality. Since $L^\infty(\partial\Omega)$ is dense in $B^p_{-1/p}(\partial\Omega)$, the existence of solutions to (5.3) with data Λ, where $\Lambda \in B^p_{-1/p}(\partial\Omega)$ and $\langle \Lambda, 1\rangle = 0$, follows from the estimate (6.5) by a simple limiting argument. This completes the proof. □

Lemma 6.2 and Theorem 6.3 lead to the following.

Theorem 6.4. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$ and $1 < p < \infty$. Then the solvability of* (5.3) *for $s = 1/p$ is equivalent to the solvability of* (5.3) *for $s = 1/q$, where $q = \frac{p}{p-1}$.*

Remark 6.5. We show in Section 5 that if Ω is a bounded convex domain in $\mathbb{R}^d$, then the Neumann problem (5.3) is solvable for $s = 1/p$ and $2 < p < \infty$. Thus, by Theorem 6.4, the Neumann problem (5.3) in convex domains with $s = 1/p$ is solvable for any $1 < p < \infty$. This completes the proof of Theorem 1.2.

Remark 6.6. Theorem 1.3 follows readily from Theorems 1.2 and 6.3.

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