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Note

Covering a square of side $n + \varepsilon$ with unit squares

Alexander Soifer[a, b]

[a] *University of Colorado at Colorado Springs, P. O. Box 7150, Colorado Springs, CO 80933, USA*
[b] *DIMACS, Rutgers University*

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Abstract

In [Covering a triangle with triangles, Amer. Math. Monthly 112 (1) (2005) 78; Cover-up, Geombinatorics XIV (1) (2004) 8–9], Conway and I showed that in order to cover an equilateral triangle of side length $n+\varepsilon$, n^2+2 unit equilateral triangles suffice while obviously n^2+1 are wanted. (The latest "triangular" results can be found in [D. Karabash, A. Soifer, On covering of trigons, Geombinatorics XV (1) (2005) 13–17].) Here I pose an analogous problem for squares and show that in order to cover a square of side length $n+\varepsilon$, $n^2+o(1)\,n+O(1)$ unit squares suffice. This problem is dual to the one solved by Erdös and Graham 30 years ago [On packing squares with equal squares, J. Combin. Theory (A) 19 (1975) 119–123], which dealt with packing unit squares in a square. And as in Erdös–Graham, in our problem a natural upper bound of $(n+1)^2$ provided by a square lattice can be much improved.

Keywords: Combinatorial; Discrete geometry; Optimal covering; Packing

Naturally, a square of side n can be covered by n^2 unit squares. When, however, the side length increases to $n+\varepsilon$, we obtain a new open problem:

The Problem. Find the smallest number $\Pi(n)$ of unit squares that can cover a square of side length $n+\varepsilon$.

This problem, in a sense, is dual to the one solved by Erdös and Graham 30 years ago [EG], which dealt with packing unit squares in a square. And as in Erdös–Graham, in our problem a natural upper bound of $(n+1)^2$ provided by a square lattice can be much improved. Let us do it.

E-mail address: asoifer@uccs.edu.
URL: http://www.uccs.edu/∼ asoifer.

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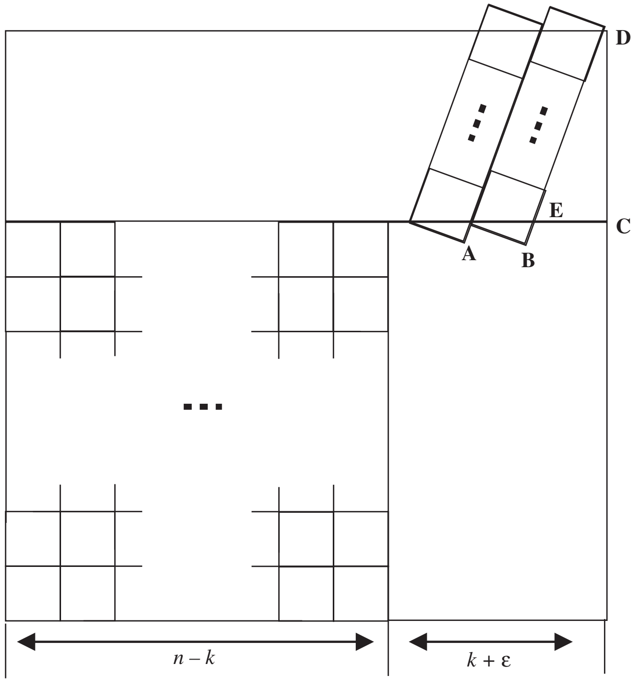

Fig. 1.

The Bound. Given a $(n+\varepsilon)\times(n+\varepsilon)$ square. To cover it with unit squares we will use a square grid in the lower left $(n-k)\times(n-k)$ subsquare, and $1\times(k+1)$ polyominos in covering one (upper) side of the remaining L-shaped molding and then doubling the result to account for the right-hand side of the molding (Fig. 1).

Let us look closer at this polyomino covering. Observe that in fact the circle on AD as the diameter would have B and C on it. The close-up is shown in Fig. 2.

Denote AE $= x$, then ED $= (k+\varepsilon)\,x$, and from the triangle ABE by Pythagoras we get a quadratic in x:

$$1+[(k+1)-(k+\varepsilon)\,x]^2=x^2,$$

i.e.,

$$\Big[(k+\varepsilon)^2-1\Big]x^2-2\,(k+1)\,(k+\varepsilon)\,x+\Big(k^2+2k+2\Big)=0.$$

Thus,

$$x=\frac{(k+1)\,(k+\varepsilon)\pm\sqrt{2k+2-2k\varepsilon-\varepsilon^2}}{\big(k^2-1\big)+2k\varepsilon+\varepsilon^2}.$$

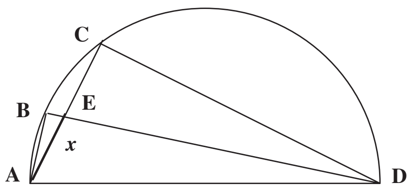

Fig. 2.

Observing that ours is the root with the minus (for with plus x becomes unseemly large for $k = 1$ and small ε), and noticing that $\sqrt{2k + 2 - 2k\varepsilon - \varepsilon^2} < \sqrt{2k+2} - \frac{k}{\sqrt{2k+2}}\varepsilon$, we get

$$x > \frac{(k+1)(k+\varepsilon) - \sqrt{2k+2} - \frac{k}{\sqrt{2k+2}}\varepsilon}{\left(k^2 - 1\right) + 2k\varepsilon + \varepsilon^2} = \frac{\left(k^2 + k - \sqrt{2k+2}\right) + \left(k + 1 + \frac{k}{\sqrt{2k+2}}\right)\varepsilon}{\left(k^2 - 1\right) + 2k\varepsilon + \varepsilon^2}$$
$$> \frac{k^2 + k - \sqrt{2k+2}}{k^2 + k\varepsilon - 1},$$

where the last inequality is true for any $k \geqslant 2$. Observe (Fig. 1) that x is precisely the "horizontal width" of a $1 \times (k+1)$ polyomino; thus, in our covering we need at most $2\left\lceil \frac{n+\varepsilon}{x} \right\rceil$ of such polyominos (we double to account for two parts of the L-shape to be covered). From this we get the desired upper bound (we can—but will not—slightly improve it by eliminating duplication in the covering of the upper right square in Fig. 1 and throwing away squares at the ends of "L" that are completely outside the square we cover):

$$\Pi(n) < (n-k)^2 + 2(k+1)\left\lceil \frac{k^2 + k\varepsilon - 1}{k^2 + k - \sqrt{2k+2}}(n+\varepsilon) \right\rceil.$$

Choosing $\varepsilon = o(k)$ and introducing the symbol $\lceil\lceil a \rceil\rceil$ to denote the lowest integer strictly greater than a, we get

$$\Pi(n) < (n-k)^2 + 2(k+1)\left\lceil\left\lceil \frac{k^2 - 1}{k^2 + k - \sqrt{2k+2}} n \right\rceil\right\rceil.$$

For $k = 4999$, for example, this gives the upper bound $\Pi(n) < n^2 + 0.04n + 4999^2 + 10000$ for $n > 4999$. In fact, it is easy to see that the upper bound we found is asymptotically equivalent to $n^2 + o(1)n + O(1)$. This upper bound is asymptotically the best possible, and so I have accomplished the stated goal. I have a conjecture for the lower bound.

Asymptotic Upper Bound Result:. $n^2 + o(1)n + O(1)$.

Asymptotic Lower Bound Conjecture:. $n^2 + O(1)$.

For small values of n the problem of determining exact values of $\Pi(n)$ is open and not easy. To colleagues interested in pursuing this direction, I can offer a few numbers they can start with: $\Pi(1) = 3$; $5 \leqslant \Pi(2) \leqslant 7$; and $10 \leqslant \Pi(3) \leqslant 14$.

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References

[CS1] J.H. Conway, A. Soifer, Covering a triangle with triangles, Amer. Math. Monthly 112 (1) (2005) 78.
[CS2] J.H. Conway, A. Soifer, Cover-up, Geombinatorics XIV (1) (2004) 8–9.
[EG] P. Erdös, R.L. Graham, On packing squares with equal squares, J. Combin. Theory (A) 19 (1975) 119–123.
[KS] D. Karabash, A. Soifer, On covering of trigons, Geombinatorics XV (1) (2005) 13–17.