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Self-orthogonal modules over coherent rings

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Abstract

Let R be a left coherent ring, S any ring and ${}_R\omega_S$ an (R,S)-bimodule. Suppose ω_S has an ultimately closed FP-injective resolution and ${}_R\omega_S$ satisfies the conditions: (1) ω_S is finitely presented; (2) The natural map $R \to \mathrm{End}(\omega_S)$ is an isomorphism; (3) $\mathrm{Ext}^i_S(\omega,\omega)=0$ for any $i \geq 1$. Then a finitely presented left R-module A satisfying $\mathrm{Ext}^i_R(A,\omega)=0$ for any $i \geq 1$ implies that A is ω-reflexive. Let R be a left coherent ring, S a right coherent ring and ${}_R\omega_S$ a faithfully balanced self-orthogonal bimodule and $n \geq 0$. Then the FP-injective dimension of ${}_R\omega_S$ is equal to or less than n as both left R-module and right S-module if and only if every finitely presented left R-module and every finitely presented right S-module have finite generalized Gorenstein dimension at most n. © 2001 Elsevier Science B.V. All rights reserved.

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1. Preliminaries

Throughout this paper, we assume that all rings are associative with identity elements, and that all modules considered are unital.

Let R be a ring and M a left (resp. right) R-module. Recall that M is called finitely presented if there is a finitely generated projective left (resp. right) R-module P and a finitely generated submodule N of P such that $P/N \cong M$. We use $\mathrm{mod}\,R$ (resp. $\mathrm{mod}\,R^{\mathrm{op}}$) to denote the category of finitely presented left (resp. right) R-modules. R is called a left (resp. right) coherent ring if every finitely generated submodule of a finitely presented left (resp. right) R-module also is finitely presented. A left (resp. right) R-module A is called FP-injective if $\mathrm{Ext}^1_R(F,A)=0$ for every finitely presented left (resp. right) R-module F. Let $\mathrm{l.FP\text{-}id}_R(A)$ (resp. $\mathrm{r.FP\text{-}id}_R(A)$) denote the smallest

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integer $n \geq 0$ such that $\text{Ext}_R^{n+1}(F, A) = 0$ for every finitely presented left (resp. right) R-module F (see [9] for a reference).

Let R and S be rings and ${}_R\omega_S$ an (R, S)-bimodule. Suppose A is a left R-module (resp. right S-module). We call $\text{Hom}_R({}_RA, {}_R\omega_S)$ (resp. $\text{Hom}_S(A_S, {}_R\omega_S)$) the dual module of A with respect to ω, and denote either of these modules by A^ω. For a homomorphism f between R-modules (resp. S^{op}-modules), we put $f^\omega = \text{Hom}(f, {}_R\omega_S)$. Let $\sigma_A : A \to A^{\omega\omega}$ via $\sigma_A(x)(f) = f(x)$ for any $x \in A$ and $f \in A^\omega$ be the canonical evaluation homomorphism. If σ_A is an isomorphism, then A is called a ω-reflexive module.

An (R, S)-bimodule ${}_R\omega_S$ is called a cotilting bimodule if it satisfies the following conditions (cf. [7]):

(C1l) ${}_R\omega$ is finitely presented;
(C1r) ω_S is finitely presented;
(C2l) The natural map $S^{\text{op}} \to \text{End}({}_R\omega)$ is an isomorphism;
(C2r) The natural map $R \to \text{End}(\omega_S)$ is an isomorphism;
(C3l) $\text{Ext}_R^i(\omega, \omega) = 0$ for any $i \geq 1$;
(C3r) $\text{Ext}_S^i(\omega, \omega) = 0$ for any $i \geq 1$;
(C4l) $\text{l.FP-id}_R(\omega) < \infty$;
(C4r) $\text{r.FP-id}_S(\omega) < \infty$.

Remark. (1) ${}_R\omega_S$ is called a faithfully balanced self-orthogonal bimodule if it satisfies conditions (C1l), (C1r), (C2l), (C2r), (C3l) and (C3r).

(2) If ${}_R\omega_S$ satisfies condition (C2r) (resp. (C2l)), then P and P^ω are ω-reflexive for every finitely generated projective left R-module (resp. right S-module) P.

We showed in [5] that if R is an artin algebra (that is, R is an algebra over a commutative artin ring T and R is finitely generated as a T-module) and ${}_R\omega_R$ is a cotilting bimodule then a module M in $\text{mod}\, R$ is ω-reflexive provided $\text{Ext}_R^i(M, \omega) = 0$ for any $i \geq 1$. In Section 2 we generalize this result and prove that if R is a left coherent ring and S any ring and an (R, S)-bimodule ${}_R\omega_S$ satisfies conditions (C1r), (C2r) and (C3r) and ω_S has an ultimately closed FP-injective resolution, then a module A in $\text{mod}\, R$ satisfying $\text{Ext}_R^i(A, \omega) = 0$ for any $i \geq 1$ implies that A is ω-reflexive. Some known results by Miyashita [8] and Iwanaga [6] are obtained as corollaries. In Section 3 we prove that if R is a left coherent ring and S is a right coherent ring and ${}_R\omega_S$ is a faithfully balanced self-orthogonal bimodule, then $\text{l.FP-id}_R(\omega) \leq n$ and $\text{r.FP-id}_S(\omega) \leq n$ if and only if every module in $\text{mod}\, R$ and every module in $\text{mod}\, S^{\text{op}}$ have finite generalized Gorenstein dimension at most n, where n is a negative integer. This result generalizes a result by Auslander and Reiten [2].

2. Dual modules

In the following, R and S are rings, ${}_R\omega_S$ is a given (R, S)-bimodule, n is a positive integer.

Suppose $A \in \operatorname{mod} R$ (resp. $\operatorname{mod} S^{\mathrm{op}}$) and there is an exact sequence $P_1 \xrightarrow{f} P_0 \to A \to 0$ with P_0, P_1 finitely generated projective. Then we have an exact sequence $0 \to A^{\omega} \to P_0^{\omega} \xrightarrow{f^{\omega}} P_1^{\omega} \to X \to 0$, where $X = \operatorname{Coker} f^{\omega}$.

Lemma 2.1. *Suppose* $\operatorname{Ext}_S^1(\omega,\omega) = 0 = \operatorname{Ext}_S^2(\omega,\omega)$.

(1) *Let A be in* $\operatorname{mod} R$ *and X in* $\operatorname{mod} S^{\mathrm{op}}$ *as above. If ${}_R\omega_S$ satisfies condition* (C2r), *then we have the following exact sequence*:

$$0 \to \operatorname{Ext}_S^1(X,\omega) \to A \xrightarrow{\sigma_A} A^{\omega\omega} \to \operatorname{Ext}_S^2(X,\omega) \to 0.$$

(2) *Let A be in* $\operatorname{mod} S^{\mathrm{op}}$ *and X in* $\operatorname{mod} R$ *as above. If ${}_R\omega_S$ satisfies condition* (C2l), *then we have the following exact sequence*:

$$0 \to \operatorname{Ext}_S^1(A,\omega) \to X \xrightarrow{\sigma_X} X^{\omega\omega} \to \operatorname{Ext}_S^2(A,\omega) \to 0.$$

Proof. (1) The proof is analogous to that of [5, Theorem 2.3]. For the sake of completeness, we give here the proof.

Suppose $A \in \operatorname{mod} R$ and suppose

$$\begin{array}{ccccccc} P_1 & \xrightarrow{f} & P_0 & \longrightarrow & A & \longrightarrow & 0 \\ & {\scriptstyle \pi_1}\searrow & & \nearrow{\scriptstyle i_1} & & & \\ & & K & & & & \end{array}$$

is a projective resolution of A in $\operatorname{mod} R$. From the exact sequence

$$\begin{array}{ccccccccc} 0 \longrightarrow A^{\omega} \longrightarrow & P_0^{\omega} & \xrightarrow{f^{\omega}} & P_1^{\omega} & \longrightarrow X \longrightarrow 0 \\ & & {\scriptstyle \pi_2}\searrow \quad \nearrow{\scriptstyle i_2} & & \\ & & C & & \end{array}$$

we have a long exact sequence $0 \to X^{\omega} \to P_1^{\omega\omega} \xrightarrow{i_2^{\omega}} C^{\omega} \to \operatorname{Ext}_S^1(X,\omega) \to 0 \to \operatorname{Ext}_S^1(C,\omega) \to \operatorname{Ext}_S^2(X,\omega) \longrightarrow 0$ and the following exact commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \xrightarrow{i_1} & P_0 & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \sigma_{P_0}} & & \downarrow{\scriptstyle \sigma_A} & & \\ 0 & \longrightarrow & C^{\omega} & \xrightarrow{\pi_2^{\omega}} & P_0^{\omega\omega} & \longrightarrow & A^{\omega\omega} & \longrightarrow & \operatorname{Ext}_S^1(C,\omega) \longrightarrow 0, \end{array} \tag{2.1}$$

where σ_{P_0} is an isomorphism and g is an induced homomorphism. By the snake lemma we have $\operatorname{Ker} \sigma_A \cong \operatorname{Coker} g$ and $\operatorname{Coker} \sigma_A \cong \operatorname{Ext}_S^1(C,\omega) \cong \operatorname{Ext}_S^2(X,\omega)$.

Consider the following diagram:

$$\begin{array}{ccccc}
P_1 & \xrightarrow{\sigma_{P_1}} & P_1^{\omega\omega} & \xrightarrow{i_2^{\omega}} & C^{\omega} \\
\downarrow{\scriptstyle \pi_1} & & & & \| \\
K & & \xrightarrow{\quad g \quad} & & C^{\omega} \\
\downarrow{\scriptstyle i_1} & & & & \downarrow{\scriptstyle \pi_2^{\omega}} \\
P_0 & & \xrightarrow{\quad \sigma_{P_0} \quad} & & P_0^{\omega\omega}
\end{array}$$

By Diagram (2.1) $\sigma_{P_0}\cdot i_1=\pi_2^{\omega}\cdot g$, so $(\sigma_{P_0}\cdot i_1)\cdot\pi_1=(\pi_2^{\omega}\cdot g)\cdot\pi_1$ and hence $\sigma_{P_0}\cdot f=\pi_2^{\omega}\cdot g\cdot\pi_1$. Since $\sigma_{P_0}\cdot f=f^{\omega\omega}\cdot\sigma_{P_1}$ and $f^{\omega\omega}=\pi_2^{\omega}\cdot i_2^{\omega}$, it follows that $\pi_2^{\omega}\cdot i_2^{\omega}\cdot\sigma_{P_1}=\pi_2^{\omega}\cdot g\cdot\pi_1$. Since π_2^{ω} is a monomorphism, $i_2^{\omega}\cdot\sigma_{P_1}=g\cdot\pi_1$. Hence $\mathrm{Im}(i_2^{\omega}\cdot\sigma_{P_1})\subseteq\mathrm{Im}\,g$ and there is an induced commutative diagram:

$$\begin{array}{ccccccccc}
 & & P_1 \xrightarrow{\sigma_{P_1}} P_1^{\omega\omega} & \xrightarrow{i_2^{\omega}} & C^{\omega} & \to & \mathrm{Ext}_S^1(X,\omega) & \to & 0 \\
 & & \downarrow{\scriptstyle \pi_1} & & \| & & \downarrow{\scriptstyle h} & & \\
0 & \longrightarrow & K & \xrightarrow{g} & C^{\omega} & \longrightarrow & \mathrm{Coker}\,g & \longrightarrow & 0 \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}$$

It follows from the snake lemma that h is an isomorphism. So $\mathrm{Ker}\,\sigma_A\cong\mathrm{Coker}\,g\cong\mathrm{Ext}_S^1(X,\omega)$ and we obtain the required exact sequence.

(2) Suppose $A\in\mathrm{mod}\,S^{\mathrm{op}}$ and suppose $P_1\xrightarrow{f}P_0\to A\to 0$ is a projective resolution of A in $\mathrm{mod}\,S^{\mathrm{op}}$. Then we have an exact sequence $0\to A^{\omega}\to P_0^{\omega}\xrightarrow{f^{\omega}}P_1^{\omega}\to X\to 0$ and the following exact commutative diagram:

$$\begin{array}{ccccccccc}
 & & P_1 & \xrightarrow{f} & P_0 & \longrightarrow & A & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \sigma_{P_1}} & & \downarrow{\scriptstyle \sigma_{P_0}} & & \downarrow & & \\
0 \to X^{\omega} & \to & P_1^{\omega\omega} & \xrightarrow{f^{\omega\omega}} & P_0^{\omega\omega} & \longrightarrow & \mathrm{Coker}\,f^{\omega\omega} & \to & 0
\end{array}$$

It is easy to see that $A\cong\mathrm{Coker}\,f^{\omega\omega}$. Noting that P_1^{ω} and P_0^{ω} are ω-reflexive, it is not difficult to see that the proof of (2) is analogous to that of (1). So we omit it. □

Theorem 2.2. *Suppose ${}_R\omega_S$ satisfies conditions* (C2r) *and* (C3r) *and* $P_n\xrightarrow{d_n}P_{n-1}\xrightarrow{d_{n-1}}\cdots\to P_1\xrightarrow{d_1}P_0\to A\to 0$ *is an exact sequence in* $\mathrm{mod}\,R$ *with all* P_i *finitely generated projective. If* $\mathrm{Ext}_R^i(A,\omega)=0$ *for any* $1\le i\le n-1$, *then we have the following exact* *sequence*:

$$0 \to \mathrm{Ext}_S^n(X,\omega) \to A \xrightarrow{\sigma_A} A^{\omega\omega} \to \mathrm{Ext}_S^{n+1}(X,\omega) \to 0$$

where $X = \mathrm{Coker}\, d_n^{\omega}$.

Proof. The case for $n=1$ follows from Lemma 2.1(1). Now suppose $n \geq 2$. Consider the given exact sequence

$$P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \to P_1 \xrightarrow{d_1} P_0 \to A \to 0,$$

where all P_i are finitely generated projective. Since $\mathrm{Ext}_R^i(A,\omega)=0$ for any $1 \leq i \leq n-1$, we have the following exact sequence:

$$0 \to A^{\omega} \to P_0^{\omega} \xrightarrow{d_1^{\omega}} P_1^{\omega} \to \cdots \to P_{n-1}^{\omega} \xrightarrow{d_n^{\omega}} P_n^{\omega} \to X \to 0, \tag{2.2}$$

where $X = \mathrm{Coker}\, d_n^{\omega}$.

By Lemma 2.1(1), there is an exact sequence

$$0 \to \mathrm{Ext}_S^1(Y,\omega) \to A \xrightarrow{\sigma_A} A^{\omega\omega} \to \mathrm{Ext}_S^2(Y,\omega) \to 0, \tag{2.3}$$

where $Y = \mathrm{Coker}\, d_1^{\omega}$. By the exactness of (2.2) and the assumption that ${}_R\omega_S$ satisfies condition (C3r), we have $\mathrm{Ext}_S^i(Y,\omega) \cong \mathrm{Ext}_S^{i+n-1}(X,\omega)$. Then we get the desired exact sequence from (2.3), which completes the proof. □

Lemma 2.3. *Suppose ${}_R\omega_S$ satisfies condition* (C2r) and $A \in \mathrm{mod}\, R$ with $\mathrm{Ext}_R^i(A,\omega)=0$ *for any* $1 \leq i \leq n$. *If* $P_{n+1} \xrightarrow{d_{n+1}} P_n \to \cdots \to P_0 \to A \to 0$ *is an exact sequence with all P_i finitely generated projective*, *then* $\mathrm{Ext}_S^i(\mathrm{Coker}\, d_{n+1}^{\omega},\omega) = 0$ *for any* $1 \leq i \leq n$.

Proof. Suppose $P_{n+1} \xrightarrow{d_{n+1}} P_n \to \cdots \to P_0 \to A \to 0$ is an exact sequence in $\mathrm{mod}\, R$ with all P_i projective. Since $\mathrm{Ext}_R^i(A,\omega)=0$ for any $1 \leq i \leq n$, $0 \to A^{\omega} \to P_0^{\omega} \to \cdots \to P_n^{\omega} \xrightarrow{d_{n+1}^{\omega}} P_{n+1}^{\omega} \to \mathrm{Coker}\, d_{n+1}^{\omega} \to 0$ is exact. Since ${}_R\omega_S$ satisfies condition (C2r), every P_i is ω-reflexive. So we have an induced exact sequence $0 \to (\mathrm{Coker}\, d_{n+1}^{\omega})^{\omega} \to P_{n+1} \xrightarrow{d_{n+1}} P_n \to \cdots \to P_0 \to A \to 0$ and hence $\mathrm{Ext}_S^i(\mathrm{Coker}\, d_{n+1}^{\omega},\omega)=0$ for any $1 \leq i \leq n$. □

Let $M \in \mathrm{mod}\, S^{\mathrm{op}}$. Suppose

$$0 \to M \xrightarrow{\delta_0} I_0 \xrightarrow{\delta_1} I_1 \xrightarrow{\delta_2} \cdots \to I_i \to \cdots \tag{2.4}$$

is an exact sequence with all I_i FP-injective S^{op}-modules. Such an exact sequence is called an FP-injective resolution of M. If there is a positive integer n, such that $\mathrm{Im}\,\delta_n$ has a decomposition $\bigoplus_{j=1}^m W_j$ with each W_j isomorphic to a direct summand of some $\mathrm{Im}\,\delta_{i_j}$ with $i_j < n$, then (2.4) is called an FP-injective resolution of M ultimately closed at n. An ultimately closed FP-injective resolution of M means an FP-injective resolution of M ultimately closed at n for some n. This notion extends the one given by Colby and Fuller [4, p. 345].

Remark. For an S^{op}-module A it is easy to see that r.FP-id$_S(A) \leq n$ if and only if there is an exact sequence $0 \to A \to I_0 \to I_1 \to \cdots \to I_n \to 0$ with all I_i FP-injective

S^{op}-modules. It is clear that such an exact sequence is an FP-injective resolution of A ultimately closed at $n+1$.

Theorem 2.4. *Let R be a left coherent ring. Suppose ${}_R\omega_S$ satisfies the conditions* (C1r), (C2r) *and* (C3r) *and ω_S has an* FP-*injective resolution ultimately closed at n. If $A \in \operatorname{mod} R$ satisfies $\operatorname{Ext}^i_R(A,\omega)=0$ for any $1 \le i \le n$, then A is ω-reflexive.*

Proof. Suppose $A \in \operatorname{mod} R$ satisfies $\operatorname{Ext}^i_R(A,\omega)=0$ for any $1 \le i \le n$. Since R is a left coherent ring, there is an exact sequence

$$P_{n+1} \xrightarrow{d_{n+1}} P_n \xrightarrow{d_n} \cdots \to P_1 \xrightarrow{d_1} P_0 \to A \to 0,$$

with all P_i finitely generated projective. Set $X = \operatorname{Coker} d^{\omega}_{n+1}$.

By Lemma 2.3, $\operatorname{Ext}^i_S(X,\omega)=0$ for any $1 \le i \le n$. Since ω_S is finitely presented and every P_i^{ω} is a direct summand of finite direct sum of copies of ω_S, every P_i^{ω} is finitely presented in $\operatorname{mod} S^{\text{op}}$. So X is finitely presented in $\operatorname{mod} S^{\text{op}}$ by [3, Proposition 1.6].

Let

$$0 \to \omega_S \xrightarrow{\delta_0} I_0 \xrightarrow{\delta_1} I_1 \xrightarrow{\delta_2} \cdots \xrightarrow{\delta_i} I_i \to \cdots$$

be an FP-injective resolution of ω_S ultimately closed at n. Then $\operatorname{Im}\delta_n = \bigoplus_{j=1}^m \operatorname{Im}\delta_{i_j}$ with $0 \le i_j \le n-1$. Since X is finitely presented in $\operatorname{mod} S^{\text{op}}$, $\operatorname{Ext}^j_S(X,I_i)=0$ for any $j \ge 1$ and $i \ge 0$. So $\operatorname{Ext}^{n+1}_S(X,\omega) \cong \operatorname{Ext}^1_S(X,\operatorname{Im}\delta_n) = \operatorname{Ext}^1_S(X,\bigoplus_{j=1}^m \operatorname{Im}\delta_{i_j}) \cong \bigoplus_{j=1}^m \operatorname{Ext}^1_S(X,\operatorname{Im}\delta_{i_j}) \cong \bigoplus_{j=1}^m \operatorname{Ext}^{i_j+1}_S(X,\omega)=0$ (since $1 \le i_j+1 \le n$). We conclude that $\operatorname{Ext}^i_S(X,\omega)=0$ for any $1 \le i \le n+1$. Similar to the above argument we show that $\operatorname{Ext}^{n+2}_S(X,\omega) \cong \bigoplus_{j=1}^m \operatorname{Ext}^{i_j+2}_S(X,\omega)=0$.

Since $\operatorname{Ext}^i_R(A,\omega)=0$ for any $1 \le i \le n$, by Theorem 2.2 we have the following exact sequence:

$$0 \to \operatorname{Ext}^{n+1}_S(X,\omega) \to A \xrightarrow{\sigma_A} A^{\omega\omega} \to \operatorname{Ext}^{n+2}_S(X,\omega) \to 0.$$

But $\operatorname{Ext}^{n+1}_S(X,\omega)=0=\operatorname{Ext}^{n+2}_S(X,\omega)$, so A is ω-reflexive. The proof is complete. □

Remark. [5, Theorem 3.8] is an immediate corollary of Theorem 2.4.

Corollary 2.5. *Under the assumptions of Theorem* 2.4, *if $A \in \operatorname{mod} R$ satisfies $\operatorname{Ext}^i_R(A,\omega)=0$ for any $0 \le i \le n$, then $A=0$.*

Proof. By Theorem 2.4. □

Theorem 2.6. *Under the assumptions of Theorem* 2.4, *suppose ${}_R\omega$ is flat. If $A \in \operatorname{mod} R$ satisfies $\operatorname{Ext}^i_R(A,R)=0$ for any $0 \le i \le n$, then $A=0$.*

Proof. The proof is analogous to that of [5, Theorem 3.10]. For the sake of completeness, we give here the proof.

Suppose $A \in \operatorname{mod} R$ satisfies $\operatorname{Ext}_R^i(A,R) = 0$ for any $0 \le i \le n$. We use $\Omega^i(A)$ to denote the ith syzygy module of A for any $i \ge 0$ and $(-)^*$ to denote $\operatorname{Hom}_R(-,R)$.

Since R is a left coherent ring and $A \in \operatorname{mod} R$, $\Omega^i(A) \in \operatorname{mod} R$. So there is an exact sequence $P_1 \to P_0 \to \Omega^i(A) \to 0$ with P_0, P_1 finitely generated projective. By [1, Theorem 2.8], for any $i \ge 0$ we have the following exact sequence

$$\operatorname{Ext}_R^i(A,R) \otimes_R \omega \to \operatorname{Ext}_R^i(A,\omega) \to \operatorname{Tor}_1^R(X,\omega),$$

where $X = \operatorname{Coker}(P_0^* \to P_1^*)$. Because ${}_R\omega$ is flat, $\operatorname{Tor}_1^R(X,\omega) = 0$. So $\operatorname{Ext}_R^i(A,\omega) = 0$ for any $0 \le i \le n$, which implies $A = 0$ by Corollary 2.5. The proof is complete. □

Recall from [4] that the strong Nakayama conjecture is true for a ring R if the condition of $\operatorname{Ext}_R^i(A,R) = 0$ for a finitely generated left R-module A and any $i \ge 0$ implies $A = 0$. We know that a left noether ring is a left coherent ring, and if R is a left noether ring then a left R-module is finitely generated if and only if it is finitely presented.

In completely similar proofs to those of Theorems 2.4 and 2.6, we get a generalization of [5, Theorem 3.10] (also cf. [4, Theorem 2]) as follows.

Corollary 2.7. *Let R be a left noether ring and S any ring. If there is an (R,S)-bimodule ${}_R\omega_S$ which satisfies conditions* (C1r), (C2r) *and* (C3r) *and ${}_R\omega$ is flat and ω_S has an ultimately closed injective resolution, then the strong Nakayama conjecture holds over R.*

Let $\mathscr{A}$ be an abelian category and $\mathscr{B}$ a full subcategory of $\mathscr{A}$. An object $X \in \mathscr{A}$ is called an embedding cogenerator for $\mathscr{B}$ if every object in $\mathscr{B}$ admits an injection to some direct product of copies of X in $\mathscr{A}$, that is, $\operatorname{Rej}_Y(X)\ (=\bigcap\{\operatorname{Ker} h \mid h : Y \to X\}) = 0$ for any $Y \in \mathscr{B}$.

We have the following result which is better than results by Miyashita [8, Corollary in Section 6] and Iwanaga [6, Theorem 2].

Proposition 2.8. *Under the assumptions of Theorem* 2.4, *suppose* $0 \to {}_R\omega \to E_0 \xrightarrow{f_1} E_1 \xrightarrow{f_2} \cdots \xrightarrow{f_i} E_i \xrightarrow{f_{i+1}} \cdots$ *is an* FP-*injective resolution of ${}_R\omega$. Then $\bigoplus_{i=0}^n E_i$ is an embedding* FP-*injective cogenerator for* $\operatorname{mod} R$.

Proof. By [9, Corollary 2.4], $\bigoplus_{i=0}^n E_i$ is an FP-injective R-module.

By Corollary 2.5, for any $0 \ne A \in \operatorname{mod} R$, $\operatorname{Ext}_R^t(A,\omega) \ne 0$ for some t with $0 \le t \le n$ (otherwise $A = 0$). From the given exact sequence

$$0 \to {}_R\omega \to E_0 \xrightarrow{f_1} E_1 \xrightarrow{f_2} \cdots \xrightarrow{f_i} E_i \xrightarrow{f_{i+1}} \cdots$$

we get the exact sequences

$$\operatorname{Hom}_R(A, \operatorname{Im} f_i) \to \operatorname{Ext}_R^i(A,\omega) \to 0,$$
$$0 \to \operatorname{Hom}_R(A, \operatorname{Im} f_i) \to \operatorname{Hom}_R(A, E_i)$$

for any $i \geq 1$. Because $\mathrm{Ext}_R^t(A, \omega) \neq 0$, $\mathrm{Hom}_R(A, \mathrm{Im}\, f_t) \neq 0$ and $\mathrm{Hom}_R(A, E_t) \neq 0$. Therefore, we conclude that $\mathrm{Hom}_R(A, \bigoplus_{i=0}^n E_i) \neq 0$ for any $0 \neq A \in \mathrm{mod}\, R$.

Let $0 \neq x \in A$. Since R is a left coherent ring and $A \in \mathrm{mod}\, R$, the finitely generated submodule Rx of A is also in $\mathrm{mod}\, R$. By the above argument we have $\mathrm{Hom}_R(Rx, \bigoplus_{i=0}^n E_i) \neq 0$. Let $0 \neq h \in \mathrm{Hom}_R(Rx, \bigoplus_{i=0}^n E_i)$. Since $\bigoplus_{i=0}^n E_i$ is FP-injective, so h can be extended to a homomorphism $\bar{h} : A \to \bigoplus_{i=0}^n E_i$ with $\bar{h}(x) = h(x) \neq 0$. Thus $\mathrm{Rej}_A(\bigoplus_{i=0}^n E_i) = 0$. We conclude that $\bigoplus_{i=0}^n E_i$ is an embedding cogenerator for $\mathrm{mod}\, R$. □

3. Generalized Gorenstein dimension

The following definitions are cited from [2]. But, R and S here are not necessarily artin algebras. In the following, we assume that R is a left coherent ring and S is a right coherent ring, and that ${}_R\omega_S$ is a faithfully balanced self-orthogonal (R, S)-bimodule.

Definition 3.1. A module M in $\mathrm{mod}\, R$ is said to have generalized Gorenstein dimension zero (with respect to ω), denoted by $\text{G-dim}_\omega(M) = 0$, if the following conditions hold:
(1) M is ω-reflexive.
(2) $\mathrm{Ext}_R^i(M, \omega) = 0 = \mathrm{Ext}_S^i(M^\omega, \omega)$ for any $i \geq 1$.

Definition 3.2. For any $n \geq 0$, M in $\mathrm{mod}\, R$ is said to have generalized Gorenstein dimension at most n (with respect to ω), denoted by $\text{G-dim}_\omega(M) \leq n$, if there is an exact sequence $0 \to M_n \to \cdots \to M_1 \to M_0 \to M \to 0$ in $\mathrm{mod}\, R$ with $\text{G-dim}_\omega(M_i) = 0$ for any $0 \leq i \leq n$.

Remark. For any $N \in \mathrm{mod}\, S^{\mathrm{op}}$, we may give a similar definition of $\text{G-dim}_\omega(N)$ as above.

In [2] Auslander and Reiten showed that if R is an artin algebra, then ω has finite injective dimension as a left R-module if and only if every module in $\mathrm{mod}\, R$ has finite generalized Gerenstein dimension. In this section we develop their arguments and generalize this result. Under our assumptions, for any $n \geq 0$, we prove that $\text{l.FP-id}_R(\omega) \leq n$ and $\text{r.FP-id}_S(\omega) \leq n$ if and only if every module in $\mathrm{mod}\, R$ and every module in $\mathrm{mod}\, S^{\mathrm{op}}$ have finite generalized Gorenstein dimension at most n.

Lemma 3.3. *For a positive integer n, the following statements are equivalent:*

(1) *Every M in $\mathrm{mod}\, R$ with $\mathrm{Ext}_R^i(M, \omega) = 0$ for any $1 \leq i \leq n$ is ω-reflexive.*
(2) *Every N in $\mathrm{mod}\, S^{\mathrm{op}}$ with $\mathrm{Ext}_S^i(N, \omega) = 0$ for any $1 \leq i \leq n$ satisfies $\mathrm{Ext}_S^i(N, \omega) = 0$ for any $i \geq 1$.*

Proof. (1)⇒(2) Let N be in $\mathrm{mod}\, S^{\mathrm{op}}$ with $\mathrm{Ext}_S^i(N, \omega) = 0$ for any $1 \leq i \leq n$ and let $P_{n+1} \xrightarrow{d_{n+1}} P_n \xrightarrow{d_n} \cdots \xrightarrow{d_1} P_0 \to N \to 0$ be a projective resolution of N in $\mathrm{mod}\, S^{\mathrm{op}}$. By symmetric conclusion of Lemma 2.3, $\mathrm{Ext}_R^i(\mathrm{Coker}\, d_{n+1}^\omega, \omega) = 0$ for any $1 \leq i \leq n$ and hence $\operatorname{Coker} d_{n+1}^{\omega}$ is ω-reflexive by (1). Then by Lemma 2.1(2), $\operatorname{Ext}_S^1(\operatorname{Coker} d_{n+1}, \omega) = 0$. But $\operatorname{Ext}_S^{n+1}(N, \omega) \cong \operatorname{Ext}_S^1(\operatorname{Coker} d_{n+1}, \omega)$, so $\operatorname{Ext}_S^{n+1}(N, \omega) = 0$.

Since $0 \to \operatorname{Im} d_1 \to P_0 \to N \to 0$ is exact, $\operatorname{Ext}_S^i(\operatorname{Im} d_1, \omega) = 0$ for any $1 \le i \le n$. Repeating the above argument we have $\operatorname{Ext}_S^{n+1}(\operatorname{Im} d_1, \omega) = 0$ and hence $\operatorname{Ext}_S^{n+2}(N, \omega) = 0$. Continuing this procedure, our assertion follows.

(2)⇒(1) Let M be in $\operatorname{mod} R$ with $\operatorname{Ext}_R^i(M, \omega) = 0$ for any $1 \le i \le n$ and let $Q_{n+1} \xrightarrow{f_{n+1}} Q_n \xrightarrow{f_n} \cdots \xrightarrow{f_1} Q_0 \to M \to 0$ be a projective resolution of M in $\operatorname{mod} R$. Then we get an exact sequence $0 \to \operatorname{Coker} f_1^{\omega} \to Q_2^{\omega} \xrightarrow{f_2^{\omega}} \cdots \xrightarrow{f_{n+1}^{\omega}} Q_{n+1}^{\omega} \to \operatorname{Coker} f_{n+1}^{\omega} \to 0$ in $\operatorname{mod} S^{\mathrm{op}}$. By Lemma 2.3, $\operatorname{Ext}_S^i(\operatorname{Coker} f_{n+1}^{\omega}, \omega) = 0$ for any $1 \le i \le n$ and thus $\operatorname{Ext}_S^i(\operatorname{Coker} f_{n+1}^{\omega}, \omega) = 0$ for any $i \ge 1$ by (2). By the last exact sequence, $\operatorname{Ext}_S^i(\operatorname{Coker} f_1^{\omega}, \omega) = 0$ for any $i \ge 1$. It follows from Lemma 2.1(1) that M is ω-reflexive. □

Lemma 3.4. *Suppose* $\text{r.FP-id}_S(\omega) < \infty$. *If a module* M *in* $\operatorname{mod} R$ *satisfies* $\operatorname{Ext}_R^i(M, \omega) = 0$ *for any* $i \ge 1$, *then* $\text{G-dim}_{\omega}(M) = 0$.

Proof. Suppose $\text{r.FP-id}_S(\omega) = n < \infty$ and suppose $N \in \operatorname{mod} S^{\mathrm{op}}$ with $\operatorname{Ext}_S^i(N, \omega) = 0$ for any $1 \le i \le n$. Then $\operatorname{Ext}_S^i(N, \omega) = 0$ for any $i \ge 1$. By Lemma 3.3, A is ω-reflexive for any A in $\operatorname{mod} R$ satisfying $\operatorname{Ext}_R^i(A, \omega) = 0$ for any $i \ge 1$.

Suppose $\cdots \to Q_n \xrightarrow{f_n} Q_{n-1} \to \cdots \to Q_0 \to M \to 0$ is a projective resolution of M in $\operatorname{mod} R$. Since $\operatorname{Ext}_R^i(M, \omega) = 0$ for any $i \ge 1$ by assumption, M is ω-reflexive and there is an induced exact sequence $0 \to M^{\omega} \to Q_0^{\omega} \to \cdots \to Q_{n-1}^{\omega} \xrightarrow{f_n^{\omega}} Q_n^{\omega} \to \cdots$ in $\operatorname{mod} S^{\mathrm{op}}$ with all Q_i^{ω} in $\operatorname{add} \omega_S$ (the full subcategory of $\operatorname{mod} S^{\mathrm{op}}$ consisting of the modules isomorphic to the direct summands of finite direct sums of copies of ω_S). Because $\text{r.FP-id}_S(\omega) = n$, $\operatorname{Ext}_S^i(M^{\omega}, \omega) \cong \operatorname{Ext}_S^{i+n}(\operatorname{Im} f_n^{\omega}, \omega) = 0$ for any $i \ge 1$. So $\text{G-dim}_{\omega}(M) = 0$. □

Now we prove the main result of this section, which is a generalization of [2, Theorem 4.4].

Theorem 3.5. *For any* $n \ge 0$, $\text{l.FP-id}_R(\omega) \le n$ *and* $\text{r.FP-id}_S(\omega) \le n$ *if and only if every module in* $\operatorname{mod} R$ *and every module in* $\operatorname{mod} S^{\mathrm{op}}$ *have finite generalized Gorenstein dimension at most* n.

Proof. "*Only if*" part. Suppose $\text{l.FP-id}_R(\omega) \le n$ and $\text{r.FP-id}_S(\omega) \le n$. Let $M \in \operatorname{mod} R$ and $0 \to K \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$ be an exact sequence in $\operatorname{mod} R$ with all P_i projective. Since $\operatorname{Ext}_R^i(K, \omega) = \operatorname{Ext}_R^{n+i}(M, \omega) = 0$ for any $i \ge 1$, $\text{G-dim}_{\omega}(K) = 0$ by Lemma 3.4. So $\text{G-dim}_{\omega}(M) \le n$. Similarly we show that $\text{G-dim}_{\omega}(N) \le n$ for any $N \in \operatorname{mod} S^{\mathrm{op}}$.

"*If*" part. Let $M \in \operatorname{mod} R$. Then $\text{G-dim}_{\omega}(M) \le n$ by assumption. So there is an exact sequence $0 \to M_n \to \cdots \to M_1 \to M_0 \to M \to 0$ in $\operatorname{mod} R$ with $\text{G-dim}_{\omega}(M_i) = 0$ for any $0 \le i \le n$, and hence $\operatorname{Ext}_R^j(M_i, \omega) = 0$ for any $0 \le i \le n$ and $j \ge 1$. Therefore, $\operatorname{Ext}_R^{n+1}(M, \omega) \cong \operatorname{Ext}_R^1(M_n, \omega) = 0$ and $\text{l.FP-id}_R(\omega) \le n$. Similarly, we show that $\text{r.FP-id}_S(\omega) \le n$. The proof is complete. □

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