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Krull relations in Hopf Galois extensions: Lifting and twisting

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1. Introduction

This paper continues our study, begun in [MS], of the relationship between the prime ideals of an algebra A and of a subalgebra R such that $R \subset A$ is a faithfully flat H-Galois extension, for some finite-dimensional Hopf algebra H. In that paper we defined three basic Krull relations, Incomparability (INC), t-Lying Over (t-LO), and Going Up (GU), analogous to the classical Krull relations for prime ideals; we also defined three new "dual" Krull relations. We say that H itself is said to have one of the Krull relations if the relation holds for all faithfully flat H-Galois extensions. We showed in [MS] that H has one of the three "dual" Krull relations if and only if the dual Hopf algebra H^* of H has the original relation (hence the name).

An important example of Hopf Galois extensions is given by Hopf crossed products $A = R_\sigma \# H$. Moreover, Galois extensions can be useful in studying crossed products, since they satisfy a "transitivity" property which crossed products lack. That is, if K is a normal Hopf subalgebra of H with Hopf quotient $\overline{H}$, then, in general, one cannot write $A = R_\sigma \# H = (R_\sigma K) \#_\tau \overline{H}$, by an example of [S2]. Another basic example of a Hopf Galois extension is given by a Hopf algebra A with a normal Hopf subalgebra R of finite index such that A is faithfully flat over R: for then $R \subset A$ is faithfully flat H-Galois.

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A fundamental question in this area is to determine which Hopf algebras satisfy the various Krull relations. As a consequence of our work in [MS], we established results for two main classes of Hopf algebras:

(1) All six Krull relations hold if H is semisolvable and semisimple.
(2) If H is pointed, then H satisfies Going Up and the three dual Krull relations.

(1) depended on our main result on Krull relations, the Transitivity Theorem, which enabled us to go from a normal Hopf subalgebra K of H and the corresponding Hopf quotient $\overline{H}$ up to H itself; it also depended on known facts about the Krull relations in smash products with group algebras kG [LP] or their duals $(kG)^*$ [CM]. (2) followed by reducing to the coradical H_0 of H, and using known facts about smash products of pointed Hopf algebras by [Ch,CRW,Q]. We note that Incomparability and Lying Over remain open for pointed Hopf algebras, even for restricted enveloping algebras in characteristic $p > 0$.

The object of this paper is to extend the work of [MS], by first proving that the Krull relations are preserved under various changes of the Hopf algebra H, and then by applying these results together with some recent constructions of Hopf algebras in order to give new examples of Hopf algebras for which some or all of the Krull relations hold.

We first prove that any of the six Krull relations can be lifted to H from a Hopf subalgebra K of H containing the coradical H_0 of H. Dually this implies that if I is a Hopf ideal of H contained in the Jacobson radical of H, then any Krull relation will lift from the quotient H/I to H. We then study twisting the Hopf algebra H, either by a Hopf 2-cocycle $\sigma : H \otimes H \to k$ or by a dual 2-cocycle $\Omega \in H \otimes H$. We show that the three dual Krull relations are preserved by twisting H to H_σ, and dually that the three basic Krull relations are preserved by twisting H to H^Ω.

As a consequence we are able to show that all six relations hold for pointed Hopf algebras which are coradically graded; such Hopf algebras include the Taft algebras, as well as the Borel subalgebras of Lusztig's Frobenius kernels $u_q(g)$, g a semisimple Lie algebra, and the finite-dimensional pointed Hopf algebras $u(\mathcal{D})$ defined in [AS] when the linking elements have trivial relations. We show that the Drinfel'd double $D(H)$ will always have the dual Krull relations provided H has all six Krull relations; in particular this is true when $H = kG$, a group algebra, or more generally if $H = k^G \#_\sigma^\tau kF$, a bicrossed product constructed from a factorizable group $L = FG$. For bismash products $H = k^G \# kF$, $D(H)$ satisfies all six of the Krull relations. Finally any triangular Hopf algebra will have the three basic Krull relations.

More specifically, in Section 2 we review the definitions of the basic Krull relations t-lying over, for some natural number t (t-LO), going up (GU), and incomparability (INC), and of the dual notions t-coLO, coGU and coINC. We define two new relations strong GU and strong coGU, which are stronger versions of going up and co going up, which we will require in the paper. We also give a precise statement of some of the major results from [MS] which we shall need.

Section 3 concerns the question mentioned above of lifting the Krull relations from a Hopf subalgebra $K \subset H$ such that K contains the coradical H_0 of H, to H itself (Theorem 3.4). More generally we consider two different Galois extensions A and B, of the same base ring R, for two different Hopf algebras H and K, and compare the Krull relations in

$R \subset A$ and $R \subset B$. The application to a quotient Hopf algebra H/I where I is a nilpotent Hopf ideal, concerning which of the Krull relations hold for H if they hold for H/I, is obtained by dualizing the coradical result (Theorem 3.5).

Section 4 is then a discussion of the examples on coradically graded Hopf algebras, using the results of Section 3.

Our second main topic, in Section 5, concerns when the Krull relations are preserved by twisting. We first review twistings of Hopf algebras and of H-comodule algebras by a Hopf 2-cocycle σ of the Hopf algebra; in this case the multiplication of H_σ is twisted but the comultiplication remains the same. Any H-comodule algebra A can also be twisted by σ. As a preliminary step, we prove that if $R \subset A$ is an H-Galois extension, then $R_\sigma \subset A_\sigma$ is an H_σ-Galois extension; moreover, H_σ-Spec $R = H$-Spec R (Theorem 5.3). We then prove that any one of coINC, t-coLO, or strong coGU is preserved under these twistings (Theorem 5.6). We also consider the dual situation, of twisting the comultiplication of H via $\Omega \in H \otimes H$; in this case H^Ω has the same multiplication as H but its comultiplication is twisted. Dualizing the previous result we see that any one of INC, t-LO, or strong GU is preserved under these twistings (Theorem 5.7). Using Theorems 5.6 and 5.7 it follows that if H is a graded Hopf algebra with identity component $K = H(0)$, then any one of the Krull relations lifts from K to H (Theorem 3.6).

In Section 6 we apply the work in Sections 3 and 5 to the examples concerning Drinfel'd doubles and twists. Here we may use some known facts about obtaining Hopf algebras through twisting, such as the results about twisting the Drinfel'd double in [DT,RS], about twisting bismash products in [BGMj], and the classification theorems of [EG1,EG2] exhibiting triangular Hopf algebras as twists.

In fact, it is possible that any finite-dimensional Hopf algebra H satisfies all of the Krull relations; no counterexamples are known. More generally it is not known whether INC is true for any finite extension $R \subset A$, although it is true if R is Noetherian [Le]. Moreover, LO can fail even for finite extensions of Noetherian rings [HO].

2. The Krull relations revisited

In this section we first review the Krull relations from [MS], and introduce new versions of several of them which we shall need in this paper. We then state more precisely some of the other results from [MS] we shall need, such as the Transitivity Theorem.

Throughout H is a finite-dimensional Hopf algebra over a field k, and $R \subset A$ denotes a faithfully flat H-Galois extension. As in [MS, 1.1, 2.3], we say that an ideal I of R is *H-stable* if $IA = AI$, and let $(I : H)$ denote the largest H-stable ideal of R in I. I is an *H-prime ideal* of R if $I \neq R$, and whenever $JK \subset I$, for J, K H-stable ideals of R, either $J \subset I$ or $K \subset I$.

To avoid confusion, we will usually write P for a prime in $\mathrm{Spec}(A)$, Q for a prime in $\mathrm{Spec}(R)$, and I for an H-prime in H-$\mathrm{Spec}(R)$. We recall [MS, Lemma 2.2]:

Lemma 2.1. (1) *The map $f : \mathrm{Spec}(R) \to H\text{-}\mathrm{Spec}(R)$ given by $Q \mapsto (Q : H)$ is well defined and surjective.*

(2) *The map* $g : \mathrm{Spec}(A) \to H\text{-}\mathrm{Spec}(R)$ *given by* $P \mapsto P \cap R$ *is well defined and surjective.*

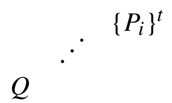
As in [MS], we say that $P \in \mathrm{Spec}(A)$ *lies over* $Q \in \mathrm{Spec}(R)$ if and only if $(Q : H) = P \cap R$. We will also say that $P \in \mathrm{Spec}(A)$ *lies over* $I \in H\text{-}\mathrm{Spec}(R)$ if and only if $I = P \cap R$. By Lemma 2.1, any $P \in \mathrm{Spec}(A)$ lies over some $Q \in \mathrm{Spec}(R)$; conversely for any $Q \in \mathrm{Spec}(R)$, there exists some $P \in \mathrm{Spec}(A)$ such that P lies over Q. Similarly any $P \in \mathrm{Spec}(A)$ lies over some $I \in H\text{-}\mathrm{Spec}(R)$; conversely for any $I \in H\text{-}\mathrm{Spec}(R)$, there exists some $P \in \mathrm{Spec}(A)$ such that P lies over I.

We note that the definition of P lying over Q reduces to the standard definition of lying over in non-commutative rings, that is that Q is minimal over $P \cap R$, under some additional assumptions; see [MS, 4.7].

We may use diagrams, as in [P], to represent many of the Krull relations. Thus, for example, the diagram in 2.2(3) means that given $Q_2 \subset Q_1$ in $\mathrm{Spec}(R)$ and $P_2 \in \mathrm{Spec}(A)$ which lies over Q_2, there exists some $P_1 \in \mathrm{Spec}(A)$ such that $P_2 \subset P_1$ and P_1 lies over Q_1. In the following definition, (1)–(3) and (1)′–(3)′ appear in [MS]. It is shown in [MS, 4.3] that (1)′–(3)′ are the duals of (1)–(3), in the sense that a condition (i) is true for H if and only if (i)′ is true for H^*. (4) and (4)′ are new; (4)′ will be useful since it is defined only in terms of R and not A.

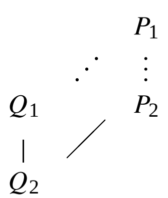
Definition 2.2 (*The Krull relations*).

(1) The H-Galois extension $R \subset A$ has *t-lying over* (t-LO) if for any $Q \in \mathrm{Spec}(R)$, there exist $P_1, \ldots, P_n \in \mathrm{Spec}(A)$, where $n \leqslant \dim H$, such that all P_i lie over Q, and such that $(\bigcap_{i=1}^{n} P_i)^t \subset (Q : H)A$:

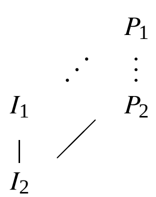


(2) $R \subset A$ has *incomparability* (INC) if for any $P_2 \subset P_1$ in $\mathrm{Spec}(A)$ with $P_2 \neq P_1$, then $P_2 \cap R \neq P_1 \cap R$.

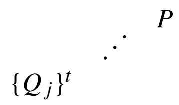
(3) $R \subset A$ has *going up* (GU) if



(4) $R \subset A$ has *strong going up* (S-GU) if

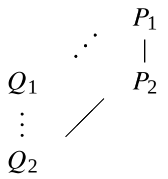


(1)′ $R \subset A$ has *t-co-lying over* (t-coLO) if for any $P \in \mathrm{Spec}(A)$, there exist $Q_1, \ldots, Q_m \in \mathrm{Spec}(R)$, where $m \leqslant \dim H$, such that P lies over all Q_j, and such that $(\bigcap_{j=1}^{m} Q_j)^t \subset P \cap R$.

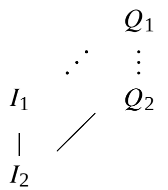


(2)′ $R \subset A$ has *co-incomparability* (coINC) if for any $Q_2 \subset Q_1$ in $\mathrm{Spec}(R)$ with $Q_2 \neq Q_1$, then $(Q_2 : H) \neq (Q_1 : H)$.

(3)′ $R \subset A$ has *co-going up* (coGU) if



(4)′ $R \subset A$ has *strong co-going up* (S-coGU) if



Note that (1)′, (2)′ and (4)′ only depend on R; although $P \in \mathrm{Spec}(A)$ appears in (1)′ it can be replaced by $P \cap R$, hence by $I \in H\text{-}\mathrm{Spec}(R)$ using Lemma 2.1.

Definition 2.3. We say the Hopf algebra H has one of the Krull relations above if for *all* faithfully flat H-Galois extensions $R \subset A$, the given Krull relation holds.

To illustrate the Krull relations, consider a smash product extension $R \subset A = R \# H$ where R is prime, or more generally H-prime. If H has t-LO and INC, then P is a minimal prime of A precisely when $P \cap R = 0$, A has $n \leqslant \dim H$ minimal primes, say $P_1, \ldots, P_n$, and if $N := \bigcap_i P_i$, then $N^t = 0$ and N is the largest nilpotent ideal of A [MS, 4.7].

We now relate the two Krull relations strong GU and strong coGU to the previous ones. We require the following result.

Theorem 2.4 [MS, Theorem 4.3]. *For each of the Krull relations* (1)–(3) *and* (1)′–(3)′, *H has a basic relation* (i) *if and only if H^* has the dual relation* (i)′.

Lemma 2.5. *For any finite-dimensional Hopf algebra H, H has strong* GU $\Leftrightarrow$ *H^* has strong* coGU. *Moreover*:

(1) *If* H *has strong* GU, *then* H *has* GU (*that is,* 2.2(4) *implies* 2.2(3)).
(1)′ *If* H *has strong* coGU, *then* H *has* coGU (*that is,* 2.2(4)′ *implies* 2.2(3)′).
(2) *If* H *has* GU, *and either strong* coGU *or* t-coLO, *then* H *has strong* GU (*that is,* 2.2(3) *together with either* 2.2(4)′ *or* 2.2(1)′ *imply* 2.2(4)).
(2)′ *If* H *has* coGU, *and either strong* GU *or* t-LO, *then* H *has strong* coGU (*that is,* 2.2(3)′ *together with either* 2.2(4) *or* 2.2(1) *imply* 2.2(4)′).

Proof. The fact that H has strong GU $\Leftrightarrow H^*$ has strong coGU follows similarly to the proof that H has GU $\Leftrightarrow H^*$ has coGU in Theorem 2.4. Thus (1) and (2) are the dual statements to (1)′ and (2)′, respectively, and so it suffices to show only (1) and (2).

(1) Assume that $Q_2 \subset Q_1$ in $\operatorname{Spec}(R)$ and that $P_2 \in \operatorname{Spec}(A)$ lies over Q_2. Let $I_i := (Q_i : H)$, for $i = 1, 2$; then $P_2 \cap R = I_2$. By strong GU, there exists $P_1 \in \operatorname{Spec}(A)$ such that $P_2 \subset P_1$ and $P_1 \cap R = I_1$. But $I_1 := (Q_1 : H)$. Thus H has GU.

(2) First assume H has GU and strong coGU. Assume that $I_2 \subset I_1$ in H-$\operatorname{Spec}(R)$ and that $P_2 \in \operatorname{Spec}(A)$ lies over I_2. By Lemma 2.1 there exists $Q_2 \in \operatorname{Spec}(R)$ with $(Q_2 : H) = I_2 = P_2 \cap R$. By strong coGU, there exists $Q_1 \in H$-$\operatorname{Spec}(R)$ such that $Q_2 \subset Q_1$ and $(Q_2 : H) = I_2$. Now use GU to find $P_1 \in \operatorname{Spec}(A)$ such that $P_2 \subset P_1$ and P_1 lies over Q_1. Then P_1 lies over I_1, and H has strong GU.

Now assume H has GU and t-coLO. Assume again that $I_2 \subset I_1$ in H-$\operatorname{Spec}(R)$ and that $P_2 \in \operatorname{Spec}(A)$ lies over I_2. By Lemma 2.1 there exists $Q \in \operatorname{Spec}(R)$ with $(Q : H) = I_1$. By t-coLO, there exist $Q_i \in H$-$\operatorname{Spec}(R)$, $i = 1, \dots, m$, such that $(Q_i : H) = I_2$ for all i and that $(\bigcap Q_i)^t \subset I_2$. Since $I_2 \subset I_1 = (Q : H) \subset Q$ and Q is prime, some Q_i, call it Q_2, is contained in Q. Now use GU to find $P_1 \in \operatorname{Spec}(A)$ such that $P_2 \subset P_1$ and P_1 lies over Q. Then P_1 lies over I_1, and H has strong GU. □

Corollary 2.6. *If* H *is pointed, then* H *has strong* GU *and strong* coGU.

Proof. As noted in the introduction, any pointed Hopf algebra has GU, t-coLO, coGU, and coINC. Thus by Lemma 2.5(2), H has strong GU. We may now use Lemma 2.5(2)′ to see that H also has strong coGU. □

For later use we note

Remark 2.7. Let $\delta : A \to A \otimes H, a \mapsto a_{(0)} \otimes a_{(1)}$, be an H-comodule algebra with coinvariant elements $R = A^{\operatorname{co} H}$, and assume that $R \subset A$ is an H-Galois extension. Then the opposite algebras $R^{\mathrm{op}} \subset A^{\mathrm{op}}$ form an H^{op}-Galois extension with comodule structure $\delta^{\mathrm{op}} : A^{\mathrm{op}} \to A^{\mathrm{op}} \otimes H^{\mathrm{op}}, a^{\mathrm{op}} \mapsto a_{(0)}^{\mathrm{op}} \otimes a_{(1)}^{\mathrm{op}}$. Thus if H satisfies any one of the Krull relations then so does H^{op}.

Proof. The Galois map $A^{\mathrm{op}} \otimes_{R^{\mathrm{op}}} A^{\mathrm{op}} \to A^{\mathrm{op}} \otimes H^{\mathrm{op}}, x^{\mathrm{op}} \otimes y^{\mathrm{op}} \mapsto (y_{(0)}x)^{\mathrm{op}} \otimes y_{(1)}^{\mathrm{op}}$, is surjective hence bijective since $A \otimes A \to A \otimes H, y \otimes x \mapsto y_{(0)}x \otimes y_{(1)}$, is surjective. □

We now state precisely the two main results of [MS] which we require here. The first is the Transitivity Theorem.

Theorem 2.8 [MS, Theorem 6.7]. *Let H be a finite-dimensional Hopf algebra, K a normal Hopf subalgebra of H and $\overline{H} := H/HK^+$.*

(1) *Assume K has s-*LO (*respectively s-*coLO) *and $\overline{H}$ has t-*LO (*respectively t-*coLO). *Then H has st-*LO (*respectively st-*coLO).
(2) *Assume K has s-*coLO *and $\overline{H}$ has t-*coLO (*respectively s-*LO *and t-*LO) *for some s and t. If K and $\overline{H}$ have* GU (*respectively* coGU), *then so does H.*
(3) *Assume $\overline{H}$ has t-*coLO (*respectively K has t-*LO) *for some t. If K and $\overline{H}$ have INC* (*respectively* coINC), *then so does H.*

As a consequence of this theorem, if both K and $\overline{H}$ have all of the Krull relations, then so does H. Since the Krull relations are known for smash products with kG [LP] or $(kG)^*$ [CM], and it suffices to prove the Krull relations for smash products, the result mentioned in the introduction for semisimple semisolvable Hopf algebras follows.

Another case to which the theorem applies is that of a tensor product of two Hopf algebras $H = K \otimes L$: for then, $K \cong K \otimes 1$ is a normal Hopf subalgebra of H with quotient $\overline{H} = H/HK^+ \cong L$. Thus for example if K satisfies s-LO and L satisfies t-LO, then H satisfies st-LO.

For our results on lifting we also need to extend the definition of H-stable to subcoalgebras of H.

Definition 2.9. Let $C \subset H$ be a subcoalgebra. Define $A(C) := \rho^{-1}(A \otimes C)$; note $A(C)$ is an R-subbimodule of A and a C-subcomodule. An ideal I in R is called *C-stable* if $IA(C) = A(C)I$.

Let $(I : C)$ denote the largest C-stable ideal in$R$ which is contained in I. A C-stable ideal I in R, $I \neq R$, is called *C-prime*, if whenever $KL \subset I$ for K, L C-stable ideals of R, then $K \subset I$ or $L \subset I$. C-Spec(R) is the set of all C-prime ideals in R.

Lemma 2.10 [MS, Lemma 3.3]. *Let $C \subset H$ be a subcoalgebra and I an ideal in R. Then $((I : C) : H) = (I : H)$. Moreover, if I is H-stable, it is C-stable.*

Theorem 2.11 [MS, Theorem 3.7]. *Let $H_0 \subset H_1 \subset \cdots \subset H_m = H$ be the coradical filtration of H and define $t := m + 1$. Then for any ideal I of R,*

$$(I : H_0)^t \subset (I : H).$$

Theorem 2.11 extends a result of [Ch] for a pointed Hopf algebra H and an H-module algebra A.

3. Galois extensions of the same base ring and lifting from the coradical

In this section we will show that if K is a Hopf subalgebra of H containing the coradical H_0, then any Krull relation may be lifted from K to H; dually, any Krull relation may be lifted to H from a quotient modulo a nilpotent Hopf ideal. These results will follow from a more general result, which may be of independent interest. That is, we consider two Hopf Galois extensions of the same base ring R, for two different Hopf algebras H and K.

Definition 3.1. Let H and K be Hopf algebras with $\dim K \leqslant \dim H$, and let R be a k-algebra. Assume that A and B are two ring extensions of R such that $R \subset A$ is faithfully flat H-Galois and that $R \subset B$ is faithfully flat K-Galois. We say that the triple (R, A, B) is (H, K)-*Krull admissible* if the following two conditions hold:

(1) for all ideals I of R, $((I : K) : H) = (I : H)$;
(2) there exists t such that for all ideals I of R, $(I : K)^t \subset (I : H)$.

Lemma 3.2. *Assume that* (R, A, B) *is* (H, K)*-Krull admissible. Then the following diagram is commutative*:

$$\begin{array}{ccc} & \operatorname{Spec}(R) & \\ \swarrow & & \searrow \\ K\text{-}\operatorname{Spec}(R) & \xrightarrow{\cong} & H\text{-}\operatorname{Spec}(R) \end{array}$$

where

$$\operatorname{Spec}(R) \to K\text{-}\operatorname{Spec}(R) \textit{ is given by } Q \mapsto (Q : K),$$
$$\operatorname{Spec}(R) \to H\text{-}\operatorname{Spec}(R) \textit{ is given by } Q \mapsto (Q : H),$$
$$\Phi : K\text{-}\operatorname{Spec}(R) \to H\text{-}\operatorname{Spec}(R) \textit{ is given by } J \mapsto (J : H).$$

Moreover, the isomorphism Φ *respects inclusions in both directions.*

Proof. First, Φ is defined on all of K-Spec(R) since $Q \mapsto (Q : K)$ is surjective by Lemma 2.1. It is well-defined and the diagram commutes by Definition 3.1(1). To see that Φ is a bijection, first note that it is surjective since $P \mapsto (P : H)$ is surjective by Lemma 2.1. To see that Φ is injective and respects both inclusions, first note that by definition, $J_1 \subset J_2 \in K$-Spec(R) implies $\Phi(J_1) \subset \Phi(J_2)$. In the reverse direction, assume that $(J_1 : H) \subset (J_2 : H)$. Then by 3.1(2),

$$J_1^t = (J_1 : K)^t \subset (J_1 : H) \subset (J_2 : H) \subset J_2.$$

Since J_2 is K-prime, it follows that $J_1 \subset J_2$. This argument also shows that Φ is injective. □

Proposition 3.3. *Assume that* (R, A, B) *is* (H, K)*-Krull admissible.*

(1) *If* $R \subset B$ *has* coINC, *then* $R \subset A$ *has* coINC.
(2) *If* $R \subset B$ *has* s-coLO, *then* $R \subset A$ *has* ls-coLO.

(3) *If* $R \subset B$ *has strong* coGU, *then* $R \subset A$ *has strong* coGU.

Proof. (1) Let $Q_1 \subset Q_2$ in $\mathrm{Spec}(R)$ with $(Q_1 : H) = (Q_2 : H)$. Then by Lemma 3.2, $(Q_1 : K) = (Q_2 : K)$. Thus $Q_1 = Q_2$ since $R \subset B$ has coINC.

(2) Let $P \in \mathrm{Spec}(A)$. We want $Q_1, \ldots, Q_m \in \mathrm{Spec}(R)$, for some $m \leqslant \dim H$, such that $(Q_j : H) = P \cap R$ for all j and $(\bigcap_{j=1}^m Q_J)^{ls} \subset P \cap R$. Now by Lemma 2.1, there exist $Q \in \mathrm{Spec}(R)$ and $\widetilde{P} \in \mathrm{Spec}(B)$ such that $(Q : H) = P \cap R$ and $(Q : K) = \widetilde{P} \cap R$. Since $R \subset B$ has s-coLO, there exist $Q_1, \ldots, Q_m \in \mathrm{Spec}(R)$, for $m \leqslant \dim H$, such that $(Q_j : K) = (Q : K) = \widetilde{P} \cap R$ for all j and $(\bigcap_{j=1}^m Q_J)^s \subset (Q : K)$. By 3.1(1), $(Q_j : H) = (Q : H) = P \cap R$, and by 3.1(2), $(Q : K)^l \subset (Q : H)$. Thus

$$\left(\bigcap_{j=1}^{m} Q_j\right)^{ls} \subset (Q : K)^l \subset (Q : H) \subset P \cap R.$$

(3) We need to complete the diagram

$$\begin{array}{ccc} & & Q_1 \\ & \cdot^{\cdot^{\cdot}} & \vdots \\ I_1 & & Q_2 \\ | & / & \\ I_2 & & \end{array}$$

where $I_1, I_2 \in H\text{-}\mathrm{Spec}(R)$ and $Q_2 \in \mathrm{Spec}(R)$ with $(Q_2 : H) = I_2$. By Lemma 3.2, there exists $J_2 \subset J_1 \in K\text{-}\mathrm{Spec}(R)$ such that $(J_i : H) = I_i$, for $i = 1, 2$. Now by Lemma 3.2, $(Q_2 : H) = I_2$ implies that $(Q_2 : K) = J_2$. Since $R \subset B$ has strong coGU, there exists $Q_1 \in \mathrm{Spec}(R)$ such that the diagram

$$\begin{array}{ccc} & & Q_1 \\ & \cdot^{\cdot^{\cdot}} & \vdots \\ J_1 & & Q_2 \\ | & / & \\ J_2 & & \end{array}$$

is complete. Hence $(Q_1 : K) = J_1$, and so by 3.1(1), $(Q_1 : H) = (J_1 : H) = I_1$ and we are done. □

Using Theorem 2.11 [MS, Theorem 3.7], we will apply the preceding proposition to our case of interest, namely to a Hopf subalgebra $K \subset H$ containing the coradical H_0 of H. We let $J(H)$ denote the Jacobson radical of H, and let t be the index of nilpotency of $J(H^*)$ (that is, t is the smallest $n \geqslant 1$ such that $J(H^*)^n = 0$). Note that the length of the coradical filtration of H is $t - 1$.

Theorem 3.4. *Let K be a Hopf subalgebra of H such that $H_0 \subset K$. Let $R \subset A$ be a faithfully flat H-Galois extension, with comodule structure map $\delta : A \to A \otimes H$, and let $B := \delta^{-1}(A \otimes K)$. Then (R, A, B) is (H, K)-Krull admissible. Thus*

(1) *K has* coINC *implies that H has* coINC;
(2) *K has s-*coLO *implies that H has st-*coLO, *where t is the index of nilpotency of $J(H^*)$;*
(3) *K has strong* coGU *implies that H has strong* coGU.

Proof. First note that $R \subset B$ is K-Galois by [S1, 3.11]. By [MS, Lemma 6.3], part (1) of 3.1 holds. Moreover, by Theorem 2.11, $(I : H_0)^t \subset (I : H)$, where t is the nilpotency index of $J(H^*)$. Since $(I : K)$ is K-stable, it is also H_0-stable by Lemma 2.10 with $C = K_0 = H_0$; thus $(I : K) \subset (I : H_0)$. It follows that $(I : K)^t \subset (I : H)$ and so 3.1(2) holds. Thus (R, A, B) is (H, K)-Krull admissible. (1)–(3) now follow from Proposition 3.3. □

The formal dual of Theorem 3.4 applies to quotients of H by a Hopf ideal contained in the radical $J(H)$.

Theorem 3.5. *Let I be a nilpotent Hopf ideal of H and let $\overline{H} = H/I$ be the quotient Hopf algebra. Then*

(1) *$\overline{H}$ has INC implies that H has INC;*
(2) *$\overline{H}$ has s-*LO *implies that H has st-*LO, *where now t is the index of nilpotency of $J(H)$;*
(3) *$\overline{H}$ has strong* GU *implies that H has strong* GU.

Proof. Since $I \subset J(H)$, H/I maps surjectively to $H/J(H)$, and so

$$H^* \supset \overline{H}^* \supset (H^*)_0 = \big(H/J(H)\big)^*.$$

Letting $K = \overline{H}^*$, we see that this theorem is precisely the dual of Theorem 3.4. The result follows by Theorem 2.4 and Lemma 2.5. □

We now apply both Theorem 3.4 and its dual to $\mathbb{N}$-graded finite-dimensional Hopf algebras. That is, $H = \bigoplus_{n \geqslant 0} H(n)$, where the grading is both as an algebra and as a coalgebra, and the antipode is a graded map; see [Sw2, p. 237]. By [Sw2, 11.1.1], $H(0) \supseteq H_0$, the coradical.

Theorem 3.6. *Let H be a finite-dimensional $\mathbb{N}$-graded Hopf algebra and let $K = H(0)$. If K has any one of the Krull relations* 2.2(1), (2), (4) *or* 2.2(1)′, (2)′, (4)′, *then so does H. Moreover, if K has all of the Krull relations, then so does H.*

Proof. The projection $\pi : H \to K$ is a surjective Hopf algebra map with nilpotent kernel $\bigoplus_{n \geqslant 1} H(n)$, and $H_0 \subset K$. The first part of the theorem now follows from Theorems 3.4 and 3.5. The second part follows from the first part together with Lemma 2.5. □

As an example of such a Hopf algebra, we could begin with any Hopf algebra H such that H_0 is a Hopf subalgebra, and consider its coradical filtration $\{H_n\}$. Let $gr(H)$ be the associated graded Hopf algebra; that is, $gr(H) = \bigoplus_{n \geqslant 0} H(n)$, where $H(n) = H_n/H_{n-1}$. Then $gr(H)$ is a graded Hopf algebra, with the same coradical as H.

Corollary 3.7. *Assume that H is pointed. Then $gr(H)$ has all of the Krull relations.*

Proof. $gr(H)$ is also pointed with $gr(H)_0 \cong H_0 = kG$ for some finite group G. Since kG has all of the Krull relations (see the discussion after 2.8), Theorem 3.6 applies. □

When H is graded and $H(n) = H_n/H_{n-1}$, H is said to be *coradically graded*. Thus Corollary 3.7 says that a coradically graded Hopf algebra satisfies all of the Krull relations.

4. Applications I: Pointed Hopf algebras

In this section we give some explicit examples of pointed graded Hopf algebras to which the results of the last section apply.

Example 4.1. Let T_n be the Hopf algebras described by Taft in [Tf]. That is, let ω be a primitive nth root of unity in k. Then

$$T_n = k\langle g, x \mid g^n = 1,\ x^n = 0,\ xg = \omega gx\rangle,$$

where g is group-like and x is a $(1, g)$-skew primitive. Clearly $H = T_n$ is pointed with coradical $H_0 = k\langle g\rangle$; moreover, since H_k is spanned by H_{k-1} together with all monomials in x of degree k, it follows that $H(k)$ is spanned by all non-zero monomials in x of degree k. Thus H is coradically graded, and so satisfies all of the Krull relations by Corollary 3.7.

In fact, we could have seen that T_n satisfies all of the Krull relations directly from [MS], since it is known that $T_n^* \cong T_n$; that is, T_n is also copointed.

Example 4.2. Let g be a semisimple Lie algebra over $\mathbb{C}$, and let $u_q(g)$ be the finite-dimensional quantum group of Lusztig, for q a primitive nth root of 1 in $\mathbb{C}$. Write $u_q(g) = \mathbb{C}\langle e_i, f_i, k_i\rangle$ where $\{e_i, f_i, k_i \mid i = 1, \ldots, n\}$ are the usual generators for $u_q(g)$ [K, IV.5.6] and let $H = u_q(g)^{\geqslant 0}$ be a Borel subalgebra. That is, $H = \mathbb{C}\langle e_i, k_i\rangle$. Then H is coradically graded with $H_0 = \mathbb{C}\langle k_i\rangle$, a group algebra. By Corollary 3.7, H has all of the Krull relations. Note that when $g = sl_2$, H is isomorphic to a Taft algebra (Example 4.1).

Example 4.3. Consider the pointed Hopf algebras $u(\mathcal{D})$ defined in [AS, 5.17] in terms of a linking datum $\mathcal{D}$ of finite Cartan type; these algebras can be considered as generalizations of $u_q(g)$. $u(\mathcal{D})$ is coradically graded if all the linking elements $\lambda_{ij} = 0$. Thus in this case $u(\mathcal{D})$ satisfies all of the Krull relations by Corollary 3.7.

Another Hopf algebra for which all of the Krull relations are satisfied is given by modified supergroup algebras, as described in [AEG]; these are based on the definition of supergroups due to Kostant [Ko]. First, a (finite-dimensional) supergroup is constructed from a finite group G and a finite-dimensional representation V of G. Let $\bigwedge V$ be the exterior algebra of V and let $\mathcal{H} = \bigwedge V \# kG$. Then $\mathcal{H}$ becomes a cocommutative Hopf superalgebra by letting V be odd, G be even, and each $x \in V$ be (graded) primitive. $\mathcal{H}$ is called a *supergroup* in [Ko], although in his formulation $\bigwedge V$ is viewed as $U(\mathfrak{g})$, where $\mathfrak{g} = V$ is an odd Lie superalgebra. Moreover, every finite-dimensional cocommutative Hopf superalgebra over $\mathbb{C}$ is of this form.

To describe the modified supergroup algebra H, consider $\mathcal{H}$ as above and assume in addition that G contains a central group-like element g such that $g^2 = 1$ and $gxg = -x$ for all $x \in V$. We define H by letting $H = \mathcal{H}$ as an algebra, but changing the comultiplication on $\mathcal{H}$ by defining $\Delta_H(x) := x \otimes 1 + g \otimes x$ for all $x \in V$, and letting $\Delta_H(y) = \Delta_{\mathcal{H}}(y)$ for all $y \in G$. With this definition, (H, Δ_H) becomes an ordinary Hopf algebra, the *modified supergroup algebra*.

Alternatively, H can be described as follows: note that $k\mathbb{Z}_2 = k\langle u\rangle$ acts on $\mathcal{H}$ via $u \cdot x = -x$ for all $x \in V$ and $u \cdot y = y$ for all $y \in G$. We may thus form the Radford biproduct $\widetilde{\mathcal{H}} = \mathcal{H} * k\mathbb{Z}_2$; it is an ordinary Hopf algebra, and H may be identified with the quotient $\widetilde{\mathcal{H}}/\widetilde{\mathcal{H}}\mathcal{L}^+$, where $\mathcal{L} = k\langle gu\rangle$.

Now $K := \bigwedge V \# k\langle g\rangle$ is a normal Hopf subalgebra of H, with quotient Hopf algebra $H/HK^+ \cong k(G/\langle g\rangle)$.

Corollary 4.4. *Let H be a modified supergroup algebra as above. Then H satisfies all of the Krull relations.*

Proof. Let K be as above; K is pointed and coradically graded and so satisfies all of the Krull relations. Since $\overline{H} = H/HK^+ \cong k(G/\langle g\rangle)$ is a group algebra, it also satisfies all the Krull relations. Thus by the Transitivity Theorem 2.8, H satisfies all of the Krull relations. □

5. The Krull relations under twistings

In this section we first consider what happens to the Krull relations when the multiplication of a Hopf algebra H is twisted by a cocycle σ, and then turn to the case when the comultiplication of H is twisted by a dual cocycle $\Omega \in H \otimes H$. Twisting H by a cocycle σ was studied in [Do].

First, recall from [Sw1] that for a Hopf algebra H, a (left) 2-*cocycle* on H is a convolution-invertible map $\sigma : H \otimes H \to k$ satisfying the equality

$$\sigma(h_{(1)}, l_{(1)})\sigma(h_{(2)}l_{(2)}, m) = \sigma(l_{(1)}, m_{(1)})\sigma(h, l_{(2)}m_{(2)}) \tag{5.1}$$

for all $h, l, m \in H$. We assume also that σ is normal, that is,

$$\sigma(h, 1) = \sigma(1, h) = \varepsilon(h) \quad \text{for all } h \in H.$$

We may now form a new Hopf algebra H_σ by leaving the coalgebra structure of H unchanged but twisting the algebra structure by σ. That is, H_σ has new multiplication

$$h \cdot_\sigma l := \sigma(h_{(1)}, l_{(1)}) h_{(2)} l_{(2)} \sigma^{-1}(h_{(3)}, l_{(3)}), \tag{5.2}$$

for all $h, l \in H$. One can also define a new antipode.

Also, given a right H-comodule algebra A, we may form the algebra A_σ, with twisted multiplication

$$a \cdot_\sigma b = \sigma^{-1}(a_{(1)}, b_{(1)}) a_{(0)} b_{(0)}$$

for all $a, b \in A$. Then A_σ is a right H_σ-comodule algebra, using the same comodule structure map as for A. We note that we need σ^{-1} here because of the mixture of a left cocycle with a right comodule.

A reference for the above facts is [KS, 10.2.3]; see also [M, Section 7.5] for a discussion of A_σ. We first show that twisting preserves Galois extensions.

Theorem 5.3. *Let $R \subset A$ be an H-extension, let σ be a cocycle on H, and consider the twisted algebra A_σ. Then $R_\sigma = R$, and an ideal I of R is H-stable if and only if it is H_σ-stable. Moreover,*

(1) *$R \subset A$ is H-Galois if and only if $R \subset A_\sigma$ is H_σ-Galois*;
(2) *$R \subset A$ is H-cleft if and only if $R \subset A_\sigma$ is H_σ-cleft*; *moreover, if $A = R \#_\tau H$, then $A_\sigma \cong R \#_{\tau^\sigma} H_\sigma$, where $\tau^\sigma = \tau * \sigma^{-1}$*;
(3) H-Spec$(R) = H_\sigma$-*Spec*(R).

Proof. First, since $R = A^{\mathrm{co}H} = A_\sigma^{\mathrm{co}H_\sigma}$, it is easy to see that $r \cdot_\sigma a = ra$ and $a \cdot_\sigma r = ar$ for any $r \in R, a \in A$. It follows that $R_\sigma = R$. Moreover, if I is any ideal of R and $AI = IA$, then clearly $A \cdot_\sigma I = I \cdot_\sigma A$. Thus the fact about stability follows.

(1) Consider the two canonical Galois maps for A and A_σ; that is, $\beta : A \otimes_R A \to A \otimes H$ via $a \otimes b \mapsto ab_{(0)} \otimes b_{(1)}$ and $\beta^\sigma : A_\sigma \otimes_R A_\sigma \to A_\sigma \otimes H_\sigma$ via $a \otimes b \mapsto a \cdot_\sigma b_{(0)} \otimes b_{(1)} = a_{(0)} b_{(0)} \otimes b_{(2)} \sigma^{-1}(a_{(1)}, b_{(1)})$.

Define $\Phi, \Psi : A \otimes H \to A \otimes H$ by

$$\Phi(a \otimes h) = a_{(0)} \otimes \sigma^{-1}(a_{(1)}, Sh_{(3)}) \sigma(h_{(1)}, Sh_{(2)}) h_{(4)}$$

and

$$\Psi(a \otimes h) = a_{(0)} \otimes \sigma(a_{(1)}, Sh_{(1)}) \sigma^{-1}(Sh_{(2)}, h_{(3)}) h_{(4)}.$$

We claim that $\Psi = \Phi^{-1}$ and that $\beta = \Phi \circ \beta^\sigma$. Thus β is bijective if and only if β^{-1} is bijective. To show this we require the cocycle condition (5.1), and, in addition, the identities

$$\sigma(h_{(1)}, Sh_{(2)}) \sigma^{-1}(Sh_{(3)}, h_{(4)}) = \varepsilon(h), \tag{5.4}$$

$$\sigma^{-1}(Sh_{(1)}, h_{(2)})\sigma(h_{(3)}, Sh_{(4)}) = \varepsilon(h). \tag{5.5}$$

Identity (5.4) appears in [BM,Do]; see also [KS]. (5.5) can be obtained from (5.4) as follows: apply (5.4) to the left cocycle σ^{-1} on H^{cop} and use that $S_{H^{\mathrm{cop}}} = \bar{S}_H$. Then

$$\sigma^{-1}\big(h_{(4)}, \bar{S}h_{(3)}\big)\sigma\big(\bar{S}h_{(2)}, h_{(1)}\big) = \varepsilon(h).$$

Now replace h by Sh and we have (5.5).

We can now show that $\Psi = \Phi^{-1}$. First,

$$\begin{aligned}
&(\Psi \circ \Phi)(a \otimes h)\\
&\quad = \Psi\big(a_{(0)} \otimes \sigma^{-1}(a_{(1)}, Sh_{(3)})\sigma(h_{(1)}, Sh_{(2)})h_{(4)}\big)\\
&\quad = a_{(0)(0)} \otimes \sigma(a_{(0)(1)}, Sh_{(4)(1)})\sigma^{-1}(Sh_{(4)(2)}, h_{(4)(3)})\sigma^{-1}(a_{(1)}, Sh_{(3)})\\
&\qquad \times \sigma(h_{(1)}, Sh_{(2)})h_{(4)(4)}\\
&\quad = a_{(0)} \otimes \sigma(a_{(1)}, Sh_{(4)})\sigma^{-1}(Sh_{(5)}, h_{(6)})\sigma^{-1}(a_2, Sh_{(3)})(h_{(1)}, Sh_{(2)})h_{(7)}\\
&\quad = a_{(0)} \otimes \sigma^{-1}(Sh_{(3)}, h_{(4)})\sigma(h_{(1)}, Sh_{(2)})h_{(5)}\\
&\quad = a \otimes h,
\end{aligned}$$

using (5.4) in the last step. Similarly, using (5.5), we see that

$$\begin{aligned}
&(\Phi \circ \Psi)(a \otimes h)\\
&\quad = \Phi\big(a_{(0)} \otimes \sigma(a_{(1)}, Sh_{(1)})\sigma^{-1}(Sh_{(2)}, h_{(3)})h_{(4)}\big)\\
&\quad = a_{(0)(0)} \otimes \sigma(a_{(1)}, Sh_{(1)})\sigma^{-1}(Sh_{(2)}, h_{(3)})\sigma^{-1}(a_{(0)(1)}, Sh_{(4)(3)})\\
&\qquad \times \sigma(h_{(4)(1)}, Sh_{(4)(2)})h_{(4)(4)}\\
&\quad = a_{(0)} \otimes \sigma(a_{(2)}, Sh_{(1)})\sigma^{-1}(Sh_{(2)}, h_{(3)})\sigma^{-1}(a_{(1)}, Sh_{(6)})\sigma(h_{(4)}, Sh_{(5)})h_{(7)}\\
&\quad = a_{(0)} \otimes \sigma(a_2, Sh_{(1)})\sigma^{-1}(a_{(1)}, Sh_{(2)})h_{(3)} = a \otimes h.
\end{aligned}$$

Thus $\Psi = \Phi^{-1}$. Finally we check that $\beta = \Phi \circ \beta^\sigma$, using (5.1):

$$\begin{aligned}
&\Phi\big(\beta^\sigma(a \otimes h)\big)\\
&\quad = \Phi\big(a_{(0)}b_{(0)} \otimes b_{(2)}\sigma^{-1}(a_{(1)}, b_{(1)})\big)\\
&\quad = a_{(0)}b_{(0)} \otimes \sigma^{-1}(a_{(0)(1)}b_{(0)(1)}, Sb_{(2)(3)})\sigma(b_{(2)(1)}, Sb_{(2)(2)})\sigma^{-1}(a_{(1)}, b_{(1)})b_{(2)(4)}\\
&\quad = a_{(0)}b_{(0)} \otimes \sigma^{-1}(a_{(1)}b_{(1)}, Sb_{(5)})\sigma(b_{(3)}, Sb_{(4)})\sigma^{-1}(a_2, b_{(2)})b_{(6)}\\
&\quad = a_{(0)}b_{(0)} \otimes \sigma^{-1}(a_{(1)}, b_{(1)}Sb_{(6)})\sigma^{-1}(b_{(2)}, Sb_{(5)})\sigma(b_{(3)}, Sb_{(4)})b_{(6)}\\
&\quad = a_{(0)}b_{(0)} \otimes \sigma^{-1}(a_{(1)}, b_{(1)}Sb_{(2)})b_{(3)}\\
&\quad = ab_{(0)} \otimes b_{(1)} = \beta(a \otimes b).
\end{aligned}$$

This proves (1).

(2) Assume that A is H-cleft, via the H-comodule map $\gamma : H \to A$ with convolution inverse γ^{-1}. We claim that A_σ is H_σ-cleft, via the same map $\gamma^\sigma = \gamma$ on vector spaces, but with convolution inverse $(\gamma^\sigma)^{-1}(h) = \gamma^{-1}(h_{(3)})\sigma(h_{(1)}, Sh_{(2)})$. First, the fact that γ is an H_σ-comodule map follows since $H = H_\sigma$ as coalgebras and $A = A_\sigma$ as comodules. Also, since γ is a comodule map, $\delta(\gamma(h)) = \gamma(h_{(1)}) \otimes h_{(2)}$ and $\delta(\gamma^{-1}(h)) = \gamma^{-1}(h_{(2)}) \otimes Sh_{(1)}$, where δ is the comodule structure map of A. It follows that

$$\delta\big((\gamma^\sigma)^{-1}(h)\big) = \gamma^{-1}(h_{(4)}) \otimes Sh_{(3)}\sigma(h_{(1)}, Sh_{(2)}).$$

Now in $\mathrm{Hom}(H, A_\sigma)$,

$$\begin{aligned}
&\gamma(h_{(1)}) \cdot_\sigma (\gamma^\sigma)^{-1}(h_{(2)}) \\
&\quad = \gamma(h_{(1)})_{(0)}(\gamma^\sigma)^{-1}(h_{(2)})_{(0)}\sigma^{-1}\big(\gamma(h_{(1)})_{(1)}, (\gamma^\sigma)^{-1}(h_{(2)})_{(1)}\big) \\
&\quad = \gamma(h_{(1)(1)})\gamma^{-1}(h_{(2)(4)})\sigma^{-1}(h_{(1)(2)}, Sh_{(2)(3)})\sigma(h_{(2)(1)}, Sh_{(2)(2)}) \\
&\quad = \gamma(h_{(1)})\gamma^{-1}(h_{(6)})\sigma^{-1}(h_{(2)}, Sh_{(5)})\sigma(h_{(3)}, Sh_{(4)}) \\
&\quad = \gamma(h_{(1)})\gamma^{-1}(h_{(2)}) = \varepsilon(h)1.
\end{aligned}$$

Since H is finite-dimensional, also $(\gamma^\sigma)^{-1}$ is the left inverse of γ, and so A_σ is H_σ-cleft.

To see that the new cocycle τ^σ is as described, first recall that cleft extensions are always crossed products. Thus $A_\sigma \cong R \#_{\tau^\sigma} H_\sigma$ for some Hopf 2-cocycle $\tau^\sigma : H_\sigma \otimes H_\sigma \to R$, where the H_σ-comodule structure on $R \#_{\tau^\sigma} H_\sigma$ is given by $id \otimes \Delta_{H_\sigma} = id \otimes \Delta_H$.

Choose $r \# g$ and $s \# h$ in $A = R \#_\tau H$, and consider their multiplication in A_σ:

$$\begin{aligned}
&(r \# g) \cdot_\sigma (s \# h) \\
&\quad = (r \# g_{(1)})(s \# h_{(1)})\sigma^{-1}(g_{(2)}, h_{(2)}) \\
&\quad = r(g_{(1)} \cdot s)\tau(g_{(2)}, h_{(1)}) \# g_{(3)}h_{(2)}\sigma^{-1}(g_{(4)}, h_{(3)}) \\
&\quad = r(g_{(1)} \cdot s)\tau(g_{(2)}, h_{(1)}) \# \sigma^{-1}(g_{(3)}, h_{(2)})\sigma(g_{(4)}, h_{(3)})g_{(5)}h_{(4)}\sigma^{-1}(g_{(6)}, h_{(5)}) \\
&\quad = r(g_{(1)} \cdot s)\tau(g_{(2)}, h_{(1)})\sigma^{-1}(g_{(3)}, h_{(2)}) \# g_{(4)} \cdot_\sigma h_{(3)}.
\end{aligned}$$

But considered as elements in $R \#_{\tau^\sigma} H_\sigma$, their product is

$$(r \# g)(s \# h) = r(g_{(1)} \cdot s)\tau^\sigma(g_{(2)}, h_{(1)}) \# g_{(3)} \cdot_\sigma h_{(2)}.$$

Thus $\tau^\sigma(g, h) = \tau(g_{(1)}, h_{(1)})\sigma^{-1}(g_{(2)}, h_{(2)})$.

Alternatively the cocycle can be expressed in terms of the cleft map γ (respectively γ^σ).

(3) The identification of the stable parts of Spec follows from the remarks at the beginning of the proof, once we know (1). □

Theorem 5.6. *Let H_σ be any cocycle twist of H. Then*:

(1) *any one of the Krull relations* coINC*,* t-coLO*, and strong* coGU *is true for H $\Leftrightarrow$ it is true for H_σ;*
(2) *if H has* coGU *and* t-LO*, then H_σ also has* coGU.

Proof. (1) follows from Theorem 5.3(3), because the three dual Krull relations coINC, t-coLO, and strong coGU are defined only in terms of ideals of R, as noted in the remark after Definition 2.2.

(2) follows from (1) and Lemma 2.5(2)$'$. □

We now consider dual cocycle twists, as in [Dr]. That is, let $\Omega \in H \otimes H$ be a dual cocycle for H. Then H^Ω has the same multiplication as H but has new comultiplication $\Delta_\Omega(h) = \Omega \Delta_H(h)\Omega^{-1}$. This construction is the formal dual of the construction of the cocycle twists in Section 4, in the following sense: if H is finite-dimensional, then $(H^*)^\Omega = (H_\sigma)^*$. For if σ is a 2-cocycle on H, then σ corresponds to an invertible element $\Omega \in H^* \otimes H^* \cong (H \otimes H)^*$, and we may twist the comultiplication of H^* by Ω. The explicit correspondence between σ and Ω is given by

$$\sigma(h,l) = \sum \Omega^1(h)\Omega^2(l).$$

Analogously if A is an H-comodule algebra, then A is an H^*-module algebra, and we can consider it either as twisted by σ or by Ω.

Using this reformulation we may state the dual version of Theorem 5.6.

Theorem 5.7. *Let H^Ω be any dual cocycle twist of H. Then*:

(1) *any one of the Krull relations* INC*,* t-LO*, and strong* GU *is true for H $\Leftrightarrow$ it is true for H^Ω;*
(2) *if H has* GU *and* t-coLO*, then H^Ω also has* GU.

Remark 5.8. In the terminology of [MS, Definition 8.8], H is called *strongly semisimple* if for all left H-module algebras A with ring of invariants $R = A^H$, and any $P \in \operatorname{Spec} A$, $P \cap R$ is a semiprime ideal of R. Similarly H is called *strongly cosemisimple* if for all right H-comodule algebras A with ring of coinvariants $R = A^{\mathrm{co}H}$, and any $P \in \operatorname{Spec} A$, $P \cap R$ is a semiprime ideal of R. Theorem 5.3(3) implies that H is strongly cosemisimple if and only if H_σ is strongly cosemisimple, and thus dually that H is strongly semisimple if and only if H^Ω is strongly semisimple.

By [MS, Theorem 8.11 and Corollary 8.14], H is strongly cosemisimple if and only if for every H-module algebra R, every H-semiprime ideal of R is semiprime; equivalently, for all H-module algebras R, the prime radical $P(R)$ is always H-stable. Thus this stability property is preserved by twisting with a cocycle σ.

Similarly from [MS, Theorem 8.10 and Corollary 8.14], H is strongly semisimple if and only if for every H-semiprime H-module algebra R, the smash product $R \# H$ is semiprime. Thus this property is preserved by twisting with a dual cocycle Ω.

6. Applications II: Triangular Hopf algebras and the Drinfeld double

We first consider triangular Hopf algebras.

Theorem 6.1. *Let k be an algebraically closed field of characteristic* 0*, and let H be a* (*finite-dimensional*) *triangular Hopf algebra. Then H has the Krull relations* INC*, t*-LO*, and strong* GU*.*

Proof. By [EG2], for any triangular Hopf algebra, the Jacobson radical $J(H)$ is a Hopf ideal. Thus $\overline{H} = H/J(H)$ is a semisimple triangular Hopf algebra. By [EG1], it follows that $\overline{H} = kG^{\Omega}$, the twist of a group algebra by a dual cocycle $\Omega \in kG \otimes kG$. Applying Theorem 5.7, we see that $\overline{H}$ has INC, t-LO, and strong GU, since kG has these three properties. The theorem now follows from Theorem 3.5. □

An alternate proof of Theorem 6.1 may be given using supergroups. For, in [EG2], it is shown that if H is triangular then it is a dual cocycle twist of a supergroup. Thus Theorem 6.1 would follow immediately from Theorem 5.7 and Corollary 4.4.

We next consider the general question of when any (or all) of the Krull relations lift from H and H^* to the Drinfeld double $D(H)$. We obtain a complete answer for bismash products of groups and for factorizable Hopf algebras.

We are able to give one general result. We use a result of Doi and Takeuchi that for any H, $D(H) = (H^{*\mathrm{cop}} \otimes H)_\sigma$ for some cocycle σ on $H^{*\mathrm{cop}} \otimes H$ [DT].

Corollary 6.2. *If H has the six Krull relations* INC*, s*-LO*,* GU*,* coINC*, t*-coLO*, and* coGU*, then the double $D(H)$ has* coINC*, st*-coLO*, and* coGU*.*

Proof. By Lemma 2.5, H also has strong GU and strong coGU. By Remark 2.7 and the discussion following the Transitivity Theorem 2.8, the tensor product $H^{*\mathrm{cop}} \otimes H$ has all the Krull relations. Apply [DT] to see that $D(H) = (H^{*\mathrm{cop}} \otimes H)_\sigma$. The result now follows by Theorem 5.6. □

We now consider bismash products. Assume that L is a factorizable group, that is $L = FG$, where F and G are subgroups of L with $F \cap G = \{1\}$. Then (F, G) form a *matched pair of groups* in the sense of [Tk], and we may thus construct the bismash product $H = k^G \# kF$; in this case $H^* = k^F \# kG$. More generally, the bicrossed product extensions $H = k^G \#_\sigma^\tau kF$ are classified by classes of pairs $[\sigma, \tau]$ in the OpExt group; see Masuoka's survey [Ma].

We have already noted that both kG and $(kG)^*$ satisfy all of the Krull relations. In addition, $(kG)^*$ always satisfies 1-LO [CM], and kG will satisfy 1-LO whenever it is semisimple (by combining [LP] with[FM]).

We will need the following result of [BGMj]: let $H = k^G \# kF$ be the bismash product for the factorizable group $L = FG$, as described above. Then $D(H) \cong D(kL)^{\Omega}$ for some dual cocycle Ω.

Theorem 6.3. *Let $L = FG$ be a factorizable group as above. Then*:

(1) *any bicrossed product $H = k^G \#_\sigma^\tau kF$ satisfies all of the Krull relations, and for any Ω, H^Ω satisfies the basic Krull relations* INC*,* t-LO*, and* GU;
(2) *if H is the bismash product, then $D(H)$ satisfies all of the Krull relations.*

If also $\operatorname{char} k = 0$ *or if* $\operatorname{char} k = p > 0$ *and p does not divide $|G|$, then in* (1) *H will satisfy* 1-LO *and* 1-coLO *and H^Ω will satisfy* 1-LO. *In* (2) *$D(H)$ will satisfy* 1-LO *and* 1-coLO.

Proof. (1) First, any bicrossed product H satisfies all the Krull relations by the Transitivity Theorem 2.8, since $K = k^G \# 1$ is a normal Hopf subalgebra of H with quotient $\overline{H} \cong kF$. Then H^Ω satisfies the basic Krull relations by Theorem 5.7.

(2) By (1), H satisfies all of the Krull relations. Thus by Corollary 6.2, $D(H)$ satisfies the dual Krull relations. By the result of [BGMj] described above, $D(H) \cong D(kL)^\Omega$ for some dual cocycle Ω. Since $D(kL)$ satisfies all of the Krull relations, $D(kL)^\Omega$ satisfies the basic Krull relations by Theorem 5.7. Thus $D(H)$ satisfies all six Krull relations.

In the case when kG is semisimple, the facts about 1-LO and 1-coLO follow from the Transitivity Theorem 2.8(1) together with (1). □

Corollary 6.4. *Let $H = (k^G \#_\sigma^\tau kF)^\Omega$, let R be an H-semiprime H-module algebra, and assume* $\operatorname{char} k = 0$ *or* $\operatorname{char} K = p > 0$ *and p does not divide $|G|$. Then $R \# H$ must be semiprime.*

Proof. H satisfies 1-LO by the theorem. Thus $R \# H$ is semiprime, by [MS, Proposition 4.5(1)]. □

Bicrossed products are an important class of Hopf algebras, as they are closely related to the group-theoretical quasi-Hopf algebras defined in [O] and studied further in [ENO,N]. By definition a quasi-Hopf algebra is *group theoretical* if its category of representations is a group theoretical category $\mathcal{C}(L, \omega, F, \alpha)$, where L is a finite group, $F \subset L$ is a subgroup, $\omega : G \times G \times G \to k^\times$ is a normalized 3-cocycle, and $\alpha : F \times F \to k^\times$ is a normalized 2-cocycle such that $\omega|_F = 1$ [O]. Here k is algebraically closed of characteristic 0. A major example is given by the quasi-Hopf version of the bicrossed products above, although in general G might not be a group and is replaced by a fixed set Q of coset representatives of F in L. It is shown in [N] that any group-theoretical quasi-Hopf algebra H is gauge-equivalent to some quasi-Hopf algebra $M = k^Q \#_\sigma^\tau kF$, where σ and τ are defined using ω. This means that $H = M^\Omega$, a twist of M, although in this case the twisting element Ω does not have to be a cocycle. It is an open question as to whether every semisimple Hopf algebra over $\mathbb{C}$ is group-theoretical [ENO].

Finally we consider factorizable Hopf algebras. See [RS] for the definition. We use the result of [RS], which says that if H is a factorizable Hopf algebra, then $D(H) \cong (H \otimes H)^\Omega$ for some dual cocycle Ω; see [S3] for another proof.

Corollary 6.5. *Assume that H is factorizable and satisfies all of the Krull relations. Then $D(H)$ satisfies all of the Krull relations.*

Proof. First, $D(H)$ satisfies the dual Krull relations by Corollary 6.2. By the Transitivity Theorem 2.8, $H \otimes H$ also satisfies all of the Krull relations. Now apply the above result of [RS] together with Theorem 5.7 to see that $D(H)$ satisfies INC, t-LO, and strong GU. □

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