LINEAR ALGEBRA AND ITS APPLICATIONS 5, 1–12 (1972)

Principal Submatrices IX: Interlacing Inequalities for Singular Values of Submatrices*

R. C. THOMPSON
University of California
Santa Barbara, California

Communicated by Alan J. Hoffman

INTRODUCTION

This paper is the ninth in a continuing series studying the submatrices of a matrix. Our main objectives in the previous papers in the series has been to examine, simultaneously, all of the k-square principal submatrices of an n-square matrix A. Usually A has been symmetric or Hermitian, and much of our effort has centered around the well-known fact asserting that the eigenvalues of an $(n-1)$-square principal submatrix of Hermitian A always interlace the eigenvalues of A. In this paper we study the singular values of the submatrices (not necessarily principal submatrices) of an arbitrary matrix A. Although we study not necessarily principal submatrices, this paper is included in the Principal Submatrices series because (as our proofs will show) the singular values of an arbitrary submatrix of matrix A may be approached through an examination of the principal submatrices of AA^*. (Here A^* is the Hermitian adjoint of A.)

We now give a brief summary of certain particular cases of our results that merit special attention. Let A be an $n \times n$ real or complex matrix, and let $\alpha_1 \geqslant \alpha_2 \geqslant \cdots \geqslant \alpha_n$ be the singular values of A. (They are defined to be the eigenvalues of the positive semidefinite matrix $(AA^*)^{1/2}$.) Let $B = A_{ij}$ be the $(n-1)$-square submatrix of A obtained by deleting row i and column j, and let $\beta_1 \geqslant \beta_2 \geqslant \cdots \geqslant \beta_{n-1}$ be the singular values of B. Our first theorem yields, as a special case, these interlacing inequalities:

* The preparation of this paper was supported by the U.S. Air Force Office of Scientific Research, under Grant 698-67.

$$\begin{aligned}
&\alpha_1 \geqslant \beta_1 \geqslant \alpha_3,\\
&\alpha_2 \geqslant \beta_2 \geqslant \alpha_4,\\
&\quad\cdots\\
&\alpha_t \geqslant \beta_t \geqslant \alpha_{t+2}, \qquad 1 \leqslant t \leqslant n-2,\\
&\quad\cdots\\
&\alpha_{n-2} \geqslant \beta_{n-2} \geqslant \alpha_n,\\
&\alpha_{n-1} \geqslant \beta_{n-1}.
\end{aligned} \tag{1}$$

That inequalities (1) are the best that can be asserted is shown by (a special case of) Theorem 2. It follows from Theorem 2 that, if arbitrary nonnegative numbers $\beta_1 \geqslant \cdots \geqslant \beta_{n-1}$ are given satisfying (1), there will always exist unitary matrices U and V such that the singular values of $(UAV)_{ij}$ are $\beta_1, \ldots, \beta_{n-1}$. (Of course, A and UAV always have the same singular values $\alpha_1, \ldots, \alpha_n$.) Thus nothing more than (1) can hold in general, when looking at a fixed submatrix. Further results can be obtained, however, by examining all the submatrices of A of fixed degree. Now let $\beta_{ij,1} \geqslant \cdots \geqslant \beta_{ij,n-1}$ denote the singular values of A_{ij}. We obtain the following estimates on the *mean square* of the tth singular value of all the $(n-1)$-square submatrices A_{ij} of A:

$$\begin{aligned}
\left(\frac{1}{n}\right)^2 \alpha_t^2 + \frac{2(n-1)}{n^2}\alpha_{t+1}^2 + \left(\frac{n-1}{n}\right)^2 \alpha_{t+2}^2 &\leqslant \frac{1}{n^2}\sum_{i,j=1}^{n} \beta_{ij,t}^2\\
&\leqslant \left(\frac{n-1}{n}\right)^2 \alpha_t^2 + \frac{2(n-1)}{n^2}\alpha_{t+1}^2 + \left(\frac{1}{n}\right)^2 \alpha_{t+2}^2,\\
&\qquad 1 \leqslant t \leqslant n-2,
\end{aligned} \tag{2}$$

$$\begin{aligned}
\left(\frac{1}{n}\right)^2 \alpha_{n-1}^2 + \frac{n-1}{n}\frac{1}{n}\alpha_n^2 &\leqslant \frac{1}{n^2}\sum_{i,j=1}^{n} \beta_{ij,n-1}^2\\
&\leqslant \left(\frac{n-1}{n}\right)^2 \alpha_{n-1}^2 + \frac{n-1}{n}\frac{1}{n}\alpha_n^2.
\end{aligned} \tag{3}$$

In (2) we have displayed convex combinations of α_t^2, α_{t+1}^2, α_{t+2}^2 which serve as upper and lower bounds for the mean square of the tth singular value ($t \leqslant n-2$) of the different $(n-1)$-square submatrices A_{ij} of A. (By (1), this mean lies between α_t^2 and α_{t+2}^2.) In (3), we have similar, though not precisely the same, convex combinations of α_{n-1}^2, α_n^2, and 0 yielding bounds for the mean square of the $\beta_{ij,n-1}$. These results, (2) and (3), will appear as special cases of Theorem 3.

Let

$$f_{ij}(\lambda) = (\lambda - \beta_{ij,1}^2) \cdots (\lambda - \beta_{ij,n-1}^2) \tag{4}$$

be the singular value polynomial of A_{ij}. This is the polynomial whose roots are the squares of the singular values of A_{ij}. Let

$$f(\lambda) = (\lambda - \alpha_1{}^2) \cdots (\lambda - \alpha_n{}^2) \tag{5}$$

be the corresponding polynomial for A. As a particular instance of Theorem 4, we obtain

$$\sum_{i=1}^{n} \sum_{j=1}^{n} f_{ij}(\lambda) = \frac{d}{d\lambda} \lambda \frac{d}{d\lambda} f(\lambda). \tag{6}$$

It is interesting to contrast formula (6) with the well-known result asserting that the sum of the characteristic polynomials of all the principal $(n - 1)$-square submatrices of A is just the derivative of the characteristic polynomial of A.

RESULTS

As we shall be studying rectangular matrices, we give first the definition of the singular values of a rectangular matrix.

DEFINITION. Let A be an $m \times n$ matrix. The singular values

$$\alpha_1 \geqslant \alpha_2 \geqslant \cdots \geqslant \alpha_{\min(m,n)} \tag{7}$$

of A are the common eigenvalues of the positive semidefinite matrices $(AA^*)^{1/2}$ and $(A^*A)^{1/2}$.

Since AA^* is m-square and A^*A is n-square, the eigenvalues of $(AA^*)^{1/2}$ and $(A^*A)^{1/2}$ do not coincide in full. However, it is well known that the nonzero eigenvalues (including multiplicities) of these two matrices always coincide. It is frequently convenient to define α_t to be zero for $\min(m, n) < t \leqslant \max(m, n)$. Then $\alpha_1{}^2 \geqslant \cdots \geqslant \alpha^2_{\max(m,n)}$, and the roots of AA^* (respectively A^*A) are the first m (respectively n) of these numbers.

We are now ready for Theorem 1.

THEOREM 1. *Let A be an $m \times n$ matrix with singular values* (7). *Let B be a $p \times q$ submatrix of A, with singular values*

$$\beta_1 \geqslant \beta_2 \geqslant \cdots \geqslant \beta_{\min(p,q)}. \tag{8}$$

Then

$$\alpha_i \geqslant \beta_i, \qquad \textit{for} \quad i = 1, 2, \ldots, \min(p, q), \tag{9}$$

$$\beta_i \geqslant \alpha_{i+(m-p)+(n-q)}, \qquad \textit{for} \quad i \leqslant \min(p + q - m, p + q - n). \tag{10}$$

Proof. For an arbitrary matrix M, let $M[i_1, \ldots, i_p | j_1, \ldots, j_q]$ denote the submatrix of M lying at the intersection of rows $i_1, \ldots, i_p$ and columns $j_1, \ldots, j_q$.

Suppose that $B = A[i_1, \ldots, i_p | j_1, \ldots, j_q]$. To simplify notation let $\omega = \{i_1, \ldots, i_p\}$ and $\tau = \{j_1, \ldots, j_q\}$ denote the sets of integers giving the rows and columns of A used to form B, and denote B by $B = A[\omega|\tau]$.

Let us view B as a submatrix of UAV, where U is an m-square unitary matrix and V is an n-square unitary matrix. In this proof we may take $U = I_m$ and $V = I_n$. (In the next theorem, U and V will become variable.) Then

$$B = U[i_1, \ldots, i_p | 1, \ldots, m] A V[1, \ldots, n | j_1, \ldots, j_q]. \tag{11}$$

Thus

$$BB^* = U[i_1, \ldots, i_p | 1, \ldots, m] XX^* U^*[1, \ldots, m | i_1, \ldots, i_p], \tag{12}$$

where

$$X = AV[1, \ldots, n | j_1, \ldots, j_q] \tag{13}$$

is $m \times q$. Thus BB^* is a principal p-square submatrix of the m-square Hermitian matrix UXX^*U^*. Let

$$x_1^2 \geqslant x_2^2 \geqslant \cdots \geqslant x_{\min(m,q)}^2 \geqslant x_{\min(m,q)+1}^2 = \cdots = x_m^2 = 0, \tag{14}$$

denote the eigenvalues of XX^*. Thus $x_1, \ldots, x_{\min(m,q)}$ are the singular values of X. From the well-known formulas linking the eigenvalues of a Hermitian matrix with the eigenvalues of a principal submatrix, we obtain

$$x_i^2 \geqslant \beta_i^2 \geqslant x_{i+m-p}^2 \quad \text{for} \quad i = 1, 2, \ldots, p. \tag{15}$$

Now $x_1^2, \ldots, x_{\min(m,q)}^2$, 0 ($q - \min(m, q)$ times) are the eigenvalues of

$$X^*X = V^*[j_1, \ldots, j_q | 1, \ldots, n] A^*AV[1, \ldots, n | j_1, \ldots, j_q]. \tag{16}$$

Thus X^*X is a principal q-square submatrix of the n-square Hermitian matrix V^*A^*AV. Hence

$$\alpha_i^2 \geqslant x_i^2 \geqslant \alpha_{i+n-q}^2 \quad \text{for} \quad i = 1, 2, \ldots, q. \tag{17}$$

Thus for $i \leqslant \min(p, q)$ we have $\alpha_i^2 \geqslant x_i^2 \geqslant \beta_i^2$, yielding (9). And for $i \leqslant \min(p + q - m, p + q - n)$ we have $\beta_i^2 \geqslant x_{i+m-p}^2 \geqslant \alpha_{i+(n-q)+(m-p)}^2$, yielding (10).

The proof of Theorem 1 is now complete. We shall present a second proof of Theorem 1 at the end of this paper.

THEOREM 2. *Let A be an $m \times n$ matrix with singular values* (7). *Let arbitrary nonnegative numbers* (8) *be given, satisfying both* (9) *and* (10). *Then m-square unitary matrix U and n-square unitary matrix V exist such that the singular values of the $p \times q$ submatrix*

$$(UAV)[i_1, \ldots, i_p | j_1, \ldots, j_q]$$

of UAV are the numbers (8).

Proof. Define β_i to be zero if $i > \min(p, q)$, and define α_i to be zero if $i > \min(m, n)$. Now define inductively nonnegative numbers $x_1, \ldots, x_{\min(m,q)}$ by

$$x_1 = \min \begin{cases} \alpha_1 \\ \beta_{1-m+p} & \text{if} \quad m - p < 1 \end{cases} \tag{18}$$

and

$$x_i = \min \begin{cases} \alpha_i \\ \beta_{i-m+p} & \text{if} \quad m - p < i, \\ x_{i-1} \end{cases}$$

$$\text{for} \quad 2 \leqslant i \leqslant \min(m, q). \tag{19}$$

(We include β_{i-m+p} in (18) and (19) only if i satisfies the indicated condition.) For all $t > \min(m, q)$, define x_t by $x_t = 0$.

It is plain that $x_1 \geqslant \cdots \geqslant x_{\min(m,q)}$. We claim that inequalities (17) are satisfied. Plainly, $x_i \leqslant \alpha_i$ for $i \leqslant \min(m, q)$, and this also holds for $\min(m, q) < i \leqslant q$ since then $x_i = 0$. We show by induction on i that the lower bounds in (17) are satisfied. To show that $x_1 \geqslant \alpha_{1+n-q}$, we must show that both of the quantities entering into the minimum in (18) exceed

α_{1+n-q}. Plainly, by (7), $\alpha_1 \geqslant \alpha_{1+n-q}$. If $m - p < 1$ (thus $m = p$), (10) tells us that $\beta_1 \geqslant \alpha_{1+n-q}$, provided $1 \leqslant \min(q, m + q - n)$. However, if $m + q - n \leqslant 0$, we have $m + 1 \leqslant 1 + n - q$ and thus automatically $0 = \alpha_{1+n-q} \leqslant \beta_1$. Hence $x_1 \geqslant \alpha_{1+n-q}$. Suppose (induction) $x_{i-1} \geqslant \alpha_{i-1+n-q}$. Let $i \leqslant \min(m, q)$. If we show that each of the three quantities entering into the minimum in (19) exceeds α_{i+n-q}, it will follow that $x_i \geqslant \alpha_{i+n-q}$. Plainly, by (7), $\alpha_i \geqslant \alpha_{i+n-q}$. If $m - p < i$, we obtain from (10) that $\beta_{i-m+p} \geqslant \alpha_{i+n-q}$, provided $i \leqslant \min(q, m + q - n)$. By induction, $x_{i-1} \geqslant \alpha_{i-1+n-q} \geqslant \alpha_{i+n-q}$ (by (7)). Thus $x_i \geqslant \alpha_{i+n-q}$, except perhaps if $i > \min(q, m + q - n)$. However, if $i > \min(q, m + q - n)$, then $i + n - q > \min(n, m)$, so that $\alpha_{i+n-q} = 0$ and hence automatically $x_i \geqslant \alpha_{i+n-q}$. Therefore $x_i \geqslant \alpha_{i+n-q}$ is established if $i \leqslant \min(m, q)$. If $i > \min(m, q)$, then $i + n - q > \min(n - q + m, n) \geqslant \min(m, n)$, so that automatically $0 = \alpha_{i+n-q} \leqslant x_i$. Therefore inequalities (17) are established.

We now claim that inequalities (15) are satisfied. By (19), $x_{i+m-p} \leqslant \beta_i$, for $i + m - p \leqslant \min(m, q)$. Thus the lower inequality in (15) is satisfied, provided $i \leqslant \min(p, p + q - m)$. If $i > \min(p, p + q - m)$, then $i + m - p > \min(m, q)$ and hence $x_{i+m-p} = 0$, so that automatically $\beta_i \geqslant x_{i+m-p}$. Thus the lower inequalities in (15) are satisfied. We show by induction on i that $x_i \geqslant \beta_i$. For $i = 1$ this follows immediately from (18), since $\alpha_1 \geqslant \beta_1$. Suppose $x_{i-1} \geqslant \beta_{i-1}$. If we show that each of the three quantities entering into the minimum in (19) exceeds β_i, we may conclude that $x_i \geqslant \beta_i$. We may assume also that $i \leqslant \min(p, q)$, since $\beta_i = 0$ ($\leqslant x_i$) for $i > \min(p, q)$. Thus (by (9)) $\alpha_i \geqslant \beta_i$. If $m - p < i$, $\beta_{i-m+p} \geqslant \beta_i$ by (8). By induction $x_{i-1} \geqslant \beta_{i-1} \geqslant \beta_i$. Hence the inequality $x_i \geqslant \beta_i$ for all $i \leqslant p$ is established.

It is a known fact (see [1]), because $x_1^2 \geqslant \cdots \geqslant x_q^2$ satisfy (17), there exists an n-square unitary matrix V such that the eigenvalues of

$$X^*X = V^*[j_1, \ldots, j_q|1, \ldots, n]A^*AV[1, \ldots, n|j_1, \ldots, j_q], \tag{20}$$

are

$$x_1^2, \ldots, x^2_{\min(m,q)}, \ldots, x_q^2. \tag{21}$$

Here

$$X = AV[1, \ldots, n|j_1, \ldots, j_q].$$

Thus XX^* has

$$x_1^2, \ldots, x^2_{\min(m,q)}, \ldots, x_m^2 \tag{22}$$

as eigenvalues. Because the inequalities (15) are satisfied, there exists an m-square unitary matrix U such that

$$UXX^*U^*[i_1,\ldots,i_p|i_1,\ldots,i_p]$$

has eigenvalues $\beta_1{}^2 \geqslant \cdots \geqslant \beta^2_{\min(p,q)} \geqslant \cdots = \beta_p{}^2$. It is now immediate that the submatrix

$$U[i_1,\ldots,i_p|1,\ldots,m]AV[1,\ldots,n|j_1,\ldots,j_q]$$

of UAV has (8) as its singular values. The proof of Theorem 2 is now finished.

We remark that the nonincreasing condition (8) is actually superfluous. More precisely, we have Theorem 2′.

THEOREM 2′. *Let arbitrary numbers* $\beta_1,\ldots,\beta_{\min(p,q)}$ *be given, such that* (9) *and* (10) *hold. Then the conclusions of Theorem* 2 *are valid.*

Proof. The proof amounts to showing that, if (9) and (10) are valid for not necessarily decreasing numbers $\beta_1,\ldots,\beta_{\min(p,q)}$, then (9) and (10) remain valid if $\beta_1,\ldots,\beta_{\min(p,q)}$ are rearranged into decreasing order. More precisely, let σ be a permutation of $1, 2,\ldots, s = \min(p, q)$ such that $\beta_{\sigma(1)} \geqslant \beta_{\sigma(2)} \geqslant \cdots \geqslant \beta_{\sigma(s)}$. If $\sigma(i) \geqslant i$, we then have $\beta_{\sigma(i)} \leqslant \alpha_{\sigma(i)} \leqslant \alpha_i$. If $\sigma(i) < i$, then for some $j < i$ we have $\sigma(j) \geqslant i$, and hence $\beta_{\sigma(i)} \leqslant \beta_{\sigma(j)} \leqslant \alpha_{\sigma(j)} \leqslant \alpha_i$. Thus $\beta_{\sigma(i)} \leqslant \alpha_i$ holds for all i. Similarly, for $i \leqslant \min(p + q - m, p + q - n)$, if $\sigma(i) \leqslant i$ then $\beta_{\sigma(i)} \geqslant \beta_i \geqslant \alpha_{i+m-p+n-q}$. If $\sigma(i) > i$, then for some $j > i$ we have $\sigma(j) \leqslant i$. But then $\beta_{\sigma(i)} \geqslant \beta_{\sigma(j)} \geqslant \alpha_{\sigma(j)+m-p+n-q} \geqslant \alpha_{i+m-p+n-q}$. Thus $\beta_{\sigma(i)} \geqslant \alpha_{i+m-p+n-q}$ for all $i \leqslant \min(p + q - m, p + q - n)$.

It now follows from Theorem 2 that we may find U and V such that the singular values of $UAV[i_1,\ldots,i_p|j_1,\ldots,j_q]$ are $\beta_{\sigma(1)},\ldots,\beta_{\sigma(\min(p,q))}$; that is, $\beta_1,\ldots,\beta_{\min(p,q)}$.

For the next theorems we let Q_{mp} denote the totality of $\binom{m}{p}$ sequences $\omega = \{i_1,\ldots,i_p\}$ of integers for which $1 \leqslant i_1 < \cdots < i_p \leqslant m$, and we let Q_{nq} denote the totality of sequences $\tau = \{j_1,\ldots,j_q\}$ of integers for which $1 \leqslant j_1 < \cdots < j_q \leqslant n$. We let

$$A[\omega|\tau] = A[i_1,\ldots,i_p|j_1,\ldots,j_q] \tag{23}$$

be the $p \times q$ submatrix of A at the intersection of the rows ω and the columns τ, and we let

$$\beta_{\omega\tau,1} \geqslant \beta_{\omega\tau,2} \geqslant \cdots \geqslant \beta_{\omega\tau,\min(p,q)}$$

be the singular values of (23). As before, we let $\beta_{\omega\tau,t} = 0$ for $t > \min(p, q)$.

THEOREM 3. *Define rational numbers* $\varphi_0, \ldots, \varphi_{m-p}$ *and* $\psi_0, \ldots, \psi_{n-q}$ *by the polynomial identities*

$$\prod_{i=p}^{m-1} \left(\frac{\lambda + i}{1 + i}\right) = \sum_{t=0}^{m-p} \varphi_t \lambda^{m-p-t}, \tag{24}$$

$$\prod_{i=q}^{n-1} \left(\frac{\lambda + i}{1 + i}\right) = \sum_{t=0}^{n-q} \psi_t \lambda^{n-q-t}. \tag{25}$$

For $i \leqslant \min(p, q)$, *define rational numbers* $d_0, \ldots, d_{\min(m+n-p-q,n-i)}$, *and* $d_0', \ldots, d'_{\min(m+n-p-q,n-i)}$ *(depending on* i*) by the polynomial identities*

$$\left(\sum_{r=0}^{\min(m-p,q-i)} \varphi_r \lambda^{m-p-r}\right)\left(\sum_{s=0}^{n-q} \psi_s \lambda^{n-q-s}\right) = \sum_{\rho=0}^{\min(m+n-p-q,n-i)} d_\rho \lambda^{m+n-p-q-\rho}, \tag{26}$$

and

$$\left(\sum_{r=0}^{\min(m-p,q-i)} \varphi_{m-p-r} \lambda^{m-p-r}\right)\left(\sum_{s=0}^{n-q} \psi_{n-q-s} \lambda^{n-q-s}\right) = \sum_{\rho=0}^{\min(m+n-p-q,n-i)} d_\rho' \lambda^{m+n-p-q-\rho}. \tag{27}$$

Then

$$\sum_{\rho=0}^{\min(m+n-p-q,n-i)} d_\rho \alpha_{i+\rho}^2 \leqslant \frac{1}{\binom{m}{p}} \frac{1}{\binom{n}{q}} \sum_{\omega \in Q_{mp}, \tau \in Q_{nq}} \beta_{\omega\tau,i}^2 \leqslant \sum_{\rho=0}^{\min(m+n-p-q,n-i)} d_\rho' \alpha_{i+\rho}^2. \tag{28}$$

Proof. Let $X_\tau = AV[1, \ldots, n | j_1, \ldots, j_q]$, and let

$$x_{\tau,1}^2 \geqslant x_{\tau,2}^2 \geqslant \cdots \geqslant x_{\tau,\min(p,q)}^2 \geqslant x_{\tau,\min(p,q)+1}^2 = \cdots = x_{\tau,m}^2 \quad (= 0)$$

be the roots of $X_\tau X_\tau^*$. Then by (15) we have

$$x_{\tau,i}^2 \geqslant \beta_{\omega\tau,i}^2 \geqslant x_{\tau,i+m-p}^2, \qquad 1 \leqslant i \leqslant p.$$

Using Theorem 16 of [1], we see that, for $i \leqslant p$ (and so for $i \leqslant \min(p, q)$),

$$\sum_{r=0}^{m-p} \varphi_r x_{\tau,i+r}^2 \leqslant \frac{1}{\binom{m}{p}} \sum_{\omega \in Q_{mp}} \beta_{\omega\tau,i}^2 \leqslant \sum_{r=0}^{m-p} \varphi_{m-p-r} x_{\tau,i+r}^2.$$

Since $x_{\tau,i+r} = 0$ whenever $i + r > \min(m, q)$, we get

$$\sum_{r=0}^{\min(m-p,q-i)} \varphi_r x_{\tau,i+r}^2 \leqslant \frac{1}{\binom{m}{p}} \sum_{\omega \in Q_{mp}} \beta_{\omega\tau,i}^2 \leqslant \sum_{r=0}^{\min(m-p,q-i)} \varphi_{m-p-r} x_{\tau,i+r}^2. \tag{29}$$

By (17), $\alpha_i \geqslant x_{\tau,i}^2 \geqslant \alpha_{i+n-q}^2$ for $1 \leqslant i \leqslant q$, and hence, by Theorem 16 of [1],

$$\sum_{s=0}^{n-q} \psi_s \alpha_{i+s}^2 \leqslant \frac{1}{\binom{n}{q}} \sum_{\tau \in Q_{nq}} x_{\tau,i}^2 \leqslant \sum_{s=0}^{n-q} \psi_{n-q-s} \alpha_{i+s}^2, \tag{30}$$

for $i \leqslant q$.

Summing (29) over τ and dividing by $\binom{n}{q}$, upon using (30) we obtain

$$\sum_{r=0}^{\min(m-p,q-i)} \varphi_r \sum_{s=0}^{n-q} \psi_s \alpha_{i+r+s}^2 \leqslant \frac{1}{\binom{m}{p}} \frac{1}{\binom{n}{q}} \sum_{\omega \in Q_{mp}, \tau \in Q_{nq}} \beta_{\omega\tau,i}^2$$

$$\leqslant \sum_{r=0}^{\min(m-p,q-i)} \varphi_{m-p-r} \sum_{s=0}^{n-q} \psi_{n-q-s} \alpha_{i+r+s}^2 \tag{31}$$

for $i \leqslant \min(p, q)$.

On the left side of (31), the coefficient of $\alpha_{i+\rho}^2$ is

$$\sum_{r=0, r+s=\rho}^{\min(m-p,q-i)} \sum_{s=0, r+s=\rho}^{n-q} \varphi_r \psi_s \quad \text{for} \quad 0 \leqslant \rho \leqslant \min(m + n - p - q, n - i).$$

However,

$$d_\rho = \sum_{r=0, r+s=\rho}^{\min(m-p,q-i)} \sum_{s=0, r+s=\rho}^{n-q} \varphi_r \psi_s \quad \text{for} \quad 0 \leqslant \rho \leqslant \min(m + n - p - q, n - i).$$

Thus the lower bound in (28) is established.

On the right side of (31) the coefficient of $\alpha_{i+\rho}^2$ is

$$\sum_{r=0,r+s=\rho}^{\min(m-p,q-i)} \sum_{s=0,r+s=\rho}^{n-q} \varphi_{m-p-r}\psi_{n-q-s}$$

$$\text{for} \quad 0 \leqslant \rho \leqslant \min(m+n-p-q, n-i).$$

However,

$$d_\rho' = \sum_{r=0,r+s=\rho}^{\min(m-p,q-i)} \sum_{s=0,r+s=\rho}^{n-q} \varphi_{m-p-r}\psi_{n-q-s}$$

$$\text{for} \quad 0 \leqslant \rho \leqslant \min(m+n-p-q, n-i).$$

The result is now at hand.

If p and q are large and i is small, so that $\min(m+n-p-q, n-i) = m+n-p-q$, formulas (28) provide convex combinations of $\alpha_i^2, \ldots, \alpha_{i+m-p+n-q}^2$ which serve as upper and lower bounds for the mean of the $\beta_{\omega\tau,i}^2$. Thus Theorem 3 provides a result sharper than can be established by applying Theorem 1, since Theorem 1 only asserts that the $\beta_{\omega\tau,i}^2$ lie between α_i^2 and $\alpha_{i+m-p+n-q}^2$. To see that in fact we have convex combinations, notice that (for these values of p, q, i),

$$\sum_{\rho=0}^{m+n-p-q} d_\rho = \sum_{\rho=0}^{m+n-p-q} \sum_{r=0,r+s=\rho}^{m-p} \sum_{s=0,r+s=\rho}^{n-q} \varphi_r\psi_s = \sum_{r=0}^{m-p} \varphi_r \sum_{s=0}^{n-q} \psi_s = 1,$$

since

$$\sum_{r=0}^{m-p} \varphi_r = 1 = \sum_{s=0}^{n-q} \psi_s.$$

Similarly

$$\sum_{\rho=0}^{m+n-p-q} d_\rho' = \sum_{\rho=0}^{m+n-p-q} \sum_{r=0,r+s=\rho}^{m-p} \sum_{s=0,r+s=\rho}^{n-q} \varphi_{m-p-r}\psi_{n-q-s}$$

$$= \sum_{r=0}^{m-p} \varphi_{m-p-r} \sum_{s=0}^{n-q} \psi_{n-q-s} = 1.$$

When p and q are small and i large, so that $\min(m+n-p-q, n-i) = n-i$, formula (28) may be regarded as providing subconvex combinations of $\alpha_i^2, \ldots, \alpha_n^2$ (convex combinations of $\alpha_i^2, \ldots, \alpha_n^2, 0$) which serve as bounds for the mean of the $\beta_{\omega\tau,i}^2$.

THEOREM 4. *Let*

$$f_{\omega,\tau}(\lambda) = (\lambda - \beta^2_{\omega\tau,1}) \cdots (\lambda - \beta^2_{\omega\tau,\min(p,q)}),$$

$$f(\lambda) = (\lambda - \alpha_1^2) \cdots (\lambda - \alpha^2_{\min(m,n)}).$$

Then

$$\sum_{\omega\in Q_{mp},\tau\in Q_{nq}} \lambda^{p-\min(p,q)} f_{\omega,\tau}(\lambda)$$

$$= \frac{1}{(m-p)!}\frac{1}{(n-q)!}\frac{d^{m-p}}{d\lambda^{m-p}}\lambda^{m-q}\frac{d^{n-q}}{d\lambda^{n-q}}\lambda^{n-\min(m,n)} f(\lambda). \tag{32}$$

Proof. Since the matrices $B_{\omega,\tau}B^*_{\omega,\tau}$ are $p \times p$ principal submatrices of the $m \times m$ matrix $X_\tau X_\tau^*$, we find [see **1**] that

$$\sum_{\omega\in Q_{mp}} (\lambda - \beta^2_{\omega\tau,1}) \cdots (\lambda - \beta^2_{\omega\tau,\min(p,q)})\lambda^{p-\min(p,q)}$$

$$= \frac{1}{(m-p)!}\frac{d^{m-p}}{d\lambda^{m-p}}(\lambda - x^2_{\tau,1}) \cdots (\lambda - x^2_{\tau,\min(m,q)})\lambda^{m-\min(m,q)}.$$

Since $X_\tau^* X_\tau$ is a principal $q \times q$ submatrix of the $n \times n$ matrix A^*A, we have

$$\sum_{\tau\in Q_{nq}} (\lambda - x^2_{\tau,1}) \cdots (\lambda - x^2_{\tau,\min(m,q)})\lambda^{q-\min(m,q)}$$

$$= \frac{1}{(n-q)!}\frac{d^{n-q}}{d\lambda^{n-q}}(\lambda - \alpha_1^2) \cdots (\lambda - \alpha^2_{\min(m,n)})\lambda^{n-\min(m,n)}.$$

Thus

$$\sum_{\omega\in Q_{mp},\tau\in Q_{nq}} f_{\omega,\tau}(\lambda)\lambda^{p-\min(p,q)}$$

$$= \frac{1}{(m-p)!}\frac{d^{m-p}}{d\lambda^{m-p}}\lambda^{m-q}\sum_{\tau\in Q_{nq}}(\lambda - x^2_{\tau,1}) \cdots (\lambda - x^2_{\tau,\min(m,q)})\lambda^{q-\min(m,q)}$$

$$= \frac{1}{(m-p)!}\frac{1}{(n-q)!}\frac{d^{m-p}}{d\lambda^{n-p}}\lambda^{m-q}\frac{d^{n-q}}{d\lambda^{n-q}}\lambda^{n-\min(m,n)} f(\lambda).$$

The proof is complete.

We now give the promised second proof of Theorem 1. For any $m \times n$ matrix A with singular values $\alpha_1 \geqslant \cdots \geqslant \alpha_{\min(m,n)}$ the roots of the $(m+n)$-square Hermitian matrix

$$M = \begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix}$$

are $\pm\alpha_1, \ldots, \pm\alpha_{\min(m,n)}$, 0 (with multiplicity $n + m - 2\min(m, n)$).

To see this, observe that

$$\begin{aligned} \det(\lambda I_{m+n} - M) &= \det \begin{bmatrix} I_m & \lambda^{-1}A \\ 0 & I_n \end{bmatrix} \det \begin{bmatrix} \lambda I_m & -A \\ -A^* & \lambda I_n \end{bmatrix} \\ &= \det \begin{bmatrix} \lambda I_m - \lambda^{-1}AA^* & 0 \\ -A^* & \lambda I_n \end{bmatrix} \\ &= \lambda^n \det(\lambda I_m - \lambda^{-1}AA^*) = \lambda^{n-m}\det(\lambda^2 I_m - AA^*). \end{aligned}$$

The principal $(p + q)$-square submatrix of M, obtained by deleting all rows and columns except rows and columns $i_1, \ldots, i_p, m + j_1, \ldots, m + j_q$, is

$$\begin{bmatrix} 0 & A[i_1, \ldots, i_p | j_1, \ldots, j_q] \\ A[i_1, \ldots, i_p | j_1, \ldots, j_q]^* & 0 \end{bmatrix}.$$

Using the inequalities connecting the eigenvalues of a $(p + q)$-principal submatrix of Hermitian matrix M with the eigenvalues of M, we obtain the inequalities (9) and (10).

REFERENCES

1 R. C. Thompson, Principal submatrices of normal and Hermitian matrices, *Illinois J. Math.* **10**(1966), 296–308.

2 R. C. Thompson and P. McEnteggert, Principal submatrices II: The upper and lower quadratic inequalities, *Linear Algebra* **1**(1968), 211–243.

3 R. C. Thompson, Principal submatrices III: Linear inequalities, *J. Res. Nat. Bur. Standards, Sect. B*, **72**, 7–22 (1968).

4 R. C. Thompson, Principal submatrices IV: On the independence of the eigenvalues of different principal submatrices, *Linear Algebra* **2**(1969), 355–374.

5 R. C. Thompson, Principal submatrices V: Some results concerning principal submatrices of arbitrary matrices, *J. Res. Nat. Bur. Standards, Section B*, **72**, 115–125 (1968).

6 R. C. Thompson, Principal submatrices VI: Cases of equality in certain linear inequalities, *Linear Algebra* **2**(1969), 375–379.

7 R. C. Thompson, Principal submatrices VII: Further results concerning matrices with equal principal minors, *J. Res. Nat. Bur. Standards*, Section B, **72**, 249–252 (1968).

8 R. C. Thompson, Principal submatrices VIII: Principal sections of a pair of forms, *Rocky Mountain J. Math.*, in press.

Received August, 1969