More on decompositions of edge-colored complete graphs

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Abstract

Let \( \mathcal{G} \) be a family of graphs whose edges are colored with elements from a set \( R \) of \( r \) colors. We assume no two vertices of \( G \) are joined by more than one edge of color \( i \) for any \( i \in R \), for each \( G \in \mathcal{G} \). \( K_n^{(r)} \) will denote the complete graph with \( r \) edges joining any pair of distinct vertices, one of each of the \( r \) colors. We describe necessary and asymptotically sufficient conditions on \( n \) for the existence of a family \( \mathcal{D} \) of subgraphs of \( K_n^{(r)} \), each of which is an isomorphic copy of some graph in \( \mathcal{G} \), so that each edge of \( K_n^{(r)} \) appears in exactly one of the subgraphs in \( \mathcal{D} \).

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1. Introduction and summary

We are concerned with extensions of the following theorem from [7].

\textbf{Theorem 1.} Let \( G \) be a graph with \( m \) edges, and let \( g \) be the greatest common divisor of the degrees of the vertices of \( G \). Then \( K_n \) admits a \( G \)-decomposition for all sufficiently large integers \( n \) for which

\[
\begin{align*}
 n - 1 &\equiv 0 \pmod{g}, \\
 n(n - 1) &\equiv 0 \pmod{2m}.
\end{align*}
\]

(1)

Here \( K_n \) denotes the complete graph on \( n \) vertices and a \( G \)-decomposition of \( K_n \) is a family \( \mathcal{D} \) of subgraphs of \( K_n \), each isomorphic to \( G \), such that each edge of \( K_n \) occurs in exactly one of the subgraphs \( C \in \mathcal{D} \).

There are applications for several types of extensions of this result. In particular, the idea of decomposing 'edge-colored' complete graphs is important.

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We consider graphs and multigraphs \( G \) whose edges have colors from a fixed set \( R \) of \( r \) colors, and families \( \mathcal{G} \) of such graphs. We consider only finite graphs and we will also assume \( \mathcal{G} \) is finite. \( K_n^{(r)} \) will denote the complete graph on \( n \) vertices with \( r \) edges joining any unordered pair \( \{x, y\} \) of distinct vertices, exactly one edge of color \( i \) for each \( i \in R \).

A \( \mathcal{G} \)-decomposition of \( K_n^{(r)} \) is a set \( \mathcal{D} \) of subgraphs of \( K_n^{(r)} \), each of which is a copy of some \( G \in \mathcal{G} \), such that each edge of \( K_n^{(r)} \) appears in exactly one member of \( \mathcal{D} \). When \( \mathcal{G} \) consists of a single graph \( G \), we speak of a \( G \)-decomposition.

To be precise, by an isomorphic copy, or simply a copy, of \( G \) in \( K_n^{(r)} \), we mean a subgraph \( C \) of \( K_n^{(r)} \) with the property that there exists a one-to-one mapping \( \phi \) from the vertices of \( G \) onto those of \( C \) such that for any two vertices \( x, y \) of \( G \), the edge of color \( i \) in \( K_n^{(r)} \) that joins \( \phi(x) \) and \( \phi(y) \) in \( K_n^{(r)} \) is present as an edge of \( C \) if and only if there is an edge of color \( i \) joining \( x \) and \( y \) in \( G \).

See [1,3] for a number of examples of problems in combinatorial designs that can either be phrased naturally in the language of decompositions of edge-colored graphs, or can be seen to be equivalent to problems about such decompositions. While applications often require the introduction of decompositions of edge-colored directed graphs, we state theorems only in the case of undirected graphs in this paper. This makes definitions, notation, and proofs very much simpler.

Given a family \( \mathcal{G} \) of edge-\( r \)-colored graphs, the set \( N \) of positive integers \( n \) so that \( K_n^{(r)} \) admits a \( \mathcal{G} \)-decomposition is easily seen to be PBD-closed in the terminology of [5]. From [6], there exist nonnegative integers \( a \) and \( b \), depending on \( \mathcal{G} \), so that the conditions
\[
\begin{align*}
n - 1 &\equiv 0 \pmod{a}, \\
n(n - 1) &\equiv 0 \pmod{b},
\end{align*}
\]
on \( n \) are both necessary and asymptotically sufficient for \( n \in N \). By asymptotically sufficient, we mean that \( N \) contains all integers \( n \) satisfying these congruences and such that \( n \geq n_0 \), where \( n_0 \) is some constant depending on \( \mathcal{G} \).

These parameters \( a \) and \( b \) are unique when we require that either \( a = b = 0 \) or that \( a \) divides \( b \) and \( b > 0 \) is even. The case \( b = 0 \) is equivalent to \( N = \{1\} \), when no nontrivial \( \mathcal{G} \)-decompositions exist. Otherwise \( N \) contains in particular all integers \( n \equiv 1 \pmod{b} \), and so is infinite.

It is our goal to compute these parameters \( a \) and \( b \) directly from \( \mathcal{G} \). For example, when \( \mathcal{G} \) contains a single graph \( G \) and \( r = 1 \), Theorem 1 says \( a \) is the g.c.d. of the degrees, and \( b \) is twice the number of edges, of \( G \). This is very simple. It is harder to determine \( a \) and \( b \) when \( |\mathcal{G}| > 1 \) or \( r > 1 \).

In the next section, we define parameters \( \alpha(\mathcal{G}) \) and \( \beta(\mathcal{G}) \) for any family \( \mathcal{G} \) of edge-colored graphs. Computation of these parameters in the general case requires linear programming and integral row operations on matrices (Hermite form), but this is still relatively simple. (We define \( \beta \) here in such a way to ensure that it is an even number, and this may differ somewhat from earlier definitions.)

The following theorem appears in Lamken and Wilson [1]. Here a simple graph is one in which there is at most one edge joining any two distinct vertices. \( K_n^{(r)} \) is not simple when \( r > 1 \), but the graphs in \( \mathcal{G} \) are required to be simple in the theorem below.

**Theorem 2.** Let \( \mathcal{G} \) be a family of simple edge-\( r \)-colored graphs. Then the conditions
\[
\begin{align*}
n - 1 &\equiv 0 \pmod{\alpha(\mathcal{G})}, \\
n(n - 1) &\equiv 0 \pmod{\beta(\mathcal{G})},
\end{align*}
\]
on \( n \) are necessary and asymptotically sufficient for the existence of a \( \mathcal{G} \)-decomposition of \( K_n^{(r)} \).

**Remark.** In [1], an additional hypothesis, that \( \mathcal{G} \) be ‘admissible’, was required. We change the definitions of \( \alpha(\mathcal{G}) \) and \( \beta(\mathcal{G}) \) slightly in this paper, so that we can drop this term in the statement of the theorem. See Section 2.

It has become clear that there are applications of \( \mathcal{G} \)-decompositions when the graphs in \( \mathcal{G} \) are not simple. See [2,3] for examples. We will require that the graphs in \( \mathcal{G} \) be colorwise-simple; here an edge-colored graph \( G \) is colorwise-simple when no two vertices of \( G \) are joined by more than one edge of any color-vertices may be joined by several edges, but those edges must have different colors. Of course, \( K_n^{(r)} \) and all of its subgraphs are colorwise-simple.

Necessary and asymptotically sufficient conditions for \( G \)-decompositions of \( K_n^{(r)} \) for certain graphs \( G \) are given in a recent paper of Mutoh [3]. We state this result below following a few definitions.
A colorset of a graph $G$ is the set of colors on the edges joining some pair of adjacent vertices (vertices joined by at least one edge). We will use $G([x, y])$ to denote the set of colors on the edges of $G$ that join adjacent vertices $x$ and $y$. If $x$ and $y$ are not adjacent in $G$, it will be convenient to let $G([x, y]) = 0$ (the number 0, not the empty set).

A family $\mathcal{F}$ of sets is tree-ordered when any two members $A, B$ of $\mathcal{F}$ are either disjoint or comparable (i.e. $A \subseteq B$ or $B \subseteq A$). (This is a different, weaker meaning of this term than in [3].)

By the colorset-family of a family $\mathcal{G}$ of edge-colored graphs, we mean the set of all colorsets of all graphs $G \in \mathcal{G}$.

**Theorem 3.** Let $\mathcal{G}$ be a family of colorwise-simple edge-$r$-colored graphs. Further assume that the colorset-family of $\mathcal{G}$ is tree-ordered. Then the conditions

$$n - 1 \equiv 0 \pmod{\alpha(\mathcal{G})},$$
$$n(n - 1) \equiv 0 \pmod{\beta(\mathcal{G})},$$

(3)
on n are necessary and asymptotically sufficient for the existence of a $\mathcal{G}$-decomposition of $K_n^{(r)}$.

Here the parameters $\alpha(\mathcal{G})$ and $\beta(\mathcal{G})$ are the same as in Theorem 2. It is clear that Theorem 3 implies Theorem 2, since all colorsets of a simple graph are singletons and so the colorset-family of a simple graph is always tree-ordered.

Mutoh’s theorem in [3] also requires as an additional hypothesis that every singleton $\{i\}, i \in R$, occurs as a colorset of $G$. But we shall show in Section 6 that Theorem 3 holds as stated.

The conditions (3) remain necessary for the existence of $\mathcal{G}$-decompositions for any family of colorwise-simple edge-colored graphs, but they fail to be asymptotically sufficient in general. To go further, we must change the definition of $\alpha(\mathcal{G})$ and $\beta(\mathcal{G})$, or more precisely, replace them with parameters $\alpha'(\mathcal{G})$ and $\beta'(\mathcal{G})$ that will be defined in Section 2. Our new result is

**Theorem 4.** Let $\mathcal{G}$ be a family of colorwise-simple edge-$r$-colored graphs. Then the conditions

$$n - 1 \equiv 0 \pmod{\alpha'(\mathcal{G})},$$
$$n(n - 1) \equiv 0 \pmod{\beta'(\mathcal{G})},$$

(4)
on n are necessary and asymptotically sufficient for the existence of a $\mathcal{G}$-decomposition of $K_n^{(r)}$.

Unfortunately, these new parameters $\alpha'(\mathcal{G})$ and $\beta'(\mathcal{G})$ can be more difficult to compute, given $\mathcal{G}$, than $\alpha(\mathcal{G})$ and $\beta(\mathcal{G})$.

We give definitions in Section 2. The proof of Theorem 4 is summarized in Section 3, but is based on a result (Theorem 7) about the existence of decompositions of certain complete multipartite graphs not proved there. Theorem 7 is proved in Section 5 following an analysis of a system of linear equations related to the decomposition problem in Section 4. At this point, the proof of Theorem 4 will be complete.

While the techniques are often similar to those used in [1,3,7,6], new ideas are required, especially in the definitions of $\alpha'$ and $\beta'$ and in Section 4. The concept of ‘resolvability’ of a set-system plays a significant role in this subject, when the graphs in $\mathcal{G}$ are not simple.

In Section 6, we compare $\beta(\mathcal{G})$ and $\beta'(\mathcal{G})$, and we derive Theorem 3 from Theorem 4.

We may ask when we can find a multiset of copies of graphs in $\mathcal{G}$ so that each edge of color $i$ of $K_n^{(r)}$ is in $\lambda_i$ copies of $\mathcal{G}$. We can view this as a decomposition problem by introducing complete graphs where any two vertices are joined by $\lambda_i$ edges of color $i, i = 1, 2, \ldots, p$. Now we need no longer require that the graphs in $\mathcal{G}$ be colorwise-simple. Necessary and asymptotically sufficient conditions for decompositions of such complete graphs are given in Section 7.

Work on balanced decompositions and decompositions of directed edge-colored complete graphs is in progress and will appear elsewhere.

2. Definitions and preliminaries

For an edge-$r$-colored graph $G$, let $\mu(G) = (m_1, m_2, \ldots, m_r)$ where $m_i$ is the number of edges of color $i$ in $G$. If there exists a $\mathcal{G}$-decomposition of $K_n^{(r)}$ in which there appear $b_G$ copies of $G$, then clearly

$$\sum_{G \in \mathcal{G}} b_G \mu(G) = \mu(K_n^{(r)}) = \frac{n(n - 1)}{2} (1, 1, \ldots, 1).$$
Thus
\[ \sum_{G \in \mathcal{G}} c_G \mu(G) = (1, 1, \ldots, 1) \]  
(5)
for some nonnegative rational numbers \( c_G \). A graph \( H \in \mathcal{G} \) will be called *useless* in \( \mathcal{G} \) when \( c_H = 0 \) whenever (5) holds with nonnegative rational numbers \( c_G, G \in \mathcal{G} \). Copies of such graphs obviously cannot occur in any \( \mathcal{G} \)-decomposition of \( K_n^{(r)} \). Let \( \mathcal{G}^* \) denote the set of graphs in \( \mathcal{G} \) that are not useless. (In [1,3], \( \mathcal{G} \) is said to be ‘admissible’ when \( \mathcal{G} = \mathcal{G}^* \).)

Then a necessary condition for the existence of a \( \mathcal{G} \)-decomposition of \( K_n^{(r)} \) is that there exist nonnegative integers so that
\[ \sum_{G \in \mathcal{G}^*} b_G \mu(G) = (t, t, \ldots, t), \]
where \( t = n(n - 1)/2 \). Let \( B \) be the set of all integers \( 2t \) so that the \( r \)-vector \((t, t, \ldots, t)\) is in the module over the integers \( \mathbb{Z} \) generated by the vectors \( \mu(G), G \in \mathcal{G}^* \). We let \( \beta(\mathcal{G}) \) denote the unique nonnegative integer that generates the ideal \( B \). (So, in particular, \( \beta(\mathcal{G}) = 0 \) when \( \mathcal{G}^* = \emptyset \).) It is clear that
\[ n(n - 1) \equiv 0 \pmod{\beta(\mathcal{G})} \]
is a necessary condition for the existence of a \( \mathcal{G} \)-decomposition of \( K_n^{(r)} \).

For a vertex \( x \) of an edge-\( r \)-colored graph \( G \), let
\[ \tau(x, G) = (\deg_1(x), \deg_2(x), \ldots, \deg_r(x)), \]
where \( \deg_j(x) \) denotes the degree of vertex \( x \) in the spanning subgraph of \( G \) determined by the edges of color \( j \), \( 1 \leq j \leq r \). Let \( A \) be the set of all integers \( t \) so that the \( r \)-vector \((t, t, \ldots, t)\) is in the module over the integers \( \mathbb{Z} \) generated by the vectors \( \tau(x, G) \), \( x \) a vertex of \( G \), \( G \in \mathcal{G}^* \). We let \( \alpha(\mathcal{G}) \) denote the unique nonnegative integer that generates the ideal \( A \). So when \( r = 1 \), \( \alpha(\mathcal{G}) \) is the greatest common divisor of all degrees of vertices of graphs in \( \mathcal{G} \). If there exists a \( \mathcal{G} \)-decomposition \( \mathcal{D} \) of \( K_n^{(r)} \) and \( u \) is any vertex of \( K_n^{(r)} \), then
\[ \sum_{C \in \mathcal{D}} \tau(u, C) = \tau(u, K_n^{(r)}) = (n - 1, n - 1, \ldots, n - 1), \]
where the sum is extended over all copies \( C \in \mathcal{D} \) that contain the vertex \( u \). Thus \( n - 1 \in A \) and
\[ n - 1 \equiv 0 \pmod{\alpha(\mathcal{G})} \]
is a necessary condition for the existence of a \( \mathcal{G} \)-decomposition of \( K_n^{(r)} \).

Note that the sum of \( \tau(x, G) \) over the vertices \( x \) of a graph \( G \) is \( 2\mu(G) \), so it follows that \( B \subseteq A \), and hence \( \alpha(\mathcal{G}) \) divides \( \beta(\mathcal{G}) \). These are the parameters \( \alpha(\mathcal{G}) \) and \( \beta(\mathcal{G}) \) for which Theorems 2 and 3 hold.

In the case that \( \mathcal{G} \) consists of a single graph \( G \), then \( G \) is useless if \( G \) does not have exactly the same number of edges of each color. If \( G \) has \( m \) edges of each color, then \( \beta(\mathcal{G}) = 2m \). It is clear that Theorem 2 implies Theorem 1.

We introduce the module \( \mathcal{M} \) over the integers \( \mathbb{Z} \) consisting of formal sums of the nonempty subsets of \( R \). For a partition \( \Pi \) of \( R \), let \( \hat{\Pi} \) denote the formal sum of the subsets in \( \Pi \). For example, when \( r = 6 \),
\[ \Pi = \{\{1, 2, 3\}, \{4\}, \{5, 6\}\}, \quad \hat{\Pi} = \{1, 2, 3\} + \{4\} + \{5, 6\} \in \mathcal{M}. \]

We say that an element of \( \mathcal{M} \) is *resolvable* if and only if it can be written as
\[ \sum_{\Pi \in \Pi} c_{\Pi} \hat{\Pi} \]
with nonnegative integer coefficients \( c_{\Pi} \), where \( \Pi \) denotes the set of all partitions of \( R \). An example of a resolvable element of \( R \) when \( r = 5 \) is
\[ 2\{a, b\} + \{a, b, c\} + \{c\} + \{c, d\} + \{d\} + \{d, e\} + 2\{e\} \]
\[ = (\{a, b\} + \{c, d\} + \{e\}) + (\{a, b\} + \{c\} + \{d, e\}) + (\{a, b, c\} + \{d\} + \{e\}). \]
For an edge-colored graph $G$, let $\hat{G}$ be the sum of all the colorsets of $G$, counted with multiplicities; that is,

$$\hat{G} = \sum_{\{x, y\}} G[\{x, y\}],$$

where the sum is extended over all unordered pairs $\{x, y\}$ of vertices of $G$. We may call $\hat{G}$ the colorset-distribution of $G$.

The following theorem is from Li Marzi et al. [2]; see Theorem 2.1 there. It is also proved in Mutoh’s paper [3]; see the proof of Theorem 3.2 there. (The terminology varies slightly.)

**Theorem 5.** Let $H$ be a colorwise-simple edge-$r$-colored graph with $m$ edges, $m > 0$, of each of $r$ different colors and whose colorset-distribution $\hat{H}$ is a resolvable element of $\mathcal{M}$. Then $K_n^{(r)}$ admits a $H$-decomposition for infinitely many values of $n$.

From Theorem 5 we may derive the following.

**Theorem 6.** Let $\mathcal{G}$ be a family of colorwise-simple edge-$r$-colored graphs. Then there exist values of $n > 1$ so that $K_n^{(r)}$ admits a $\mathcal{G}$-decomposition if and only if there exist nonnegative integers $b_G$ so that

$$\sum_{G \in \mathcal{G}} b_G \hat{G}$$

is a resolvable element of $\mathcal{M}$.

**Proof.** Suppose $\sum_{G \in \mathcal{G}} b_G \hat{G}$ is a resolvable element of $\mathcal{M}$, with nonnegative integers $b_G$. Let $H$ be the disjoint union of $b_G$ copies of $G$, $G \in \mathcal{G}$. The colorset-distribution of $H$ is $\sum_{G \in \mathcal{G}} b_G \hat{G}$. By Theorem 5, there are values of $n > 1$ so that $K_n^{(r)}$ admits an $H$-decomposition. Then $K_n^{(r)}$ admits a $\mathcal{G}$-decomposition.

Suppose there exists a $\mathcal{G}$-decomposition of $K_n^{(r)}$ in which there appear $b_G$ copies of $G$. By the definition of decomposition, there is for every pair $\{x, y\}$ of distinct vertices of $K_n^{(r)}$ a partition $H_{\{x, y\}}$ of $R$ whose sets are the colorsets $C[\{x, y\}]$ as $C$ ranges over the copies in the decomposition that contain edges joining $x$ and $y$. Then, in $\mathcal{M}$,

$$\sum_{G \in \mathcal{G}} b_G \hat{G} = \sum_{C \in \mathcal{G}} \hat{C} = \sum_{C \in \mathcal{G}} \sum_{\{x, y\}} C[\{x, y\}] = \sum_{\{x, y\}} \hat{H}_{\{x, y\}},
\tag{6}$$

where the last sum is extended over the $n(n - 1)/2$ unordered pairs of vertices of $K_n^{(r)}$. □

Eq. (6) is very important and motivates much of the new material in this paper. Consider equations in $\mathcal{M}$ of the form

$$\sum_{G \in \mathcal{G}} b_G \hat{G} = \sum_{\Pi \in \Pi} a_{\Pi} \hat{H}, \quad b_G \geq 0, \quad a_{\Pi} \geq 0,
\tag{7}$$

where the coefficients $b_G$ and $a_{\Pi}$ are nonnegative integers for all $G \in \mathcal{G}$ and $\Pi \in \Pi$.

Let $H \in \mathcal{G}$. If $b_H = 0$ in every expression of the form (7), we call $H$ unusable. Then, in view of (6), copies of $H$ cannot appear in any $\mathcal{G}$-decomposition of $K_n^{(r)}$. For example, when $\mathcal{G} = \{G\}$ consists of a single graph, $G$ is not unusable if and only if $t \hat{G}$ is resolvable for some positive integer $t$. Let $\mathcal{G}^{\#}$ be the set of graphs $H \in \mathcal{G}$ that are not unusable.

Let $\Sigma \in \Pi$. If $a_{\Sigma} = 0$ in every expression of the form (7), we call $\Sigma$ irrelevant. Then, in view of (6), $\Sigma$ can never be equal to a partition $H_{\{x, y\}}$ of $R$ arising from a pair of vertices and a $\mathcal{G}$-decomposition. For example, suppose $r = 6$ and $\mathcal{G}$ consists of a single graph $G$ with

$$\hat{G} = \{a, b\} + \{b, c\} + \{a, c\} + \{a', b'\} + \{b', c'\} + \{c', a'\} + \{a, a'\} + \{b, b'\} + \{c, c'\}.$$
There are four partitions of \( R \) into colorsets, one of which is irrelevant, namely \( \{a', a\}, \{b', b\}, \{c, c\} \). Let \( \Pi^\# \) be the set of not irrelevant partitions of \( R \).

(Determination of whether a graph is unusable or a partition is irrelevant in a specific example can be done by solving a linear programming problem, but we must, of course, first make a list of all partitions of \( R \) into colorsets.)

Fix \( r \) and \( \mathcal{G} \). Let \( \mathcal{M} \) be the submodule of \( \mathcal{A} \) generated by \( \hat{G}, G \in \mathcal{G}^\# \). The set \( B' \) of integers of the form \( 2\sum_{\Pi \in \Pi^\#} a_{\Pi} \hat{\Pi} \) where the coefficients \( a_{\Pi} \) are integers (positive, negative, or zero) such that

\[
\sum_{\Pi \in \Pi^\#} a_{\Pi} \hat{\Pi} \in \mathcal{M}
\]

is an ideal in \( \mathbb{Z} \). The unique nonnegative generator of this ideal will be denoted by \( \beta' (\mathcal{G}) \). If there exists a \( \mathcal{G} \)-decomposition of \( K_n^{(r)} \), then (6) shows that \( n(n-1) \in B' \). So

\[
n(n-1) \equiv 0 \pmod{\beta'(\mathcal{G})}
\]

is a necessary condition for the existence of such a decomposition.

For each vertex \( v \) of an edge-colored graph \( G \), let \( \hat{v} \) be the sum of all colorsets that appear at \( v \), i.e.

\[
\hat{v} = \sum_u G[\{v, u\}],
\]

where the sum is extended over all vertices \( u \) of \( G \), \( u \neq v \) (or over those vertices \( u \) adjacent to \( v \)). We may call \( \hat{v} \) the \textit{colorset-type} of \( v \) in \( G \).

Fix \( r \) and \( \mathcal{G} \). Let \( \mathcal{F} \) be the submodule of \( \mathcal{A} \) generated by \( \hat{u}, v \in V(G), G \in \mathcal{G}^\# \). The set \( A' \) of integers of the form \( \sum_{\Pi \in \Pi^\#} a_{\Pi} \hat{\Pi} \) where the coefficients \( a_{\Pi} \) are integers such that

\[
\sum_{\Pi \in \Pi^\#} a_{\Pi} \hat{\Pi} \in \mathcal{F}
\]

is an ideal in \( \mathbb{Z} \). The unique nonnegative generator of this ideal will be denoted by \( \alpha' (\mathcal{G}) \). Suppose there exists a \( \mathcal{G} \)-decomposition of \( K_n^{(r)} \) and let \( u \) be any vertex of \( K_n^{(r)} \). The sum of the colorsets \( C[\{u, y\}] \) over all copies \( C \) in the decomposition that contain edges incident with \( u \) and over the vertices \( y, y \neq u \), of \( K_n^{(r)} \) is the sum \( \sum_{y \neq u} \hat{\Pi}[u, y] \) (in the notation of the proof of Theorem 6) and is also clearly in \( \mathcal{F} \). Thus \( n-1 \in A' \). So

\[
n - 1 \equiv 0 \pmod{\alpha'(\mathcal{G})}
\]

is a necessary condition for the existence of such a decomposition.

The sum of \( \hat{v} \) over vertices \( v \) of \( G \) is \( 2\hat{G} \), so \( 2\mathcal{E} \subseteq \mathcal{F} \), and we see that \( \alpha'(\mathcal{G}) \) divides \( \beta'(\mathcal{G}) \). These are the parameters \( \alpha'(\mathcal{G}) \) and \( \beta'(\mathcal{G}) \) for which Theorem 4 holds.

When \( \beta'(\mathcal{G}) > 0 \), both \( \mathcal{G}^\# \) and \( \Pi^\# \) are nonempty. For \( H \in \mathcal{G}^\# \), there is an instance of (7) with \( b_H > 0 \). For \( \Sigma \in \Pi^\# \), there is an instance of (7) with \( a_{\Sigma} > 0 \). By adding such instances, we find an expression

\[
\sum_{G \in \mathcal{G}^\#} b_G^* \hat{G} = \sum_{\Pi \in \Pi^\#} a_{\Pi}^* \hat{\Pi}
\]

in which all coefficients \( b_G^* \) and \( a_{\Pi}^* \) are positive integers. This will be important in Section 4.

3. Summary of the proof of Theorem 4

We use the following result, which will be proved in Section 5. Here \( M_{q,r}^{(r)} \) denotes the complete multipartite graph with \( nq \) vertices that are partitioned into \( n \) ‘groups’ of size \( q \), and where vertices in different groups are joined by \( r \) edges, one of each of the \( r \) colors (and no edges join vertices in the same group). \( \mathcal{G} \)-decompositions of \( M_{q,r}^{(r)} \) are defined analogously to those of \( K_n^{(r)} \).
Theorem 7. Let $\mathcal{G}$ be a family of colorwise-simple edge-$r$-colored graphs and let $n$ be a positive integer. Assume that each graph in $\mathcal{G}$ has at most $n - 2$ vertices and that
\[
\begin{align*}
  n - 1 &\equiv 0 \pmod{\mathcal{G}(\mathcal{G})}, \\
  n(n - 1) &\equiv 0 \pmod{\mathcal{G}(\mathcal{G})}.
\end{align*}
\] (9)

Then for every sufficiently large prime power $q$ and every $d \geq n^2$, there exists a $\mathcal{G}$-decomposition of $M_{q^d,n}^{(r)}$.

Proof of Theorem 4. Assume $\mathcal{G}(\mathcal{G}) > 0$.

Let $N$ be the set of positive integers $n$ so that $K_n^{(r)}$ admits a $\mathcal{G}$-decomposition. As remarked in Section 1, $N$ is PBD-closed in the terminology of [5]. By Theorem 5, $N$ contains integers $t > 1$. The main result of [5] is that $N$ is eventually periodic with some period $\rho > 1$. This means that if $t_0 \in N$, then $N$ contains all sufficiently large (with respect to $N$ and $t_0$) integers $t \equiv t_0 \pmod{\rho}$.

We may assume $\rho$ is a multiple of $\mathcal{G}(\mathcal{G})$. By definition, $1 \in N$, so $N$ contains all sufficiently large integers $t \equiv 1 \pmod{\rho}$.

Since every integer $n$ satisfying the congruences (9) is congruent modulo $\rho$ to one of the finitely many integers $n$, $0 < n < \rho$, that satisfy the congruences (9), it will suffice to show that for each integer $n$ that satisfies the congruences (9), there exists $n_0 \in N$ with $n_0 \equiv n \pmod{\rho}$.

Given $n$ satisfying the congruences (9), first choose $n_1 \equiv n \pmod{\rho}$ large enough so that no graph in $\mathcal{G}$ has more than $n_1 - 2$ vertices. By Theorem 7, for every sufficiently large prime power $q$, $M_{q^d,n_1}^{(r)}$ admits a $\mathcal{G}$-decomposition, where here we take $d = n_1^2$. Choose $q \equiv 1 \pmod{\rho}$ large enough so that $K_{q^d,n_1}^{(r)}$ admits a $\mathcal{G}$-decomposition. The complete multipartite graph $M_{q^d,n_1}^{(r)}$ has $n_2 = q^d n_1$ vertices. A $\mathcal{G}$-decomposition of $K_{q^d,n_1}^{(r)}$ may be obtained by starting with the graphs in a $\mathcal{G}$-decomposition of $M_{q^d,n_1}^{(r)}$ and including the graphs of a $\mathcal{G}$-decomposition of the complete edge-$r$-colored graph with vertex set $U$ for each of the $n_1$ groups $U$ of the complete multipartite graph, which decompositions exists since $|U| = q^d$. Thus $n_2 \in N$, and $n_2 \equiv n \pmod{\rho}$. □

4. A system of equations

Let $\mathcal{G}$ be a family of colorwise-simple edge-$r$-colored graphs. Let $\mathcal{G}#$ and $\Pi#$ be as in Section 2. Let $\mathcal{G}$ be the set of all colorsets that appear in partitions $\Pi \in \Pi#$. Let $\mathcal{G}_n$ denote the set of all copies in $K_n^{(r)}$ of graphs in $\mathcal{G}#$. Let $E$ be the set of $n(n - 1)/2$ unordered pairs of vertices of $K_n^{(r)}$.

We introduce a (large but sparse) system of linear equations. There is to be one variable $z_C$ for each $C \in \mathcal{G}_n$ and one variable $u_{[x,y]}$ for every partition $\Pi \in \Pi#$. Let every $\{x, y\} \in E$. For each unordered pair $\{x, y\} \in E$, we require that the following holds:
\[
\sum_{C \in \mathcal{G}_n} z_C C([x,y]) = \sum_{\Pi \in \Pi#} u_{\Pi,[x,y]} \hat{\Pi} \quad \text{in} \quad \mathcal{H} \quad \text{with} \quad \sum_{\Pi \in \Pi#} u_{\Pi,[x,y]} = 1.
\] (10)

This is equivalent to $|\mathcal{G}| + 1$ linear constraints on the variables, since in addition to the sum on the right, the coefficient of a colorset $S \in \mathcal{G}$ must agree on both sides of the equation in $\mathcal{H}$. Thus (10) for all $\{x, y\}$ is equivalent to a system of linear equations with $|\mathcal{G}_n| + |E| \cdot |\Pi#|$ variables and $|E| \cdot (|\mathcal{G}| + 1)$ linear equations.

Some motivation for consideration of this system of equations is that it has a solution in 0’s and 1’s if there exists a $\mathcal{G}$-decomposition $\mathcal{D}$ of $K_n^{(r)}$. We have, as in (6),
\[
\sum_{C \in \mathcal{D}} C([x,y]) = \hat{\Pi}_{[x,y]} \quad \text{in} \quad \mathcal{H}
\] (11)

for some partition $\Pi_{[x,y]}$ of $R$ into colorsets. Then a solution of (10) is given by setting
\[
z_C = \begin{cases} 1 & \text{if } C \in \mathcal{D}, \\ 0 & \text{otherwise,} \end{cases}
\]
Theorem 8. Let $\mathcal{G}$ be a family of colorwise-simple edge-$r$-colored graphs and let $n$ be a positive integer. Assume that each graph in $\mathcal{G}$ has at most $n - 2$ vertices and that

$$n - 1 \equiv 0 \pmod{x'(\mathcal{G})},$$
$$n(n - 1) \equiv 0 \pmod{\beta'(\mathcal{G})}.$$ (12)

Then there exist integers (positive, negative, or zero) $z_C$ and $u_{\Pi,[x,y]}$, so that (10) holds for every unordered pair $\{x, y\}$ of vertices of $K_n^{(r)}$.

For the proof, we require the following two lemmas. We write $a \equiv b$ to mean $a$ and $b$ differ by an integer. In particular $a \equiv 0$ if and only if $a$ is an integer.

Lemma 9. Let $M$ be a rational matrix $M$ and $b$ a rational column vector of the same height. The system of linear equations $Mx = b$ has an integral solution $x$ if and only if for rational row vectors $y$ of the appropriate length,

$$yM \equiv 0 \quad \text{implies} \quad yb \equiv 0.$$

See, e.g. [4].

Lemma 10. If a rational valued function $\sigma$ on the unordered pairs of points of a set $X$, $|X| \geq 3$, satisfies

$$\sigma(\{a, b\}) + \sigma(\{c, d\}) \equiv \sigma(\{a, c\}) + \sigma(\{b, d\})$$ (13)

for all choices of four distinct $a, b, c, d \in X$, then there exists a function $\tau$ on $X$ so that

$$\sigma(\{x, y\}) \equiv \tau(x) + \tau(y)$$ (14)

for all distinct $x, y \in X$. If $\tau'$ is another function with the same property, then

$$\tau' \equiv \tau + \frac{\ell}{2},$$

where $\ell$ is 0 or 1; that is, either $\tau'(z) \equiv \tau(z)$ for all $z \in X$ or $\tau'(z) \equiv \tau(z) + 1/2$ for all $z \in X$.

Proof. This is similar to Lemma 5.3 in [1], and can be proved in several ways. We include a proof of the existence of $\tau$ by induction on $|X|$, and omit the remainder of the proof.

For $|X| = 3$, when the condition (13) is vacuous, the assertion is easy. Given $\sigma$ for $X$ of cardinality $\geq 4$, fix $s \in X$ and find a rational valued function $\tau$ on $X - \{s\}$ so that (14) holds for all $x, y \in X - \{s\}$. Pick any $t \in X - \{s\}$ and extend $\tau$ to all of $X$ by defining $\tau(s) = \sigma(\{s, t\}) - \tau(t)$. It remains to show $\sigma(\{s, u\}) \equiv \tau(s) + \tau(u)$ for $u \neq s, t$. Pick $v \in X$ distinct from $s, t, u$. Then

$$\sigma(\{s, u\}) \equiv \sigma(\{s, t\}) + \sigma(\{u, v\}) - \sigma(\{t, v\}) \equiv (\tau(s) + \tau(t)) + (\tau(u) + \tau(v)) - (\tau(t) + \tau(v)) = \tau(s) + \tau(u). \quad \Box$$

Proof of Theorem 8. We assume $\beta'(\mathcal{G}) > 0$, as otherwise there is nothing to prove.
In the notation of Lemma 9, the matrix $M$ of our system of equations has its columns indexed by the union of $\mathcal{C}_n$ and $E \times \mathcal{F}$. The rows correspond to the union of $E$ and $E \times \mathcal{F}$. The entries of $M$ are as follows:

<table>
<thead>
<tr>
<th>Row</th>
<th>Column</th>
<th>Entry</th>
</tr>
</thead>
<tbody>
<tr>
<td>${x, y}$</td>
<td>$C$</td>
<td>0</td>
</tr>
<tr>
<td>${x, y}$</td>
<td>$([u, v], \Pi)$</td>
<td>1 if ${x, y} = {u, v}$; 0 otherwise</td>
</tr>
<tr>
<td>${x, y}, S$</td>
<td>$C$</td>
<td>1 if $C[{x, y}] = S$; 0 otherwise</td>
</tr>
<tr>
<td>${x, y}, S$</td>
<td>$([u, v], \Pi)$</td>
<td>$-1$ if ${x, y} = {u, v}$ and $S \in \Pi$; 0 otherwise</td>
</tr>
</tbody>
</table>

The vector $b$ in Lemma 9 has entries 1 corresponding to $\{x, y\} \in E$ and 0’s in coordinates corresponding to $\{x, y\}, S \in E \times \mathcal{F}$.

To show the system has an integral solution with Lemma 9, let one rational number for each row of $M$ be given (for the coordinates of $y$). We call these numbers $f_S(\{x, y\})$ and $f_\Pi(\{x, y\})$. The assumption $yM = 0$ means

$$\sum_{\{x, y\}} f_C(\{x, y\}) = 0 \quad \text{for each } C \in \mathcal{C}_n,$$

and

$$f_S(\{x, y\}) = \sum_{\Pi \in \mathcal{F}} f_\Pi(\{x, y\}) \quad \text{for each } \{x, y\} \text{ and each } \Pi \in \mathcal{F}.$$

And we must show that under these assumptions

$$\sum_{\{x, y\}} f_S(\{x, y\}) = 0. \quad (17)$$

To clarify the notation in (15), $f_C(\{x, y\})$ is to be understood as 0 when $x$ and $y$ are not adjacent vertices of $C$, i.e. when $C[\{x, y\}]$ is not a colorset but is 0.

As suggested by our notation, we may consider each vertex $S \in \mathcal{F}$ as a function from $E$ to the rationals. We claim that, for any colorset $S \in \mathcal{F}$, $g_S$ satisfies the hypothesis (13) of Lemma 10 with $X$ the vertex set of $K_n^{(r)}$.

Fix a colorset $S_0$ and let $a, b, c, d$ be distinct elements of $X$. Choose a copy $C_1$ of some graph in $\mathcal{G}^n$ so that $a, b$ are not vertices of $C_1$, while $C_1[\{c, d\}] = S_0$. Let $C_2, C_3, C_4$ be, respectively, the subgraphs of $K_n^{(r)}$ obtained from $C_1$ by applying the three permutations (involutions) $(ad), (bc)$, and $(ad)(bc)$ to $C_1$.

We have

$$f_{C_1[\{c, d\}]}(\{c, d\}) = f_{S_0}(\{c, d\}), \quad f_{C_2[\{a, c\}]}(\{a, c\}) = f_{S_0}(\{a, c\}),$$

$$f_{C_3[\{b, d\}]}(\{b, d\}) = f_{S_0}(\{b, d\}), \quad f_{C_4[\{a, b\}]}(\{a, b\}) = f_{S_0}(\{a, b\}),$$

but $f_{C_1[\{x, y\}]}(\{x, y\}) = 0$ for all other incidences of $\{x, y\} \in \{a, b, c, d\}$. We can also see, for example, that for $u \notin \{a, b, c, d\},$

$$f_{C_1[\{c, u\}]}(\{c, u\}) = f_{C_2[\{c, u\}]}(\{c, u\}) \quad \text{and} \quad f_{C_3[\{c, u\}]}(\{c, u\}) = f_{C_4[\{c, u\}]}(\{c, u\}) = 0. \quad (18)$$

From (15), we have

$$\sum_{\{x, y\} \in E} \left( f_{C_1[\{x, y\}]}(\{x, y\}) + f_{C_2[\{x, y\}]}(\{x, y\}) - f_{C_3[\{x, y\}]}(\{x, y\}) - f_{C_4[\{x, y\}]}(\{x, y\}) \right) = 0. \quad (18)$$

For almost all $\{x, y\}$, the corresponding summand above is 0. By analysis of cases, (18) simplifies to

$$f_{S_0}(\{a, b\}) + f_{S_0}(\{c, d\}) - f_{S_0}(\{a, c\}) - f_{S_0}(\{b, d\}) = 0,$$

and this establishes our claim.

By Lemma 10, for each colorset $S \in \mathcal{F}$, there exists a function $g_S$ on the vertices $X$ so that

$$f_S(\{x, y\}) \equiv g_S(x) + g_S(y). \quad (19)$$
Since (see (8)) there exists at least one partition \( \Pi \in \Pi^\# \), and \( \sigma = f_S(\{x, y\}) \) satisfies (13) for each \( S \in \Pi \), (16) implies that the function \( \sigma = f_\ast \) satisfies (13) as well. Thus there exists a function \( g_\ast \) on the vertices of \( K \) such that

\[
f_\ast(\{x, y\}) \equiv g_\ast(x) + g_\ast(y).
\]

For any \( \Sigma \in \Pi^\# \), (16) shows that

\[
f_\ast(\{x, y\}) \equiv \left( \sum_{S \in \Sigma} g_S(x) \right) + \left( \sum_{S \in \Sigma} g_S(y) \right),
\]

so by the second part of Lemma 10,

\[
\sum_{S \in \Sigma} g_S \equiv g_\ast + \frac{\ell_{S}}{2},
\]

where \( \ell_S \) is 0 or 1.

We now show that if \( \hat{v} = \sum_{S \in \mathcal{G}} c_S S \) is a colorset-type of a vertex \( v \) of a graph in \( \mathcal{G}^\# \), then

\[
\sum_{S} c_S g_S(a) \equiv \sum_{S} c_S g_S(b)
\]

for vertices \( a, b \in X \). To see this, let \( a \) and \( b \) be given and choose a copy \( C_5 \in \mathcal{C}_n \) so that \( a \) is a vertex of \( C_5 \) and \( \sum_{S \in \mathcal{G}} c_S[a, u] = \sum_{S \in \mathcal{G}} c_S S \), but \( b \) is not a vertex of \( C_5 \). Let \( C_6 \) be the image of \( C_5 \) under the permutation \((ab)\).

By (15),

\[
\sum_{\{x, y\} \in E} (f_{C_5}(\{x, y\}) - f_{C_6}(\{x, y\})) \equiv 0.
\]

Every summand above vanishes (modulo integers), except possibly for \( |\{x, y\} \cap \{a, b\}| = 1 \). The summand corresponding to \( \{x, y\} = \{a, u\}, u \neq a, b \), is, using (19), \( \equiv g_S(a) + g_S(u) \) where \( S = C_5[a, u] \), while the summand corresponding to \( \{x, y\} = \{b, u\} \) is \( \equiv -(g_S(b) + g_S(u)) \) for the same colorset \( S \). Thus (21) simplifies to (20).

Fix \( x_0 \in X \). It follows from (20) that if \( \sum_{S} c_S S \) is any element of the module \( \mathcal{V} \), then

\[
\sum_{S} c_S g_S(x) \equiv \sum_{S} c_S g_S(x_0)
\]

for \( x \in X \). By the definition of \( \mathcal{V} \), there exist integers \( d_I \) with \( \sum_{I \in \Pi^\#} d_I = n - 1 \) and

\[
\sum_{I \in \Pi^\#} d_I \hat{I} = \sum_{I \in \Pi^\#} d_I \sum_{S \in \mathcal{V}} S \in \mathcal{V}.
\]

Then (22) gives

\[
\sum_{I \in \Pi^\#} d_I \sum_{S \in \mathcal{V}} g_S(x) \equiv \sum_{I \in \Pi^\#} d_I \sum_{S \in \mathcal{V}} g_S(x_0),
\]

\[
\sum_{I \in \Pi^\#} d_I \left( g_\ast(x) + \frac{\ell_I}{2} \right) \equiv \sum_{I \in \Pi^\#} d_I \left( g_\ast(x_0) + \frac{\ell_I}{2} \right),
\]

\[
(n - 1)g_\ast(x) \equiv (n - 1)g_\ast(x_0)
\]

for all \( x \in X \).
For each $G \in \mathcal{G}$, let $C$ be a copy in $\mathcal{C}_n$. From (15),

$$0 \equiv \sum_{\{x,y\}} f_{C\{x,y\}}(\{x,y\}) \equiv \sum_{\{x,y\}} g_{C\{x,y\}}(x) + g_{C\{x,y\}}(y) \equiv \sum_{x \in V(C)} \sum_{y \sim x} g_{C\{x,y\}}(x),$$

(24)

where $y \sim x$ means $y$ adjacent to $x$ in $C$ and $V(C)$ is the vertex set of $C$. Each inner sum on the right of (24) is of the form $\sum_{S \in \mathcal{S}} g_{S}(x)$ where $\sum_{S \in \mathcal{S}} S$ is the colorset-type of some vertex of $G$, and by (22), it does not depend on $x$. So

$$0 \equiv \sum_{x \in V(C)} \sum_{y \sim x} g_{C\{x,y\}}(x_0) = 2 \sum_{\{x,y\} \in E} g_{C\{x,y\}}(x_0).$$

Of course, $\hat{C} = \sum_{\{x,y\} \in E} C\{x,y\}$ is one of the elements that generates the module $s$. It follows that for any $\sum_{S \in \mathcal{S}} S \in s$,

$$0 \equiv 2 \sum_{S \in \mathcal{S}} g_{S}(x_0).$$

By the definition of $\beta'$, there exist integers $d_{\Pi}$ with $\sum_{\Pi \in \Pi^n} d_{\Pi} = n(n - 1)/2$ and $\sum_{\Pi \in \Pi^n} d_{\Pi} \hat{\Pi} \in s$. Then

$$0 \equiv 2 \sum_{\Pi \in \Pi^n} d_{\Pi} \sum_{S \in \Pi} g_{S}(x_0) \equiv 2 \sum_{\Pi \in \Pi^n} d_{\Pi} \left( g_*(x_0) + \frac{\ell_{\Pi}}{2} \right) \equiv n(n - 1)g_*(x_0).$$

(25)

Finally, consider the quantity on the left of (17). From (23) and (25),

$$\sum_{\{x,y\} \in E} f_*([x,y]) \equiv \sum_{\{x,y\} \in E} (g_*(x) + g_*(y)) \equiv (n - 1) \sum_{x \in X} g_*(x) \equiv n(n - 1)g_*(x_0) \equiv 0.$$

This is (17), and the proof is complete. □

Now consider the equations

$$\sum_{C \in \mathcal{C}_n} z_C C\{x,y\} = \sum_{\Pi \in \Pi^n} u_{\Pi\{x,y\}} \hat{\Pi} \text{ in } \mathcal{M} \text{ with } \sum_{\Pi \in \Pi^n} u_{\Pi\{x,y\}} = \lambda.$$ 

(26)

The variables are the same as in (10), and the only difference is the occurrence of $\lambda$ on the far right.

**Theorem 11.** Let $\mathcal{G}$ be a family of edge-$r$-colored colorwise-simple graphs and let $n$ be given so that each graph in $\mathcal{G}$ has at most $n - 2$ vertices. Assume

$$n - 1 \equiv 0 \pmod{\lambda'}(\mathcal{G}),$$

$$n(n - 1) \equiv 0 \pmod{\beta'}(\mathcal{G}).$$

(27)

For all sufficiently large integers $\lambda$, there exist nonnegative integers $z_C$ and $u_{\Pi\{x,y\}}$, so that (26) holds for every unordered pair $\{x,y\}$ of vertices of $K_n^{(r)}$.

**Proof.** It will suffice to show that (26) has a solution in integers when $\lambda = 1$ (this was done in Theorem 8) and that (26) has a solution in which all variables are positive integers for some positive integer $\lambda = \lambda_0$. Then certain nonnegative integral linear combinations of these solutions will provide nonnegative solutions for all large $\lambda$. 
To obtain a positive solution for some $\lambda$, start with (8):

$$
\sum_{G \in \mathcal{G}} b_G^* \hat{G} = \sum_{\Pi \in \Pi^d} a_{\Pi}^* \hat{\Pi}.
$$

Here all coefficients $b_G^*$ and $a_{\Pi}^*$ are positive integers. For each $G \in \mathcal{G}$, we choose one copy of $G$ in $K_n^{(r)}$; for notational convenience, we will assume each $G$ is already a subgraph of $K_n^{(r)}$. If we apply each permutation of the vertex set of $K_n^{(r)}$ to (8) and sum the resulting equations, we see

$$
n! \sum_{\Pi \in \Pi^d} a_{\Pi}^* \hat{\Pi}
$$

is an integral linear combination

$$
\sum_{C \in \mathcal{C}_n} \sum_{[x,y] \in C} c_C C_1[x,y]
$$

in which every copy $C \in \mathcal{C}_n$ has a positive coefficient. Moreover, by symmetry, each sum

$$
\sum_{C \in \mathcal{C}_n} c_C C_1[x,y]
$$

will be the same; it will not depend on $[x,y]$. Hence, for each $[x,y] \in E$,

$$
\sum_{C \in \mathcal{C}_n} c_C C_1[x,y] = 2(n-2)! \sum_{\Pi \in \Pi^d} a_{\Pi}^* \hat{\Pi},
$$

and this provides the desired positive solution of (26). $\square$

5. Decompositions of some complete multipartite graphs

Let $\mathcal{G}$ be a family of colorwise-simple edge-$r$-colored graphs and let an integer $n$ be given. Assume that each graph in $\mathcal{G}$ has at most $n-2$ vertices and that

$$
n-1 \equiv 0 \pmod{\lambda'(\mathcal{G})},
$$

$$
n(n-1) \equiv 0 \pmod{\beta'(\mathcal{G})}.
$$

Theorem 7 asserts that for every sufficiently large prime power $q$ and every $d \geq n^2$, there exists a $\mathcal{G}$-decomposition of $M_n^{(r)}$.  

**Proof of Theorem 7.** We use the same technique as in similar constructions in [1,3,7,6].

Let $q$ be a prime power that is large enough so that there exists a nonnegative integers $z_C$ and $u_{\Pi \setminus \{x,y\}}$ so that (26) for all $\{x,y\}$ holds with $\lambda = q$. Let $X$ be the vertex set of $K_n^{(r)}$.

We start with a $d$-dimensional vector space $W$ over the field $GF(q)$ of $q$ elements, a nonzero linear functional $\ell$, and linear transformations $T_x, x \in X$, from $W$ to itself with the properties: $U_x,y = (T_x - T_y)^{-1}$ exists whenever $x \neq y$ in $X$, and for any $n(n-1)/2$ scalars $\rho([x,y])$, there exist vectors $w(x), x \in X$ so that

$$
\ell(U_{x,y}(w(x) - w(y))) = \rho([x,y])
$$

for all $\{x,y\} \in E$.

(The expression on the left remains the same when $x$ and $y$ are interchanged.) Such transformations exist when $d \geq n^2$; see [6]. The vertices of $M_n^{(r)}$ will be $X \times W$ and the $n$ groups will be $\{x\} \times W$, $x \in X$.

Let $\mathcal{F} = \{F_1, F_2, \ldots, F_N\}$ be a sequence (multiset) of graphs $C$ where $C \in \mathcal{C}_n$ appears with multiplicity $z_C$.

We wish to choose scalars $\rho_i([x,y])$, $\{x,y\} \in E$, $i = 1, 2, \ldots, N$, so that: for any $\{x,y\} \in E$ and any edge $e$ of $K_n^{(r)}$ joining $x$ and $y$, no matter what its color, the list of values $\rho_i([x,y])$ over those indices $i$ such that $e$ is an edge of $F_i$ includes every element of $GF(q)$ exactly once. If that is done, then for any fixed $\{x,y\}$, the (nonempty)
colorsets $F_i[[x, y]]$ as $i$ ranges over those indices such that $\rho_i([x, y]) = \gamma$ will form a partition $\Pi_{[x, y], \gamma}$ of $R$ for each $\gamma \in GF(q)$, and we have $\sum_{i=1}^{N} F_i[[x, y]] = \sum_{\gamma \in GF(q)} \hat{\Pi}_{[x, y], \gamma}$. Conversely, if $\sum_{i=1}^{N} \hat{F}_i[[x, y]]$ can be written as the sum of $q$ elements $\hat{\Pi}$ in $\mathcal{M}$ (which is exactly the equation in (26) when $\lambda = q$), then we can choose the scalars $\rho_i([x, y])$ as required. (The value of $\rho_i([x, y])$ is completely arbitrary if $x$ and $y$ are not adjacent vertices of $F_i$; it may be any scalar.)

Next, for each $i = 1, 2, \ldots, N$, we choose vectors $w_i(x), x \in X$, so that

$$\ell(U_{x, y}(w_i(x) - w_i(y))) = \rho_i([x, y]) \quad \text{for all } [x, y] \in E.$$ 

Then, for each $i = 1, 2, \ldots, N$, we apply the mappings

$$x \mapsto (x, w_i(x) + T_x(u) + v), \quad v \in W, \quad u \in \text{kernel}(\ell)$$

to $F_i$ to obtain $q^{2d-1}$ copies of $F_i$ in $\mathcal{M}_{q^d,n}^{(r)}$. We claim that the collection $\mathcal{D}$ of the total of $Nq^{2d-1}$ copies is a $\mathcal{G}$-decomposition of $\mathcal{M}_{q^d,n}^{(r)}$.

To check this claim, consider an edge $e$ joining vertices $(x, w)$ and $(y, w')$ of $\mathcal{M}_{q^d,n}^{(r)}$ that are in different groups, i.e. $x \neq y$. Find the unique $i$ so that $F_i$ contains the edge of the same color as $e$ joining $x$ and $y$ in $K^{(r)}_{q^d,n}$ and so that $\rho_i([x, y])$ is the scalar $\ell(U_{x, y}(w - w'))$; that is, so that

$$\ell(U_{x, y}(w_i(x) - w_i(y))) = \ell(U_{x, y}(w - w')).$$

Then there is a unique $u \in \text{kernel}(\ell)$ so that

$$U_{x, y}(w_i(x) - w_i(y)) + u = U_{x, y}(w - w'),$$

which is equivalent to

$$(w_i(x) - w_i(y)) + (T_x(u) - T_y(u)) = (w - w').$$

So there is a unique $v \in W$ so that

$$\begin{cases} w_i(x) + T_x(u) + v = w, \\
 w_i(y) + T_y(u) + v = w', 
\end{cases}$$

and then the image of $F_i$ under the corresponding mapping in (28) will contain the edge $e$. A careful look at the argument will show that only one of the copies in $\mathcal{D}$ contains this edge. □

6. **Comparison of $\beta$ and $\beta'$**

The equation

$$\sum_{G \in \mathcal{G}} b_G \hat{G} = \sum_{\Pi \in \Pi} a_{\Pi} \hat{\Pi}$$

implies

$$\sum_{G \in \mathcal{G}} b_G \mu(G) = \left(\sum_{\Pi \in \Pi} a_{\Pi}\right) (1, 1, \ldots, 1).$$

It is easy to see, then, according to our definitions in Section 2, that $\mathcal{G}^\# \subset \mathcal{G}^*$ and that the ideal $B'$ is a subset of the ideal $B$, so that $\beta(\mathcal{G})$ divides $\beta'(\mathcal{G})$. Similarly, one can see that $\alpha(\mathcal{G})$ divides $\alpha'(\mathcal{G})$.

A multiset $\mathcal{A}$ of nonempty subsets of $R$ is regular (with degree, or replication number, $d$) when each element of $R$ appears in the same number $d$ (counting multiplicities) of members of $\mathcal{A}$. (So $\mathcal{A}$ is a partition of $R$ if and only if it is regular of degree 1.) The multiset $\mathcal{A}$ is resolvable when $\mathcal{A}$ is the union of some number of partitions of $R$. Every resolvable multiset $\mathcal{A}$ is regular.
Lemma 12. If a multiset $A$ of subsets of $R$ is regular and tree-ordered, then $A$ is resolvable.

Proof. Given $A$, regular with positive degree, choose a family $B$ of pairwise disjoint members of $A$ so that $\bigcup B$ is as large as possible. If possible, consider $x \in R$ with $x \notin \bigcup B$. Some set $A \in A$ will contain $x$. Let $B_0$ be the set of $B \in B$ with $B \cap A = \emptyset$. The other sets in $B$ are contained in $A$. Then $B' = \{A\} \cup B_0$ is a family of pairwise disjoint members of $A$ with $\bigcup B'$ strictly larger than $\bigcup B$. We conclude that $B$ is a partition of $R$. Delete these sets from $A$, to get a multiset of subsets of $R$ of degree one less than that of $A$. Continue until $A$ has been partitioned into partitions. □

Proof of Theorem 3. Let $G$ be a family of colorwise-simple edge-colored graphs. To prove Theorem 3, we show that, in the colorwise-simple case, $\beta'(G) = \beta(G)$ and $\alpha'(G) = \alpha(G)$. If $\beta(G) = 0$, there are no decompositions and nothing to prove, so we assume $\beta(G) > 0$.

Suppose
\[ \sum_{G \in G} b_G \mu(G) = d(1, 1, \ldots, 1) \] (29)
with $d$ and $b_G$, $G \in G$, nonnegative. Then the multiset consisting of $b_G$ copies of each colorset $S$ of $G$ for all $G \in G$ is regular of degree $d$ and, by assumption, tree-ordered. Lemma 9 implies that $\sum_{G \in G} b_G \hat{G}$ is a resolvable element of $M$, i.e.
\[ \sum_{G \in G} b_G \hat{G} = \sum_{i=1}^d \hat{I}_i \] (30)
for some partitions $\Pi_i$. This implies $2d$ is in the ideal $B'$.

In particular, if $G \in G^*$, then $b_G > 0$ in some instance of (29), and (30) implies $G \in G^*$. So $G^* = G^*$ for a tree-ordered family $G$.

Adding instance of (29) in which $b_G > 0$ for $G \in G^*$, we find an expression
\[ \sum_{G \in G^*} b_G^* \mu(G) = d^*(1, 1, \ldots, 1) \] (31)
in which $b_G^* > 0$ for all $G \in G^*$. By definition of $\beta(G)$,
\[ \sum_{G \in G^*} c_G \mu(G) = \frac{1}{2} \beta(G)(1, 1, \ldots, 1) \] (32)
for some integers $c_G$. We may assume that the coefficients in (31) are large enough so that the sum of (31) and (32) still has all coefficients nonnegative. Then both $2d^*$ and $2d^* + \beta(G)$ belong to $B'$; hence their difference $\beta(G)$ belongs to $B'$ and then $\beta'(G)$ divides $\beta(G)$.

A similar argument shows that $\alpha(G) = \alpha'(G)$. □

We remark that an analogous argument shows that $\beta(G) = \beta'(G)$ and $\alpha(G) = \alpha'(G)$ also holds when the colorset-family of $G$ is the edge set of a bipartite graph or multigraph. For such a structure, we know that regularity implies resolvability.

We conclude this section with some examples where $\beta' \neq \beta$.

Suppose $r = 3$ and $\hat{G} = \{1, 2\} \cup \{1, 3\} \cup \{2, 3\}$. Any two colorsets meet, so it is clear that no multiple $t \hat{G}$ is resolvable. Hence $\beta'(\hat{G}) = 0$ and there exist no $G$-decompositions of $K_n^{(3)}$ for $n > 1$. But $\mu(G) = (2, 2, 2)$, so $\beta(\hat{G}) = 4$.

We give one example where $\beta'(\hat{G}) > 0$ and $\beta'(\hat{G}) \neq \beta(\hat{G})$. There are surely many others.

Let $r = 10$ and let $G$ consist of a single graph $G$ whose colorsets are the 15 edges of the Petersen graph with vertex set $R$, considered as 2-element subsets of the colors. Since the Petersen graph is regular of degree 3, each color appears on three edges of $G$, and $\mu(G)$ is the constant vector $(3, 3, \ldots, 3)$ of length 10; hence $\beta(G) = 6$.
There are exactly six partitions \( \Pi \) of \( R \) into colorsets (six one-factors in the Petersen graph). The columns of the matrix below correspond to the 15 colorsets (edges of the Petersen graph), and the rows correspond to the partitions \( \Pi_1, \ldots, \Pi_6 \) of \( R \) into colorsets (one-factors).

\[
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1
\end{pmatrix}.
\]

The rows sum to the vector of all 2’s; that is, \( \hat{\Pi}_1 + \cdots + \hat{\Pi}_6 = 2\hat{G} \). It is elementary to check that no integral linear combination of the rows of this matrix is the vector of all 1’s (in particular, the Petersen graph has no one-factorization). Thus \( \beta'(\mathcal{G}) = 12 \).

The parameters \( \alpha' \) and \( \alpha'' \) depend on the additional structure of the graph \( G \). If we assume that \( G \) is the disjoint union of 15 digons (one digon for each of the 15 colorsets), then \( \alpha(\mathcal{G}) = \alpha''(\mathcal{G}) = 1 \). This follows from the existence of a single one-factor of the Petersen graph. For this graph \( G \), necessary and asymptotically sufficient conditions on \( n \) for the existence of a \( G \)-decomposition of \( K_n^{(10)} \) are \( n \equiv 0, 1, 4, \text{ or } 9 \mod 12 \).

7. Extension to the case \( \lambda > 1 \)

For any vector of positive integers \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p) \), let \( K_n^{[\lambda]} \) be the complete graph on \( n \) vertices with \( \lambda_i \) edges of color \( i \) joining any pair of vertices, \( i = 1, 2, \ldots, p \). We will define the parameters \( \alpha' \) and \( \beta' \) following the statement of our result.

**Theorem 13.** Let \( \mathcal{H} \) be a family of (not necessarily colorwise-simple) edge-\( p \)-colored graphs and let \( \lambda \) be a \( p \)-tuple of positive integers. Then the conditions

\[
\begin{align*}
n - 1 & \equiv 0 \pmod{\alpha'(\mathcal{H}, \lambda)}, \\
n(n - 1) & \equiv 0 \pmod{\beta'(\mathcal{H}, \lambda)},
\end{align*}
\]

on \( n \) are necessary and asymptotically sufficient for the existence of a \( \mathcal{H} \)-decomposition of \( K_n^{(\lambda)} \).

Let \( P \) be a set of \( p \) colors, and let \( P^{[\lambda]} \) be the multiset of colors where color \( i \) appears \( \lambda_i \) times. We call a multiset \( \Phi \) of nonempty multisets of elements of \( P \) a \( \lambda \)-partition if the union of multisets in \( \Phi \) gives \( P^{[\lambda]} \). \( \Phi \) will denote the formal sum of the multisets in \( \Phi \), counted with their multiplicities, in the module \( \mathcal{H}^* \) of all formal sums of nonempty multisets of elements from \( P \). For example, if \( p = 3 \) and \( \lambda = (1, 2, 5) \), then \( \Phi = \{\{1, 3, 3\}, \{2, 3\}, \{3\}, \{3\}, \{2\}\} \) is a \( \lambda \)-partition, and

\[
\hat{\Phi} = \{1, 3, 3\} + \{2, 3\} + 2\{3\} + \{2\}.
\]

\( H[[x, y]] \) will denote the multiset of colors that appear on the edges of \( H \) that join vertices \( x \) and \( y \), and we interpret \( H[[x, y]] = 0 \) when \( x \) and \( y \) are not adjacent in \( H \), in particular when not both are vertices of \( H \). We define

\[
\hat{H} = \sum_{[u, v]} H[[u, v]],
\]

where the sum is over all unordered pairs \( [u, v] \) of vertices of \( H \). Consider equations of the form

\[
\sum_{H} b_H \hat{H} = \sum_{\Phi} a_\Phi \hat{\Phi};
\]

where \( a_\Phi \) and \( b_H \) are integers. Let \( \mathcal{H}^\land \) be the set of graphs \( L \in \mathcal{H} \) for which there exists a solution of (34) with all coefficients nonnegative and \( b_L > 0 \). Let \( \Phi^\land \) be the set of \( \lambda \)-partitions \( \Theta \) for which there exists a solution of (34) with all coefficients nonnegative and \( a_\Theta > 0 \).
Now consider equations of the form
\[ \sum_{H \in \mathcal{H}^\diamondsuit} b_H \hat{H} = \sum_{\phi \in \Phi^\diamondsuit} \alpha_\phi \hat{\phi} \]  
(35)
with integer coefficients. Let $B^\diamondsuit$ be the ideal in $\mathbb{Z}$ consisting of integers $2t$ so that $t = \sum_{\phi \in \Phi^\diamondsuit} \alpha_\phi$ for some integers $\alpha_\phi$ and $b_H$ so that (35) holds. Let $b'(\mathcal{H}, \lambda)$ be the unique nonnegative generator of this ideal.

Let $A^\diamondsuit$ be the ideal in $\mathbb{Z}$ consisting of integers $t$ of the form $t = \sum_{\phi \in \Phi^\diamondsuit} \alpha_\phi$ such that $t = \sum_{\phi \in \Phi^\diamondsuit} \alpha_\phi \hat{\phi}$ belongs to the module generated by all
\[ \sum_{v \neq u} H([u, v]), \]  
(36)
where $u$ is a vertex of $H$ and $H \in \mathcal{H}^\diamondsuit$. Let $\alpha'(\mathcal{H}, \lambda)$ be the unique nonnegative generator of this ideal.

**Proof of Theorem 13.** As in the colorwise-simple case, it is easy to see that $\alpha'(\mathcal{H}, \lambda)$ divides $b'(\mathcal{H}, \lambda)$, and that (33) is a necessary condition for the existence of a $\mathcal{H}$-decomposition of $K_n^{[\lambda]}$. We assume $b'(\mathcal{H}, \lambda) > 0$.

For the asymptotic sufficiency, we proceed in a manner suggested by the proof of Theorem 13.1 in [1]. Let $r = \lambda_1 + \lambda_2 + \cdots + \lambda_p$. From the graphs in $\mathcal{H}$ we construct a family $\mathcal{G}$ of colorwise-simple edge-$r$-colored graphs by introducing $\lambda_i$ distinct ‘shades’ $i', i'', i''', \ldots$ of each color $i$; $R$ will denote this set of $r$ new colors (shades).

For every graph $H$ in $\mathcal{H}$, we put in $\mathcal{G}$ all colorwise-simple edge-$r$-colored graphs that can be obtained by replacing the color on an edge of color $i$ in $H$ by a shade of $i'$; such graphs $G$ will be called refinements of $H$. (There will be at least one such graph unless some pair $x, y$ of vertices in $H$ is joined by more than $\lambda_i$ edges of color $i$ for some $i$.)

To complete the proof it is enough to show that (33) implies
\[ \begin{align*}
 n - 1 & \equiv 0 \text{ (mod } \alpha'(\mathcal{G})), \\
n(n - 1) & \equiv 0 \text{ (mod } b'(\mathcal{G})).
\end{align*} \]  
(37)

By Theorem 4, for $n$ sufficiently large and satisfying (37), there exists an $\mathcal{G}$-decomposition of $K_n^{[r]}$, where $R$ is the set of colors for the edges. Then by returning the shade of each edge of $K_n^{[r]}$ to its original color, and also the shades of each edge of the graphs in the decomposition to their original colors, we obtain a $\mathcal{G}$-decomposition of $K_n^{[\lambda]}$.

Let $\mathcal{G}^\#, \Pi^\#, B^\#$, and $A^\#$ be defined as in Section 2, with respect to the family $\mathcal{G}$ and the set $R$ of $r$ colors.

We first show that any refinement $G$ of a graph $H \in \mathcal{H}^\diamondsuit$ is in $\mathcal{G}^\#$, and that any refinement $\Pi$ of a $\lambda$-partition $\Phi \in \Phi^\diamondsuit$ is in $\Pi^\#$.

If the equation
\[ \sum_{i=1}^{n} \hat{H}_i = \sum_{j=1}^{m} \hat{\phi}_j \text{ in } \mathcal{H}' \]  
(38)
holds and $\Pi_j$ is a refinement of $\Phi_j$ for $j = 1, 2, \ldots, m$, then there exist refinements $G_i$ of $H_i$, $i = 1, 2, \ldots, n$, so that
\[ \sum_{i=1}^{n} \hat{G}_i = \sum_{j=1}^{m} \hat{\Pi}_j \text{ in } \mathcal{H}. \]

The $G_i$’s are not necessarily unique; we need only transfer the shaded colorsets to the left-hand side of the equation in any order. It is clear that each refinement $\Pi_j$ is in $\Pi^\#$.

But not every refinement of a graph on the left-hand side of (38) is obtained (immediately) in this way. For example, suppose $n = 1, m = 2, \lambda = (1, 2)$,
\[ \hat{H}_1 = \{1, 2\} + \{2, 2\} + \{1\} + \{2\}, \quad \Phi_1 = \{1, 2\} + \{2\}, \quad \Phi_2 = \{1\} + \{2, 2\}. \]  
(39)
No refinements of the $\lambda$-partitions on the right (to obtain two partitions of the shades $\{1', 2', 2''\}$) will yield $\hat{G}_1 = \{1', 2'\} + \{2', 2''\} + \{1'\} + \{2'\}$ on the left. In such a case, we may add several copies of (38) and choose different
refinements of the $\Phi_i$’s on the right of (38); for example, there are refinements $\Pi_1, \Pi'_1$ of $\Phi_1$ and refinements $\Pi_2, \Pi'_2$ of $\Phi_2$ so that

$$\hat{G}_1 + \hat{G}_2 = \hat{\Pi}_1 + \hat{\Pi}'_1 + \hat{\Pi}_2 + \hat{\Pi}'_2$$

(40)

with $\hat{G}_2 = \{1', 2''\} + \{2', 2''\} + \{1'\} + \{2''\}$. This equation shows that the refinement $G_1$ of $H_1$ is not unusable. Since the graphs in $G$ are colorwise-simple, the same idea shows that any refinement of any $H_i$ on the left of (38) is in $G^\#$.

The conditions (33) will imply (37) if and only if $B^\Diamond \subseteq B^\#$ and $A^\Diamond \subseteq A^\#$. Let $2t \in B^\Diamond$ be given. Say

$$t = \sum_{\Phi \in \Phi^\Diamond} a_\Phi,$$

(41)

where (35) holds for some $b_H, H \in \mathcal{H}^\Diamond$. We want to write

$$t = \sum_{\Pi \in \Pi^\#} c_\Pi,$$

where

$$\sum_{\Pi \in \Pi^\#} c_\Pi \hat{\Pi} = \sum_{G \in G^\#} d_G \hat{G},$$

(42)

which will imply that $2t \in B^\#$. (42) is equivalent to a system of linear equations in integer variables $c_\Pi$ and $d_G$ in which, in addition to the first equation, there is one linear constraint for each colorset $S$ in the set $\mathcal{S}$ of all colorsets of graphs in $G^\#$.

In the notation of Lemma 9, the matrix $M$ of our system of equations has its columns indexed by the union of $G^\#$ and $\Pi^\#$. The rows correspond to the colorsets in $\mathcal{S}$ and one other symbol, say ‘∗’. Let $m_S(G)$ be the number of times $S$ occurs as a colorset of $G$. The entries of $M$ are as follows:

<table>
<thead>
<tr>
<th>Row</th>
<th>Column</th>
<th>Entry</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S$</td>
<td>$G$</td>
<td>$m_S(G)$</td>
</tr>
<tr>
<td>$S$</td>
<td>$\Pi$</td>
<td>$-1$ if $S \in \Pi$; 0 otherwise</td>
</tr>
<tr>
<td>∗</td>
<td>$G$</td>
<td>0</td>
</tr>
<tr>
<td>∗</td>
<td>$\Pi$</td>
<td>1</td>
</tr>
</tbody>
</table>

The vector $b$ in Lemma 9 has entries 0 corresponding to $S \in \mathcal{S}$ and a $t$ in the coordinate labeled ∗.

To show the system has an integral solution with Lemma 9, let one rational number for each row of $M$ be given (for the coordinates of $y$). We call these numbers $y_S, S \in \mathcal{S}$, and $y_\ast$. The assumption $yM \equiv 0$ means

$$\sum_{S \in \mathcal{S}} m_S(G)y_S = \sum_{\{u,v\}} y_{G[\{u,v\}]} \equiv 0 \quad \text{for every } G \in G^\#.$$  

(43)

and

$$y_\ast \equiv \sum_{S \in \Pi} y_S \quad \text{for every } \Pi \in \Pi^\#.$$  

(44)

And we must show that under these assumptions $ty_\ast \equiv 0$.

Suppose $S_1, S_2 \in \mathcal{S}$ have the property that when shades are replaced by their original colors, the same multiset $T$ of elements of $P$ results. Say $T$ is a colormultiset $H[\{u,v\}]$ for some $H \in \mathcal{H}^\Diamond$. Consider graphs $G_1, G_2, H^\#$ that are both refinements of $H$ and have the same colormultisets except that $G_1[\{u,v\}] = S_1$ and $G_2[\{u,v\}] = S_2$. Eq. (43) implies $y_{S_1} \equiv y_{S_2}$. Thus we may define rationals $z_T$ (up to integers) where $T$ is any colormultiset by $z_T \equiv y_S$ where $S$ is any refinement of $T$, and we see

$$\sum_{\{u,v\}} z_{H[\{u,v\}]} \equiv 0 \quad \text{for any } H \in \mathcal{H}^\Diamond.$$  

(45)
Then (44) implies
\[ y_* \equiv \sum_{T \in \Phi} z_T \quad \text{for every } \Phi \in \Phi^\diamond. \]  
(46)

We replace in (35) every colormultiset \( T \) by the rational number \( z_T \) to find
\[ \sum_{H \in \mathcal{H}^\diamond} b_H \sum_{\{u,v\}} z_{H([u,v])} = \sum_{\Phi \in \Phi^\diamond} a_\Phi \sum_{T \in \Phi} z_T. \]  
(47)

The left-hand side of the above equation is \( \equiv 0 \) by (45), while the right-hand side is \( \equiv ty_* \) by (46). Thus (42) has an integral solution and \( B^\diamond \subseteq B^\# \).

The proof that \( A^\diamond \subseteq A^\# \) is similar, and we omit the details. \( \square \)

References

[2] E.M. Li Marzi, C.C. Lindner, F. Rania, R.M. Wilson, \{2, 3\}-perfect \( m \)-cycle systems are equationally defined for \( m = 5, 7, 8, 9 \) and 11 only, J. Combin. Des., to appear.