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Note

On a link between Dirichlet kernels and central multinomial coefficients

Michelle Rudolph-Lilith *, Lyle E. Muller [1]

Unité de Neurosciences, Information et Complexité (UNIC), CNRS, 1 Ave de la Terrasse, 91198 Gif-sur-Yvette, France

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ABSTRACT

The central coefficients of powers of certain polynomials with arbitrary degree in x form an important family of integer sequences. Although various recursive equations addressing these coefficients do exist, no explicit analytic representation has yet been proposed. In this article, we present an explicit form of the integer sequences of central multinomial coefficients of polynomials of even degree in terms of finite sums over Dirichlet kernels, hence linking these sequences to discrete nth-degree Fourier series expansions. The approach utilizes the diagonalization of circulant Boolean matrices, and is generalizable to all multinomial coefficients of certain polynomials with even degree, thus forming the base for a new family of combinatorial identities.

1. Introduction

Let $k \in \mathbb{N}$, $k \geq 1$ and

$$P_{2k}(x) = 1 + x + x^2 + \cdots + x^{2k} \tag{1}$$

be a finite polynomial of even degree in x. Using the multinomial theorem and collecting terms with the same power in x, the nth power of $P_{2k}(x)$ with $n \in \mathbb{N}$, $n \geq 1$ is then given by

$$P_{2k}(x)^n = \left(1 + x + x^2 + \cdots + x^{2k}\right)^n = \sum_{l=0}^{2kn} p_{l,2k}^{(n)} x^l, \tag{2}$$

where $p_{l,2k}^{(n)}$ denotes the multinomial coefficient (e.g., see [4, Definition B, p. 28]) given by

$$p_{l,2k}^{(n)} = \sum_{n_i} \binom{n}{n_0, n_1, \ldots, n_{2k}} \tag{3}$$

$\forall l \in [0, 2kn]$, where in the last equation the sum runs over $n_i \in [0, n]\, \forall i \in [0, 2k]$ with $n_0 + n_1 + \cdots + n_{2k} = n$ and $n_1 + 2n_2 + \cdots + 2kn_{2k} = l$. The central $(2k+1)$-nomial coefficients $M^{(2k,n)}$, e.g., the central trinomial $(k = 1)$, pentanomial

* Corresponding author.
E-mail address: rudolph@unic.cnrs-gif.fr (M. Rudolph-Lilith).

[1] Present address: CNL, Salk Institute for Biological Studies, La Jolla CA, USA.

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($k = 2$) or heptanomial ($k = 3$) coefficients, are then given by

$$M^{(2k,n)} = p^{(n)}_{\lfloor 2kn/2 \rfloor, 2k} \equiv p^{(n)}_{kn,2k}. \tag{4}$$

Eqs. (3) and (4) provide a definition of the central coefficients in terms of sums over products of binomial coefficients. However, their explicit calculation constitutes a numerically non-trivial problem, especially for large k and n, as it involves sums over specific partitions of integer numbers. Although various recursive equations addressing these coefficients do exist, and the Almkvist–Zeilberger algorithm [2] allows for a systematic derivation of recursions for multinomial coefficients in the general case, no explicit analytical representation of these coefficients has yet been proposed in the literature.

Mathematically, the central multinomial coefficients $M^{(2k,n)}$ are linked to the number of closed walks of length n in random graphs, and recently an approach has been proposed which translates this combinatorially hard problem into one of taking powers of a specific type of circulant Boolean matrices [8, Equations (7) and (8)]. In this paper, we detail this approach, and prove a simple relation between central $(2k + 1)$-nomial coefficients defined in (4) and finite sums over Dirichlet kernels of fractional angles (for Dirichlet kernels, e.g., see [6], and Chapter I, §29 in [3]). This explicit analytical representation not only allows for a fast numerical evaluation of central multinomial coefficients, but also for an explicit construction of the whole class of sequences of central multinomial coefficients (see the On-Line Encyclopedia of Integer Sequences, OEIS, [10]; e.g., OEIS A002426, A005191, A025012, A025014).

2. A trace formula

Let $N = 2kn + 1$ with $n, k \in \mathbb{N}$. Consider the $N \times N$ circulant matrix

$$\mathbf{A}_{N,2k} = \text{circ}\big\{(1, \overbrace{1, \ldots, 1}^{2k}, 0, \ldots, 0)\big\} = \text{circ}\left\{\left(\sum_{l=0}^{2k} \delta_{j,1+l \bmod N}\right)_j\right\}. \tag{5}$$

Multiplying $\mathbf{A}_{N,2k}$ by a vector $\mathbf{x} = (1, x, x^2, \ldots, x^{2kn})$ will yield the original polynomial as the first element in the resulting vector $\mathbf{A}_{N,2k}\mathbf{x}$. Similarly, taking the n'th ($n' \le n$) power of $\mathbf{A}_{N,2k}$ and multiplying the result with $\mathbf{x}$ will yield $P_{2k}(x)^{n'}$ as first element. Thus $\mathbf{A}^{n'}_{N,2k}$ will contain the sequence of multinomial coefficients $p^{(n')}_{l,2k}$ in its first row. Moreover, as the power of a circulant matrix is again circulant, this continuous sequence of non-zero entries in a given row will shift by one column to the right on each subsequent row, and wraps around once the row-dimension N is reached. This behavior will not change even if one introduces a shift by m columns of the sequence of 1's in $\mathbf{A}_{N,2k}$, as this will correspond to simply multiplying the original polynomial by x^m. Such a shift, however, will allow, when correctly chosen, to bring the desired central multinomial coefficients on the diagonal of $\mathbf{A}^{n'}_{N,2k}$.

We can formalize this approach in the following.

Lemma 1. *Let*

$$\mathbf{A}^{(m)}_{N,2k} = \text{circ}\left\{\left(\sum_{l=0}^{2k} \delta_{j,1+(m+l) \bmod N}\right)_j\right\} \tag{6}$$

with $m \in \mathbb{N}_0$ be circulant Boolean square matrices of dimension $N = 2kn + 1$ with $k, n \in \mathbb{N}$. The central $(2k + 1)$-nomial coefficients are given by

$$M^{(2k,n)} = \frac{1}{N}\text{Tr}\left(\mathbf{A}^{(N-k)}_{N,2k}\right)^n, \tag{7}$$

where $\text{Tr}(\mathbf{A})$ denotes the trace of matrix $\mathbf{A}$.

Proof. Let $\mathbf{B} = \text{circ}\big\{(0, 1, 0, \ldots, 0)\big\}$ be a $N \times N$ cyclic permutation matrix. Note that $\mathbf{B}$ has the following properties:

$$\begin{aligned} &\mathbf{B}^0 \equiv I = \mathbf{B}^N \\ &\mathbf{B}^m = \mathbf{B}^r \quad \text{with } r = m \mod N \\ &\mathbf{B}^n\mathbf{B}^m = \mathbf{B}^{(nm) \mod N}, \end{aligned} \tag{8}$$

where $\mathbf{I}$ denotes the identity matrix of order N. The set of powers of the cyclic permutation matrix, $\{\mathbf{B}^m\}$, $m \in [0, N]$, then acts as a basis for the circulant matrices $\mathbf{A}^{(m)}_{N,2k}$ defined in (6) (e.g., see [11, Section 1.10] and [5] for a thorough introduction into circulant matrix algebra).

Let us first consider the case $m = 0$. It can easily be shown that

$$\mathbf{A}^{(0)}_{N,2k} = \mathbf{I} + \sum_{l=1}^{2k} \mathbf{B}^l.$$

Applying the multinomial theorem and ordering with respect to powers of $\mathbf{B}$, we have for the nth power of $\mathbf{A}_{N,2k}^{(0)}$

$$\begin{aligned}\left(\mathbf{A}_{N,2k}^{(0)}\right)^n &= \sum_{n_0+n_1+\cdots+n_{2k}=n} \binom{n}{n_0, n_1, \ldots, n_{2k}} \mathbf{I}^{n_0} \mathbf{B}^{n_1+2n_2+\cdots+2kn_{2k}} \\ &= b_0^{(n)}\mathbf{I} + \sum_{l=1}^{2kn} b_l^{(n)}\mathbf{B}^l \\ &= \mathrm{circ}\left\{\left(b_0^{(n)}, b_1^{(n)}, \ldots, b_{2kn}^{(n)}\right)\right\}, \end{aligned} \tag{9}$$

where $b_l^{(n)} = p_{l,2k}^{(n)}$.

For $m > 0$, we have

$$\mathbf{A}_{N,2k}^{(m)} = \mathbf{B}^m + \sum_{l=1}^{2k} \mathbf{B}^{m+l} = \mathbf{B}^m \left(\mathbf{I} + \sum_{l=1}^{2k} \mathbf{B}^l\right),$$

and obtain, with (9), for the nth power

$$\left(\mathbf{A}_{N,2k}^{(m)}\right)^n = \mathbf{B}^{mn}\left(b_0^{(n)}\mathbf{I} + \sum_{l=1}^{2kn} b_l^{(n)}\mathbf{B}^l\right). \tag{10}$$

Using (8), the factor $\mathbf{B}^{mn}$ shifts and wraps all rows of the matrix to the right by mn columns, so that

$$\begin{aligned}\left(\mathbf{A}_{N,2k}^{(m)}\right)^n &= b_0^{(n,m)}\mathbf{I} + \sum_{l=1}^{2kn} b_l^{(n,m)}\mathbf{B}^l \\ &= \mathrm{circ}\left\{\left(b_0^{(n,m)}, b_1^{(n,m)}, \ldots, b_{2kn}^{(n,m)}\right)\right\},\end{aligned}$$

where $b_l^{(n,m)} = b_{(l-nm) \bmod N}^{(n)}$ with $b_l^{(n)} = p_{l,2k}^{(n)}$ given by (3).

Using the standard enumeration of matrix elements starting with 1, for $m = 0$, the desired central multinomial coefficients can be found in row 1, column $\lfloor N/2 \rfloor + 1 = kn + 1$, i.e. $M^{(2k,n)} = b_{kn}^{(n)}$. Observing that

$$(l - n(N-k)) \bmod N \equiv (l + kn) \bmod N,$$

a shift by $m = N - k$ yields

$$M^{(2k,n)} = b_{(l+kn) \bmod N}^{(n)} = b_l^{(n,N-k)}. \tag{11}$$

That is, setting $l = 0$, the central $(2k+1)$-nomial coefficients $M^{(2k,n)}$ reside on the diagonal of $\left(\mathbf{A}_{N,2k}^{(N-k)}\right)^n$. Taking the trace of $\left(\mathbf{A}_{N,2k}^{(N-k)}\right)^n$ thus proves (7). □

With Lemma 1, the sequences of central $(2k+1)$-nomial coefficients $M^{(2k,n)}$ are given in terms of the trace of powers of the $N \times N$ circulant Boolean matrix

$$\begin{aligned}\mathbf{A}_{N,2k}^{(N-k)} &= \mathrm{circ}\{(1, \overbrace{1, \ldots, 1}^{k}, 0, \ldots, 0, \overbrace{1, \ldots, 1}^{k})\} \\ &= \mathrm{circ}\left\{\left(\sum_{l=0}^{k} \delta_{j,1+l} + \sum_{l=0}^{k-1} \delta_{j,N-l}\right)_j\right\}.\end{aligned} \tag{12}$$

This translates the original combinatorial problem into one of matrix algebra, specifically the problem of finding powers of circulant matrices, which can be solved in an analytically exact fashion.

3. A sum formula

Not only can circulant matrices be represented in terms of a simple base decomposition using powers of cyclic permutation matrices, but circulant matrices also allow for an explicit diagonalization (e.g., see [11, Section 1.6]). The latter will be utilized to prove

Lemma 2. *Let $N = 2kn + 1$ with $k, n \in \mathbb{N}$. The sequence of central $(2k+1)$-nomial coefficients $M^{(2k,n)}$ is given by*

$$M^{(2k,n)} = \frac{1}{N}\left\{(2k+1)^n + \sum_{l=1}^{2kn}\left(\frac{\sin\left[\frac{(2k+1)}{N}l\pi\right]}{\sin\left[\frac{1}{N}l\pi\right]}\right)^n\right\}. \tag{13}$$

Proof. As $\mathbf{A}_{N,2k}^{(N-k)}$, Eq. (12), is a circulant matrix, we can utilize the circulant diagonalization theorem [11, Section 1.6] to calculate its nth power. This theorem states that all circulants $\mathbf{C}$ constructed from an arbitrary N-dimensional vector $\mathbf{c} = (c_1, c_2, \ldots, c_N)$ are diagonalized by the same unitary matrix $\mathbf{U}$. That is,

$$\mathbf{C} = \mathrm{circ}(c_1, c_2, \ldots, c_N) = \mathbf{U}\mathbf{E}\mathbf{U}^{-1}, \tag{14}$$

where $\mathbf{U}$ denotes the unitary matrix with components

$$u_{rs} = \frac{1}{\sqrt{N}}\exp\left[-\frac{2\pi i}{N}(r-1)(s-1)\right] \tag{15}$$

and $\mathbf{E} = \mathrm{diag}[E_r(\mathbf{C})]$ the diagonal matrix with eigenvalues

$$E_r(\mathbf{C}) = \sum_{j=1}^{N} c_j \exp\left[-\frac{2\pi i}{N}(r-1)(j-1)\right], \tag{16}$$

$r, s \in [1, N]$. In components, (14) reads

$$c_{ij} = \sum_{r,s=1}^{N} u_{ir} e_{rs} u_{sj}^*, \tag{17}$$

where u_{rs}^* denotes the complex conjugates of u_{rs} and $e_{rs} = \delta_{rs} E_r(\mathbf{C})$.

With (12) and (16), the eigenvalues of $\mathbf{A}_{N,2k}^{(N-k)}$ are

$$\begin{aligned} E_r(\mathbf{A}_{N,2k}^{(N-k)}) &= \sum_{j=1}^{N}\left\{\sum_{l=0}^{k}\delta_{j,1+l} + \sum_{l=0}^{k-1}\delta_{j,N-l}\right\} e^{-2\pi i(r-1)(j-1)/N} \\ &= \sum_{l=0}^{k} e^{-2\pi i(r-1)l/N} + \sum_{l=0}^{k-1} e^{-2\pi i(r-1)(N-l-1)/N} \\ &= 1 + \sum_{l=1}^{k}\left\{e^{-2\pi i(r-1)l/N} + e^{-2\pi i(r-1)(N-l)/N}\right\}, \end{aligned}$$

where in the first sum we split off the $l = 0$ term and in the second sum changed the summation variable $l \to l+1$. Utilizing Euler's formula and observing that r is an integer number, we further obtain

$$\begin{aligned} E_r(\mathbf{A}_{N,2k}^{(N-k)}) &= 1 + \sum_{l=1}^{k}\left\{\cos\left[\tfrac{(r-1)l}{N}2\pi\right] + \cos\left[-\tfrac{(r-1)l}{N}2\pi + 2\pi(r-1)\right]\right. \\ &\qquad \left. - i\sin\left[\tfrac{(r-1)l}{N}2\pi\right] - i\sin\left[-\tfrac{(r-1)l}{N}2\pi + 2\pi(r-1)\right]\right\} \\ &= 1 + 2\sum_{l=1}^{k}\cos\left[\tfrac{(r-1)l}{N}2\pi\right] \\ &= \begin{cases} 1 + 2\sin\left[\dfrac{k(r-1)}{N}\pi\right]\cos\left[\dfrac{(1+k)(r-1)}{N}\pi\right]\Big/\sin\left[\dfrac{r-1}{N}\pi\right], & \text{if } r > 1; \\ 2k+1, & \text{if } r = 1. \end{cases} \end{aligned}$$

Finally, using the product-to-sum identity for trigonometric functions [1, 4.3.31–33], the last equation can be simplified, yielding

$$E_r(\mathbf{A}_{N,2k}^{(N-k)}) = \begin{cases} \sin\left[\dfrac{2k+1}{N}(r-1)\pi\right]\Big/\sin\left[\dfrac{1}{N}(r-1)\pi\right], & \text{if } r > 1; \\ 2k+1, & \text{if } r = 1. \end{cases} \tag{18}$$

With Eqs. (15), (17) and (18), one obtains for the elements of the nth power of $\mathbf{A}_{N,2k}^{(N-k)}$

$$\begin{aligned}\left(\mathbf{A}_{N,2k}^{(N-k)}\right)_{pq}^{n} &= \sum_{r=1}^{N} u_{pr} E_r^n u_{rq}^* \\ &= \frac{1}{N}\left\{E_1^n + \sum_{r=2}^{N} E_r^n e^{-2\pi i (r-1)(p-q)/N}\right\} \\ &= \frac{1}{N}\left\{(2k+1)^n + \sum_{r=1}^{2kn}\left(\frac{\sin\left[\frac{2k+1}{N} r\pi\right]}{\sin\left[\frac{1}{N} r\pi\right]}\right)^n e^{-2\pi i r (p-q)/N}\right\}.\end{aligned}$$

Taking the trace, finally, proves (13). □

Lemma 2 can now be utilized to establish a direct link between central $(2k+1)$-nomial coefficients and the nth-degree Fourier series approximation of a function via the Dirichlet kernel $D_k[\theta]$ (e.g., see [6] and [3]), the main result of this study:

Proposition 1. *Let $N = 2kn + 1$ with $k, n \in \mathbb{N}$. The sequence of central $(2k+1)$-nomial coefficients $M^{(2k,n)}$ is given by*

$$M^{(2k,n)} = \frac{1}{N}\left\{(2k+1)^n + \sum_{l=1}^{2kn}\left(D_k\left[\tfrac{1}{N} 2l\pi\right]\right)^n\right\}. \quad (19)$$

Proof. Using the trigonometric representation of Chebyshev polynomials of the second kind,

$$U_{2k}[\cos(\alpha)] = \frac{\sin[(2k+1)\alpha]}{\sin[\alpha]},$$

Eq. (13) takes the form

$$M^{(2k,n)} = \frac{1}{N}\left\{(2k+1)^n + \sum_{l=1}^{2kn}\left(U_{2k}\left[\cos\left(\tfrac{1}{N} l\pi\right)\right]\right)^n\right\}.$$

Observing that $U_{2k}[\cos(l\pi/N)] \equiv D_k[2l\pi/N]$ thus proves (19). □

4. Concluding remarks

Eqs. (13) and (19) are remarkable in several respects. First, the trigonometric terms in (13) show a striking similarity to the famous Kasteleyn product formula for the number of tilings of a $2n \times 2n$ square with 1×2 dominos (e.g., [7, Equation (13)], and [9] for context). Secondly, both (13) and (19) provide general, explicit representations of the sequences of central $(2k+1)$-nomial coefficients in terms of a linearly growing, but finite, sum over analytic constructs, thus effectively translating a combinatorial problem into an analytical one. Note specifically that $k = 1$, $n \in \mathbb{N}$ yields the sequence of central trinomial coefficients (OEIS A002426), $k = 2$ the sequence of central pentanomial coefficients (OEIS A005191), and $k = 3$ the sequence of central heptanomial coefficients (OEIS A025012).

Finally, utilizing trigonometric identities and identities obeyed by Chebyshev polynomials, these explicit representations may help to formulate general recurrences not just for central multinomial coefficients of a given sequence, but between different sequences. Moreover, by using different shift parameters m (see proof of Lemma 1), each $(2k+1)$-nomial coefficient could potentially be represented in a similarly explicit analytical form, thus allowing for a fast numerical calculation of arbitrary $(2k+1)$-nomial coefficients.

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