JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **201,** 516–531 (1996)
ARTICLE NO. 0271

Stability of Strong Detonation Travelling Waves to Combustion Model

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Submitted by William F. Ames

Received February 3, 1995

1. INTRODUCTION

In [8], from the one-dimensional combustion equations written in Lagrangian coordinates for the simple reactant to product mechanism, Majda obtain the following simplified combustion model:

$$\begin{aligned} (u + qz)_t + f(u)_x &= \beta u_{xx} \\ z_t &= -kH(u)z. \end{aligned} \tag{1.1}$$

It is hoped that this qualitative model retains most of the essential features of the Lagrangian equations, except the species diffusion (see [1, 11]).

* Supported in part by ONR under Grant N000014-91-J-1291 and WVU Senate Grant.
† In part supported by Italian Consiglio Nazionale delle Ricerche under the Grant n.211.01.26 and ONR under Grant N00014-91-J-1291.

0022-247X/96 $18.00

In the present paper, we consider the following model:

$$\begin{aligned} (u+qz)_t + f(u)_x &= \epsilon u_{xx} \\ z_t &= \nu z_{xx} - kH(u)z. \end{aligned} \tag{1.2}$$

We suppose

(H.1) $0 < \gamma_0 \leq f''(u) \leq M_1$, $f'(u) > 0$, $f'''(u) \leq 0$ $\forall u \in (-\infty, +\infty)$.

(H.2) $0 \leq H'(u) \leq M_1$, $H(u) = 0$ for $u \leq 0$; $H(u) = 1$ for $u \geq \eta_0 > 0$.

With rescaling to x and t, we might assume $\epsilon = 1$, k is replaced by another constant $B > 0$, and ν is replaced by d. We always assume $d > 0$. We rewrite (1.2) as

$$\begin{aligned} (u+qz)_t + f(u)_x &= u_{xx} \\ z_t &= dz_{xx} - BH(u)z. \end{aligned} \tag{1.3}$$

For the given $u^+ < 0 < u_* < u^*$ with

$$c = \frac{f(u^*) - f(u_*)}{u^* - u_*} = \frac{f(u^*) - f(u)}{u^* - (u^+ + q)}$$

and $u^+ + q \geq 0$ Li and Tan [6] prove that there exists a constant $B_0 > 0$; when $\beta k < B_0$, (1.1) admits a strong detonation travelling wave which connects $(u^*, 0)$ and $(u^+, 1)$. When $\beta k = B_0$, (1.1) admits a weak detonation travelling wave which connects $(u_*, 0)$ and $(u^+, 1)$; if $\beta k > B_0$, there is no travelling wave solution with speed c that connects the state $(u^+, 1)$ at $\xi = +\infty$.

Recently, in [7], Liu and Ying consider (1.1) for the case $u^+ + q \leq 0$. They first prove the existence of the travelling wave solutions and then they prove that the travelling wave is stable if $q > 0$ is sufficiently small.

In [5], Larrouturou studies the travelling wave solutions to (1.3). He also obtains the existence for both the weak and the strong detonations. His results are described in terms of q. That is, for fixed $B > 0$, he obtained a q_0 which depends on B: when $q > q_0$, (1.3) admits a strong detonation; when $q = q_0$, (1.3) admits a weak detonation. His results (for the strong detonation) can be described in terms of B.

THEOREM 1.1 (Li and Tan [6]). *Assume* (H.1) *and* (H.2) *and* $u^+ < 0 < \eta_0 < u^-$, $z^+ = 1$, $z^- = 0$ *hold. Let* $c = [f(u^-) - f(u)]/[u^- - (u^+ + q)]$. *If* $f'(u^+) < c < f'(u^-)$, *there exists a* $B_0 > 0$; *when* $0 < B \leq B_0$, (1.3) *admits a travelling wave* $((\phi, \zeta)(x - ct)$ *which satisfies*

$$(\phi, \zeta)(-\infty) = (u^-, z^-), \qquad (\phi, \zeta)(+\infty) = (u^+, z^+). \tag{1.4}$$

Moreover, the strong detonation profile above has a nonmonotone spike in the ϕ-profile; i.e., $\exists \xi_0 \in (-\infty, +\infty)$, when $\xi < \xi_0$, $\phi(\xi)$ is monotone increasing; when $\xi > \xi_0$, $\phi(\xi)$ is monotone decreasing.

The goal of this paper is to obtain the stability of the strong detonation of (1.3) under the assumption that $B > 0$ is sufficiently small. The definition of stability follows from Sattinger [12]. That is, we proceed to prove the solution of the Cauchy problem for (1.3) satisfies

$$\lim_{t\to\infty} \| u(\cdot, t) - \phi(\cdot - ct + \gamma) \| = 0 \tag{1.5}$$

$$\lim_{t\to\infty} \| z(\cdot, t) - \zeta(\cdot - ct + \gamma) \| = 0, \tag{1.6}$$

provided the initial perturbation, $u_0(x) - \phi(x)$ and $z_0(x) - \zeta(x)$, is sufficiently small in some sense. Here γ is a proper constant related to the initial perturbation.

Under the moving frame $\xi = x - ct$, (1.3) can be rewritten as

$$\begin{aligned} (u + qz)_t &= u_{\xi\xi} - f(u)_\xi + c(u + qz)_\xi \\ z_t &= dz_{\xi\xi} + cz_\xi - BH(u)z. \end{aligned} \tag{1.7}$$

The travelling wave solution of (1.3) is the steady state of (1.7) which is a C^2 function satisfying (1.4). Under the assumption of Theorem 1.1, one can easily prove that the travelling wave $(\phi, \zeta)(\xi) \to (u^\pm, z^\pm)$ exponentially fast as $\xi \to \pm\infty$.

Analogous to [4], our main result in this paper is described in terms of the weighted L_∞ norm

$$\| u(\xi) \|_{w_1} = \| w_1(\xi) u(\xi) \|_\infty$$

with $w_1(\xi) = \cosh(\eta\xi)$, where $\eta > 0$ is a proper constant.

THEOREM 1.2. *If $(u^\pm, z^\pm)$ and $f(u)$ satisfy the assumption in Theorem 1.1, then there exist $M > 0$, $B_1 > 0$, $\delta > 0$, when $0 < B \le B_1$,*

$$\| u_0(x) - \phi(x) \|_{w_1^3} \le \delta,$$

$$\| z_0(x) - \zeta(x) \|_{w_1^2} \le \delta, \qquad \| z_0'(x) - \zeta'(x) \|_{w_1} \le \delta,$$

and $(\phi(\xi), \zeta(\xi))$ *is suitably translated*; *we have*

$$\|u(\xi,t) - \phi(\xi)\|_{w_1} \le \frac{M}{1+t}$$

$$\|z(\xi,t) - \zeta(\xi)\|_{w_1} \le \frac{M}{1+t}$$

$$\left\|\partial_\xi(z(\xi,t) - \zeta(\xi))\right\|_{w_1} \le \frac{M}{1+t}.$$

In the proof of Theorem 1.2, we shall follow Goodman's integrand argument (see [2]) as well as Jones *et al*. [4] and Sattinger's weighted spectrum analysis (see [12]). In it, we introduce a new variable $v = u + qz$ and the perturbation is taken to be

$$V(\xi,t) = \int_{-\infty}^{\xi} (u + qz - (\phi + q\zeta))(s,t)\,ds$$

and we require the initial data satisfy

$$\int_{-\infty}^{+\infty} \big(u_0(x) + qz_0(x) - (\phi(x) + q\zeta(x))\big)\,dx = 0$$

which can be accomplished by a suitable choice of translation for the travelling wave.

The paper is organized as follows: In Section 2, we present some preliminaries. In Section 3, we will analyse the spectrum and obtain the resolvent estimates for the linearized operators. And, finally, in Section 4 we prove Theorem 1.2.

2. PRELIMINARIES

As we have seen in Section 1, in the moving frame, (1.3) becomes

$$\begin{aligned} (u+qz)_t &= u_{\xi\xi} - f(u)_\xi + c(u+qz)_\xi \\ z_t &= dz_{\xi\xi} + cz_\xi - BH(u)z. \end{aligned} \tag{2.1}$$

Denote $u_1 = u - \phi$, $w = z - \zeta$, $v = u_1 + qw$; we get

$$\begin{aligned} v_t &= (u_1)_{\xi\xi} - f(u)_\xi + f(\phi)_\xi + cv_\xi \\ \omega_t &= d\,\omega_{\xi\xi} + c\,\omega_\xi - BH(u)z + BH(\phi)\zeta. \end{aligned} \tag{2.2}$$

Let $V(\xi,t)=\int_{-\infty}^{\xi}v(s,t)\,ds$, we have $u_1=V_\xi-q\omega$. Integrating the first equation of (2.2), we obtain

$$\begin{aligned} V_t &= L_1V+h_1(V_\xi,\omega,\omega_\xi)\\ \omega_t &= L_2\,\omega+h_2(V_\xi,\omega,\zeta), \end{aligned} \tag{2.3}$$

where

$$L_1=\partial_\xi^2+(c-f'(\phi))\,\partial_\xi$$

$$L_2=d\,\partial_\xi^2+c\,\partial_\xi-BH(\phi)$$

$$h_1(V_\xi,\omega,\omega_\xi)=-f(u)+f(\phi)-f'(\phi)(u-\phi)+qf'(\phi)\,\omega-q\,\omega_\xi$$

$$h_2(V_\xi,\omega,\zeta)=-B(H(u)-H(\phi))\,\zeta.$$

Since $|\phi|\le C_1$ (independent of $B>0$), $0\le\zeta\le1$, by virtue of (H.1) and (H.2), one can easily get

$$\begin{aligned} |h_1| &\le C_2\left(|V_\xi|^2+|\omega|+|\omega|^2+|\omega_\xi|\right)\\ |h_2| &\le C_2B\left(|V_\xi|+|\omega|\right) \end{aligned} \tag{2.4}$$

Here C_2 is independent of $B>0$.

Following the notations in [4], let $BU(R,R)$ represent the Banach space of bounded and uniformly continuous functions that are bounded under the supremum norm.

Let $w_1(\xi)=\cosh(\eta\xi)$; $\eta>0$ is a constant to be determined. Define

$$\|v\|_{w_1}=\|w_1(\xi)v(\xi)\|_\infty$$

$$\|v\|_1=\|v\|_{w_1}+\|v_\xi\|_{w_1}.$$

And consider the Banach space

$$B_{w_1}=\{v\in BU(R,R);\|v\|_1<+\infty\}.$$

From Pego [9], as well as Pazy [8], we know the following:

LEMMA 2.1. *For each* $\eta>0$, $i=1,2$, *the operator* L_i *on* B_{w_1} *with domain* $D(L_i)$ *is a closed, densely defined operator. For some* α_i, β_i *real, with* $0<\alpha_i<\pi/2$, *the sector*

$$S_{\alpha_i,\beta_i}=\left\{\lambda\in C;\mathrm{Re}(\lambda-\beta_i)\ge-\cos\alpha_i\left|(\lambda-\beta_i)\right|\right\}$$

is in the resolvent set of L_i. *And for any such sector* S_{α_i, β_i},

$$\text{(a)} \quad \| R(\lambda, L_i) v \|_{w_1} \leq \frac{C}{|(\lambda - \beta_i)|} \|v\|_{w_1}$$

$$\text{(b)} \quad \| R(\lambda, L_i) v \|_1 \leq \frac{C}{|(\lambda - \beta_i)|} \|v\|_1$$

$$\text{(c)} \quad \| R(\lambda, L_i) v \|_1 \leq \frac{C}{|(\lambda - \beta_i)|^{1/2}} \|v\|_{w_1}$$

hold for all $\lambda \in S_{\alpha_i, \beta_i}$ *and* $v \in B_{w_1}$.

By this lemma, L_i generates an analytic semigroup $S_i(t)$ on the space B_{w_1}. So for each $(V_0, \omega_0) \in B_{w_1}$, the solution to (2.4) can be achieved via the variation of constant formula

$$V(t) = S_1(t)V_0 + \int_0^t S_1(t - \tau) h_1(\tau)\, d\tau$$

$$\omega(t) = S_2(t)\,\omega_0 + \int_0^t S_1(t - \tau) h_2(\tau)\, d\tau.$$

The solution can be extended as long as $\|V\|_{w_1} < +\infty$, $\|\omega\|_{w_1} < +\infty$.

3. RESOLVENT ESTIMATES

Consider the linear operator L:

$$Lu = u'' - 2bu' + qu, \qquad ' = d/d\xi \tag{3.1}$$

The resolvent equation for (3.1) is

$$u'' - 2bu' + qu - \lambda u = g(\xi). \tag{3.2}$$

Performing the transformation $u(\xi) = v(\xi)\exp(B(\xi))$ with $B(\xi) = \int_0^\xi b(s)\, ds$, we get

$$(\mathscr{M} - \lambda) v = e^{-B} g, \tag{3.3}$$

where $\mathscr{M} v = v'' + (b' - b^2 + q)v$.

$$p(\xi) = b'(\xi) - b^2(\xi) + q(\xi),$$

$$p_{\pm} = \lim_{\xi \to \pm\infty} p(\xi), \qquad q_{\pm} = \lim_{\xi \to \pm\infty} q(\xi).$$

In [12], Sattinger proves the following.

LEMMA 3.1 (Sattinger [12]). *We consider $\mathscr{M}$ as an linear operator on the Banach space L_∞. The eigenvalues of $\mathscr{M}$ in the interval $\lambda > \bar{p}$ ($\bar{p} = \max(p_+, p_-)$) are confined to the interval $\bar{p} < \lambda < p_1$ ($p_1 = \sup_{\xi \in R} p(\xi)$) and form a discrete set of points on the real axis which can cluster only at $\lambda = \bar{p}$. $\mathscr{M}$ has a continuum of eigenvalues in the semiinfinite interval $-\infty < \lambda < \bar{p}$.*

Let $w = 1 + e^{-B(\xi)}$. By $\sigma_w(L)$ we mean the spectrum of L relative to the Banach space B_w. Assume

$$\lim_{\xi \to \pm\infty} 2b(\xi) = k^{\pm} \tag{3.4}$$

$$\int_{-\infty}^{0} |p(\xi) - p^-| < +\infty, \qquad \int_0^{+\infty} |p(\xi) - p^+| < +\infty, \tag{3.5}$$

$$\int_{-\infty}^{0} \left| b(\xi) - \frac{k^-}{2} \right| < +\infty, \qquad \int_0^{+\infty} \left| b(\xi) - \frac{k^+}{2} \right| < +\infty, \tag{3.6}$$

Sattinger also proves the following.

THEOREM 3.2 (Sattinger [12]). *Under the above assumption on b and q, we may draw the following conclusions about the resolvent operator $(\lambda - L)^{-1}$, considered as a transformation on B_w:*

(I) *If $k^+ > 0$, $k^- > 0$, we have $\rho_w(L) \cap \mathscr{P}^+ = \rho_\infty(\mathscr{M}) \cap \mathscr{P}^+$.*

(II) *If $k^+ < 0$, $k^- < 0$, we have $\rho_w(L) \cap \mathscr{P}^- = \rho_\infty(\mathscr{M}) \cap \mathscr{P}^-$.*

(III) *If $k^+ < 0$, $k^- > 0$, we have $\rho_w(L) = \rho_\infty(\mathscr{M})$.*

(IV) *If $k^+ > 0$, $k^- < 0$, we have $\rho_w(L) \cap (\mathscr{P}^+ \cap \mathscr{P}^-) = \rho_\infty(\mathscr{M}) \cap (\mathscr{P}^+ \cap \mathscr{P}^-)$,*

where $\rho_w(L)$ is the resolvent set of L relative to the Banach space B_w, $\rho_\infty(\mathscr{M})$ is the resolvent set of $\mathscr{M}$ relative to the Banach space L_∞. $\mathscr{P}^+$ and $\mathscr{P}^-$ are exterior to the parabolas

$$\rho_\pm = \frac{(k^\pm)^2}{2(1 + \cos\theta)} \tag{3.7}$$

where $\lambda - p_\pm = \rho_\pm e^{i\theta} < -\pi < \theta < \pi$. The parabolas (3.7) meet the real axis at $\lambda = q_\pm$ and extend to infinity in the left half-plane.

THEOREM 3.3. *Let L_1, L_2 be defined as in (2.4). Under the assumption of Theorem 1.1, there exist constants $\eta > 0$ and $\delta_1 > 0$ which are independent of B, when $0 < B \le B_0$; we have*

$$\sup\{\operatorname{Re}\lambda;\ \lambda \in \sigma_{w_1}(L_i)\} \le -\delta_1 < 0$$

for $i = 1, 2$.

Proof. For operator L_2, without loss of generality, we assume $d = 1$. And so

$$L_2 u = u'' + cu' - BH(\phi)u. \tag{3.8}$$

Comparing to (3.2), we have $2b = -c < 0$, $k^+ = k^- = -c/2 < 0$. That is just the case (II) in Theorem 3.2. The correspondent $\mathscr{P}^-$ intersects the real axis at $\lambda = q^- = -B$ and extends to infinity in the left half-plane. By Lemma 3.1, we know

$$\sigma_\infty(\mathscr{M}) \cap \mathscr{P}^- = \emptyset.$$

So, we have $\mathscr{P}^- \subset \rho_w(L_2)$ (for $w = 1 + e^{c\xi/2}$).

If we choose $\eta \geq c/2$, we also have $\mathscr{P}^- \subset \rho_{w_1}(L_2)$.

Now we analyse the solutions of the ordinary differential equations

$$u'' + cu' - (\lambda + BH(\phi))u = 0$$

with $\lambda \notin \mathscr{P}^-$. When $\xi \to \pm\infty$, we have the corresponding characteristic equations

$$\gamma^2 + c\gamma - \left(\lambda + BH(\phi_\pm)\right) = 0; \tag{3.9}$$

here we denote $\phi_\pm = u^\pm$. (3.9) may also be written as

(i) for $\xi \to +\infty$,

$$\gamma^2 + c\gamma - \lambda = 0 \tag{3.10}$$

(ii) for $\xi \to -\infty$,

$$\gamma^2 + c\gamma - \lambda - B = 0. \tag{3.11}$$

One can find that there exists $\delta_0 > 0$ (independent of B). When B is small and $\lambda \in (\mathscr{P}^-)^c \cap \{\lambda; \operatorname{Re}\lambda > -\delta_0\}$, (3.10) admits no solution which satisfies $\operatorname{Re}\lambda \leq -\eta$, and (3.11) admits no solution which satisfies $\operatorname{Re}\lambda \geq \eta$. Combining this result with $\mathscr{P}^- \subset \rho_{w_1}(L_2)$, we get

$$\sup\left\{\operatorname{Re}\lambda; \lambda \in \sigma_{w_1}(L_2)\right\} \leq -\delta_1 < 0,$$

where $\delta_1 > 0$ is independent of B: $0 < B \leq B_0$.

For L_1, it satisfies $L_1 V = V'' + (c - f'(\phi))V'$. We have

$$2b = -c + f'(\phi), \qquad q = 0,$$

$$k^+ = -\frac{c - f'(\phi_+)}{2} < 0, \qquad k^- = -\frac{c - f'(\phi_-)}{2} > 0$$

which corresponds to case (III) in Theorem 3.2. So we have

$$\rho_w(L_1) = \rho_\infty(\mathscr{M}_1)$$

where $w(\xi) = 1 + \exp(\int_0^\xi (c - f'(\phi))\, ds)$.

The corresponding

$$p_\pm = q_\pm - (b_\pm)^2 = -\tfrac{1}{4}(c - f'(\phi_\pm))^2$$

$$\bar{p} = -\tfrac{1}{4}\min\Big\{(c - f'(\phi_+))^2, (c - f'(\phi_-))^2\Big\} = -\delta_2 < 0$$

$$p(\xi) = b'(\xi) - b^2(\xi) = f''(\phi)\phi' - (c - f'(\phi))^2.$$

To prove the theorem, it suffices to prove that there exists $\delta_3 > 0$ which is independent of $B > 0$, s.t.,

$$p(\xi) \le -\delta_3 < 0 \qquad \forall \xi \in (-\infty, +\infty).$$

To this end, we integrate the equation of ϕ,

$$\phi'' - f'(\phi)\phi' + c(\phi + q\zeta)' = 0,$$

to get

$$\begin{aligned} \phi' &= f(\phi) - f(u^+) - c(\phi - u^+ + q(\zeta - 1)) \\ &= (\phi + q\zeta - (u^+ + q))\left(\frac{f(\phi) - f(u^+)}{\phi + q\zeta - u^+ - q} - c\right). \end{aligned}$$

For the strong detonation profile shown in Theorem 1.1, we know there exists a $u_1 \in (u_*, u^-)$, s.t. $c = f'(u_1)$ (see Fig. 3.1)

In the neighborhood of u_1, we have

$$\frac{f(\phi) - f(u^+)}{\phi + q\zeta - u^+ - q} - c \le \frac{f(\phi) - f(u^+)}{\phi - u^+ - q} - c + \epsilon \le -\delta_3.$$

So, in the neighborhood N_0 of $u = u_1$, we have

$$p(\xi) \le -\delta_3 < 0.$$

When $\xi > \xi_0$, we have $\phi' \le 0$. And when $\phi \notin N_0$, we have $(c - f'(\phi))^2 \ge \delta_3 > 0$. Thus we get $p(\xi) \le -\delta_3 < 0\ \forall \xi \ge \xi_0$. Here ξ_0 is the maximum point as described in Theorem 1.1.

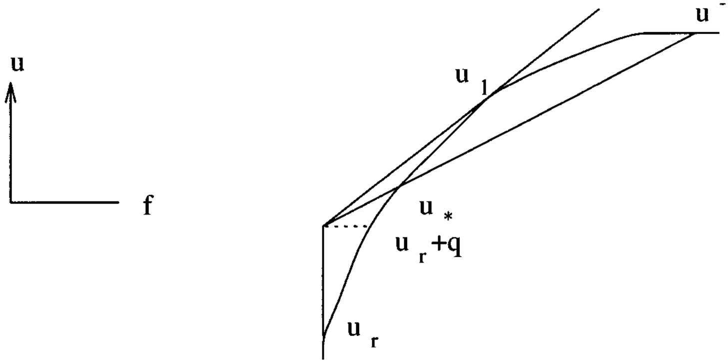

FIGURE 3.1

Now we turn to consider the case of $\xi < \xi_0$. In this case, $\phi' \geq 0$ and so $\phi(\xi) \geq u^-$. Hence,

$$\begin{aligned} p(\xi) &= f''(\phi)\big(f(\phi) - f(u^+) - c(\phi + q\zeta - u^+ - q)\big) - \big(c - f'(\phi)\big)^2 \\ &\leq f''(\phi)\big(f(\phi) - f(u^+)\big) - cf''(\phi)(\phi - u^+ - q) - \big(c - f'(\phi)\big)^2. \end{aligned}$$

Let $F(\phi) = f''(\phi)(f(\phi) - f(u^+)) - cf''(\phi)(\phi - u^+ - q) - (c - f'(\phi))^2$. We have

$$\begin{aligned} F(u^-) &\leq -\delta_1 < 0 \\ F'(\phi) &= f'''(\phi)(\phi - u^+ - q)\left(\frac{f(\phi) - f(u^+)}{\phi - u^+ - q} - c\right) \\ &\quad + f''(\phi)\big(f'(\phi) - c + 2(c - f'(\phi))\big) \\ &\leq -f''(\phi)\big(f'(\phi) - c\big) \leq 0. \end{aligned}$$

So we have $F(\phi) \leq -\delta_3$, that is, $\forall \xi \leq \xi_0$, we have $p(\xi) \leq -\delta_3 < 0$.

Combining it with Lemma 3.1 and Theorem 3.2, we know

$$\sup\big\{\mathrm{Re}\,\lambda;\ \lambda \in \sigma_{w_1}(L_1)\big\} \leq -\delta_3 < 0.$$

Choosing $\eta \geq \frac{1}{2}\max\{c, |f'(u^+) - c|, |f'(u^-) - c|\}$, for the corresponding w_1, we know Theorem 3.3 is true.

Following the semigroup theory in Pazy [9] and Henry [3], we know that L_i generates an analytic semigroup $S_i(t)$ which satisfies

$$\| S_i(t) \|_{w_1} \le M e^{-\mu t} \tag{3.12}$$

$$\| L_i S_i(t) \|_{w_1} \le \frac{1}{t} M e^{-\mu t}. \tag{3.13}$$

Here M and $\mu > 0$ are constants independent of $0 < B \le B_0$.

But on the other hand, $\omega = \zeta(\xi)$ is a nonzero solution of

$$L_2 \omega = 0. \tag{3.14}$$

Using the Wronskian determinant, we can obtain another solution $\omega = \omega_2$ which is linearly independent of $\omega = \zeta(\xi)$. By virtue of Cramer's rule, one can easily get the expression for the solution of

$$L_2 \omega = v. \tag{3.15}$$

Similar to the discussion in [4], we can prove that

$$\| \partial_\xi \omega \|_{w_1} \le M_1 \| v \|_{w_1}. \tag{3.16}$$

Here M_1 is also independent of B with $B > 0$ sufficiently small. (3.16) might be rewritten as

$$\| \partial_\xi R(0, L_2) \|_{w_1} \le M_1. \tag{3.17}$$

Similarly, for L_1 we have

$$\| \partial_\xi R(0, L_1) \|_{w_1} \le M_1. \qquad (3.17)$$

Combining (3.17), (3.18) with (3.13), we know

$$\| \partial_\xi S_i(t) \|_{w_1} \le \frac{1}{t} M e^{-\mu t} \qquad \text{for } t \ge \tfrac{1}{2}. \tag{3.19}$$

Since $S_i(t)$ is an analytic semigroup, for $0 < t \leq \frac{1}{2}$, we also have

$$\left\| \partial_\xi S_i(t) \right\|_{w_1} \leq M/\sqrt{t} \tag{3.20}$$

$$\left\| \partial_\xi S_i(t) v \right\|_{w_1} \leq \frac{M_2}{1+t}\left(\|v\|_{w_1^2} + \|v_\xi\|_{w_1} \right) \qquad \text{for } t > 0. \tag{3.22}$$

Here M_2 might depend on $B > 0$.

4. STABILITY FOR THE TRAVELLING WAVES

In this section we proceed to prove Theorem 1.2. From (2.13), (2.14) we know

$$\partial_\xi V(t) = \partial_\xi S_1(t) V_0 + \int_0^t \partial_\xi S_1(t-\tau) h_1(\tau)\, d\tau \tag{4.1}$$

$$\omega(t) = S_2(t)\,\omega_0 + \int_0^t S_2(t-\tau) h_2(\tau)\, d\tau \tag{4.2}$$

$$\partial_\xi \omega(t) = \partial_\xi S_2(t)\,\omega_0 + \int_0^t \partial_\xi S_2(t-\tau) h_2(\tau)\, d\tau. \tag{4.3}$$

When $t \geq \frac{1}{2}$, from (4.1)–(4.3) we get

$$\begin{aligned} \left\| \partial_\xi V(t) \right\|_{w_1} \leq{} & \frac{M_2}{1+t}\left(\|V_0\|_{w_1} + \|\partial_\xi V_0\|_{w_1} \right) \\ & + \int_0^{t-1/2} \frac{M}{t-\tau} e^{-\mu(t-\tau)} \Big(\left\| V_\xi(\tau) \right\|^2_{w_1^{1/2}} + \left\| \omega_\xi(\tau) \right\|_{w_1} \\ & \qquad + \left\| \omega(\tau) \right\|^2_{w_1^{1/2}} + \left\| \omega(\tau) \right\|_{w_1} \Big)\, d\tau \\ & + \int_{t-1/2}^t \frac{M}{\sqrt{t-\tau}} \Big(\left\| V_\xi(\tau) \right\|^2_{w_1^{1/2}} + \left\| \omega_\xi(\tau) \right\|_{w_1} \\ & \qquad + \left\| \omega(\tau) \right\|^2_{w_1^{1/2}} + \left\| \omega(\tau) \right\|_{w_1} \Big)\, d\tau \end{aligned} \tag{4.4}$$

$$\left\| \omega(t) \right\|_{w_1} \leq M e^{-\mu t} \|\omega_0\|_{w_1} + MB \int_0^t e^{-\mu(t-\tau)} \Big(\left\| V_\xi(\tau) \right\|_{w_1} + \left\| \omega_\xi(\tau) \right\|_{w_1} \Big)\, d\tau \tag{4.5}$$

$$\begin{aligned}\left\| \omega_\xi(t) \right\|_{w_1} \leq{}& \frac{M_2}{1+t}\left(\|\omega_0\|_{w_1} + \|\partial_\xi \omega_0\|_{w_1} \right) \\ &+ \int_0^{t-1/2} \frac{M}{t-\tau} e^{-\mu(t-\tau)} \left(\left\| V_\xi(\tau) \right\|_{w_1} + \left\| \omega_\xi(\tau) \right\|_{w_1} \right) d\tau \\ &+ MB \int_{t-1/2}^{t} \frac{1}{\sqrt{t-\tau}} \left(\left\| V_\xi(\tau) \right\|_{w_1} + \left\| \omega_\xi(\tau) \right\|_{w_1} \right) d\tau \end{aligned} \tag{4.6}$$

Here and the following M is the general constant independent of $0 < B \leq B_0$. Let

$$\begin{aligned} \rho_1(t) &= \sup_{0 \leq \tau \leq t} (1+\tau) \left\| V_\xi(\tau) \right\|_{w_1} \\ \rho_2(t) &= \sup_{0 \leq \tau \leq t} (1+\tau) \left\| \omega(\tau) \right\|_{w_1} \\ \rho_1(t) &= \sup_{0 \leq \tau \leq t} (1+\tau) \left\| \omega_\xi(\tau) \right\|_{w_1}. \end{aligned}$$

From (4.4), we obtain

$$\begin{aligned} \left\| V_\xi(t) \right\|_{w_1} \leq{}& \frac{M_2}{1+t}\left(\|V_0\|_{w_1} + \|\partial_\xi V_0\|_{w_1} \right) \\ &+ M\left(\rho_1^2(t) + \rho_2^2(t) \right) \int_0^{t-1/2} \frac{e^{-\mu(t-\tau)}}{(t-\tau)(1+\tau)^2} d\tau \\ &+ M\left(\rho_2(t) + \rho_3(t) \right) \int_0^{t-1/2} \frac{e^{-\mu(t-\tau)}}{(t-\tau)(1+\tau)^2} d\tau \\ &+ M\left(\rho_1^2(t) + \rho_2^2(t) \right) \int_{t-1/2}^{t} \frac{1}{\sqrt{t-\tau}(1+\tau)^2} d\tau \\ &+ M\left(\rho_2(t) + \rho_3(t) \right) \int_{t-1/2}^{t} \frac{1}{\sqrt{t-\tau}(1+\tau)^2} d\tau \\ \leq{}& \frac{M_2}{1+t}\left(\|V_0\|_{w_1} + \|\partial_\xi V_0\|_{w_1} \right) \\ &+ \frac{M}{1+t}\left(\rho_1^2(t) + \rho_2^2(t) + \rho_2(t) + \rho_3(t) \right). \end{aligned} \tag{4.7}$$

When $0 < t < \frac{1}{2}$, we have

$$\begin{aligned} \left\| V_\xi(t) \right\|_{w_1} \le \frac{M_2}{1+t}\left(\|V_0\|_{w_1} + \|\partial_\xi V_0\|_{w_1}\right) \\ + M\left(\rho_1^2(t) + \rho_2^2(t)\right) \int_0^t \frac{1}{\sqrt{t-\tau}(1+\tau)^2}\, d\tau \\ + M\left(\rho_2(t) + \rho_3(t)\right) \int_0^t \frac{1}{\sqrt{t-\tau}(1+\tau)^2}\, d\tau \\ \le \frac{M_2}{1+t}\left(\|V_0\|_{w_1} + \|\partial_\xi V_0\|_{w_1}\right) \\ + \frac{M}{1+t}\left(\rho_1^2(t) + \rho_2^2(t) + \rho_2(t) + \rho_3(t)\right) \end{aligned} \tag{4.8}$$

In the above, for $t \ge \frac{1}{2}$, $\tau \in (0, t - \frac{1}{2})$, we have employed the following inequalities:

(1) $$\frac{1}{t-\tau}\frac{1}{1+\tau} \le \frac{C_1}{1+t}$$

(2) $$\int_0^{t-1/2} \frac{1}{\sqrt{t-\tau}(1+\tau)^j}\, d\tau \le \frac{C_1}{1+t}, \qquad j = 1, 2,$$

(3) $$\int_0^{t-1/2} \frac{e^{-\mu(t-\tau)}}{(t-\tau)(1+\tau)^2}\, d\tau \le \frac{C_1}{1+t} e^{-\mu t} \int_0^{t-1/2} e^{-\mu\tau}\, d\tau.$$

Combining (4.7), (4.8), we get

$$\rho_1(t) \le M_2\left(\|V_0\|_{w_1} + \|\partial_\xi V_0\|_{w_1}\right) + M\left(\rho_1^2(t) + \rho_2^2(t) + \rho_2(t) + \rho_3(t)\right). \tag{4.9}$$

Similarly, from (4.6) we have

$$\rho_3(t) \le M_2\left(\|\omega_0\|_{w_1} + \|\partial_\xi \omega_0\|_{w_1}\right) + MB\left(\rho_2(t) + \rho_1(t)\right). \tag{4.10}$$

By (4.5), we obtain

$$\rho_2(t) \le M\|\omega_0\|_{w_1} + MB\left(\rho_2(t) + \rho_1(t)\right). \tag{4.11}$$

Let $0 < B < B_1 = \min(B_0, 1/2M)$; from (2.11) we know

$$\rho_2(t) \le M\|\omega_0\|_{w_1} + MB\rho_1(t). \tag{4.12}$$

Substituting (4.10), (4.12) into (4.9) we get

$$\rho_1(t) \le M_2\big(\|V_0\|_{w_1} + \|\partial_\xi V_0\|_{w_1} + \|\omega_0\|_{w_1} + \|\partial_\xi \omega_0\|_{w_1}\big) + M\rho_1^2(t) + MB\rho_1(t).$$

For $0 < B < B_1 = \min(B_0, 1/2M)$, we have

$$\rho_1(t) \le M_2\big(\|V_0\|_{w_1} + \|\partial_\xi V_0\|_{w_1} + \|\omega_0\|_{w_1} + \|\partial_\xi \omega_0\|_{w_1}\big) + M\rho_1^2(t). \quad (4.13)$$

Since $\int_{-\infty}^{+\infty} V_0(\xi)\,d\xi = 0$, similar to Lemma A.3 in [4], we have

$$\|V_0\|_{w_1^2} \le C_2\|\partial_\xi V_0\|_{w_1^3}. \quad (4.14)$$

So (4.13) can be rewritten as

$$\rho_1(t) \le M_2\big(\|\partial_\xi V_0\|_{w_1^3} + \|\omega_0\|_{w_1} + \|\partial_\xi \omega_0\|_{w_1^2}\big) + M\rho_1^2(t). \quad (4.15)$$

From (4.15), we know that there exist constants $\delta_0 > 0$, $M > 0$; when

$$\|\partial_\xi V_0\|_{w_1^3} + \|\omega_0\|_{w_1} + \|\partial_\xi \omega_0\|_{w_1^2} \le \delta_0,$$

we have

$$\rho_1(t) \le M \qquad \forall t \ge 0.$$

Returning to (4.10), (4.12), we get

$$\rho_2(t) \le M, \qquad \rho_3(t) \le M \qquad \forall t \ge 0,$$

which completes the proof of Theorem 1.2.

Remark. From the expression in (2.13), we have

$$\|V(t)\|_{w_1} \le M_2 e^{-\mu t}\|V_0\|_{w_1} + M\int_0^t e^{-\mu(t-\tau)}\Big(\|V_\xi(\tau)\|^2_{w_1^{1/2}} + \|\omega_\xi(\tau)\|_{w_1} + \|\omega(\tau)\|^2_{w_1^{1/2}} + \|\omega(\tau)\|_{w_1}\Big)\,d\tau.$$

From Theorem 1.2, we know

$$\|V(t)\|_{w_1} \le M < +\infty \qquad \forall t \ge 0,$$

which implies the global existence of the solutions of the Cauchy problem in the space B_{w_1} for (2.3) if the initial perturbation satisfies the conditions in Theorem 1.2.

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