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Deformations of Lie Algebras

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0. INTRODUCTION

The set of Lie algebra products on a fixed n-dimensional vector space V over a field F is the algebraic subset $\mathscr{L}_n$ of the affine variety $\mathrm{Hom}(\Lambda^2 V, V)$ consisting of alternating bilinear maps which satisfy the Jacobi identity. We assume the field F is algebraically closed and has characteristic 0. In the following, we refer to elements of $\mathscr{L}_n$ as "Lie algebras." The isomorphism class of the Lie algebra $\mu \in \mathscr{L}_n$ is the orbit $O(\mu)$ in $\mathscr{L}_n$ under the following $GL(n, F)$-action on $\mathscr{L}_n$:

$$(g \cdot \mu)(x, y) = g(\mu(g^{-1}x, g^{-1}y)). \tag{0.1}$$

We say that a Lie algebra μ' is a *degeneration* of a Lie algebra μ (or μ *degenerates* to μ') if μ' is in the boundary of the orbit of $O(\mu)$, i.e., in the complement of $O(\mu)$ in the Zariski closure of $O(\mu)$. For example, let μ and μ' be the 4-dimensional Lie algebras given by

$$\begin{aligned} &\mu(e_1, e_2) = e_3 && \mu'(e_1, e_2) = e_3 \\ &\mu(e_1, e_3) = e_4 && \mu'(e_i, e_j) = 0 \text{ otherwise} \\ &\mu(e_i, e_j) = 0 \text{ otherwise.} && \end{aligned} \tag{0.2}$$

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For each nonzero element t of F, let $g_t \in GL(n, F)$ be diagonal with diagonal entries $(1, 1, 1, t)$. Then $g_t \cdot \mu$ is given by

$$\begin{aligned}(g_t \cdot \mu)(e_1, e_2) &= e_3 \\ (g_t \cdot \mu)(e_1, e_3) &= te_4 \\ (g_t \cdot \mu)(e_i, e_j) &= 0 \qquad \text{otherwise.}\end{aligned} \tag{0.3}$$

From the parameterization by $t \in F \backslash \{0\}$ of the subset $g_t \cdot \mu$ of $O(\mu)$, one sees that every polynomial function vanishing on $O(\mu)$ also vanishes on the Lie algebra resulting from replacing t with 0, i.e., on μ'. Therefore μ' belongs to the Zariski closure of $O(\mu)$ and hence to the boundary of $O(\mu)$.

Degeneration of Lie algebras has been the focus of various research projects in mathematics and in physics for more than thirty years. For instance, Nijenhuis and Richardson studied the relationship between cohomology and degeneration [14], Vergne studied the geometry of the variety of nilpotent Lie algebras [16], Carles studied the geometry of the variety $\mathscr{L}_n$ of Lie algebras for small n [2], and Seeley has recently obtained results on degenerations of nilpotent Lie algebras [15]. In the physics literature, the concept of degeneration (referred to as "contraction" in physics) first appeared 40 years ago [11].

The results in this paper were motivated by the following conjecture:

Conjecture. Every nilpotent Lie algebra of dimension two or more is the degeneration of some other Lie algebra.

A Lie algebra μ cannot be a degeneration of another Lie algebra if $H^2(\mu, \mu) = 0$ [14]. For example, if μ is semisimple then $H^2(\mu, \mu) = 0$, and so a semisimple Lie algebra is not a degeneration of another Lie algebra. Support for the Conjecture comes from the fact that a nilpotent Lie algebra of dimension greater than 1 always has non-vanishing 2-cohomology [3]. A consequence of this conjecture would be that every nilpotent Lie algebra of dimension greater than 1 is a degeneration of some non-nilpotent Lie algebra. The conjecture fails for solvable Lie algebras; the non-abelian 2-dimensional Lie algebra is a solvable Lie algebra which is a degeneration of no other Lie algebra.

Characterizing orbit closure as in [8], one sees that every degeneration of a Lie algebra μ to a Lie algebra μ' gives rise to a deformation of μ'. Thus the existence of degenerations to μ implies the existence of non-trivial deformations of μ. In this paper, we construct "linear" deformations, i.e., deformations of a Lie algebra μ of the form $\mu + t\psi$. Throughout the paper we consider Lie algebras of dimension greater than or equal to 2. We show that the following Lie algebras (of dimension two or more) have non-trivial deformations:

(1) any nilpotent Lie algebra of nilpotency class 1 or 2

(2) any graded nilpotent Lie algebra

(3) any free nilpotent Lie algebra

(4) any nilpotent Lie algebra with a codimension one ideal K which satisfies (1), (2), or (3)

(5) any direct product of a nilpotent Lie algebra and a nontrivial abelian Lie algebra

(6) any nilpotent Lie algebra with a codimension one ideal K such that $\mathrm{Der}(K)$ contains a non-nilpotent element

(7) any nilpotent Lie algebra with a codimension one ideal K whose center $Z(K)$ is not contained in its derived subalgebra $[K, K]$

(8) any nilpotent Lie algebra with a codimension 2 ideal H and a codimension 1 ideal K containing H such that $\mathrm{Cent}_K(H)$ is not contained in H

(9) any solvable Lie algebra L of solvability length s with a codimension 1 ideal K and a derivation $\delta \in \mathrm{Der}(K)$ such that the sth derived subspace of $\mathrm{im}\,\delta$ is nonzero

(10) any Lie algebra L with a codimension one ideal K such that the center $Z(K)$ of K is not contained in $[L, L]$ and $Z(L) \cap K$ is not contained in $[K, K]$.

Note that every solvable Lie algebra has a codimension one ideal, so these conditions are not as restrictive as they might appear at first reading. Which of these deformations correspond to degenerations remains an open question.

1. Construction of Linear Deformations

For the purposes of this paper, we use Gerstenhaber's definition of deformation [7]:

1.1. Definition. If V is a vector space over F and $F((t))$ is the power series field over F, let $V_{F((t))} = V \otimes_F F((t))$. If $\mu \in \mathrm{Hom}(\Lambda^2 V, V)$ is a Lie product on V, then μ defines a Lie product on $V_{F((t))}$ by the natural extension $\mu(v \otimes f, w \otimes g) = \mu(v, w) \otimes fg$. If μ_t is a Lie product on $V_{F((t))}$ such that

$$\mu_t = \mu + t\varphi_1 + t^2\varphi_2 + \cdots \tag{1.1}$$

with $\varphi_i \in \mathrm{Hom}(\Lambda^2 V, V)$, then μ_t is a *deformation* of μ. The deformation μ_t is *trivial* if there is an automorphism of $V_{F((t))}$ of the form

$$g_t = \mathrm{Id} + tg_1 + t^2 g_2 + \cdots \tag{1.2}$$

with $g_i \in \mathrm{End}(V)$ such that $\mu_t(x, y) = g_t(\mu(g_t^{-1}x, g_t^{-1}y))$.

A more general definition of deformation appears in [5], and it is under this more general definition that one can realize every degeneration as a deformation (see [6]). The realization of a degeneration as a deformation is easily seen in the example given by (0.2) and (0.3). In the Lie product given by (0.3), let "t" denote a variable instead of an element of the field F; then (0.3) is the deformation of μ' given by $\mu' + t(\mu - \mu')$.

For clarity and ease of notation, we will use either [,] or μ to denote the Lie product of a Lie algebra L. We use the standard definitions for Lie algebra cohomology (see, for instance, [10]). For $\sigma, \tau \in \mathrm{Hom}(\Lambda^2 V, V)$, we use $\sum_{\mathrm{cyc}} \tau(\sigma(x, y), z)$ to denote

$$\tau(\sigma(x, y), z) + \tau(\sigma(y, z), x) + \tau(\sigma(z, x), y).$$

Then $\mu \in \mathrm{Hom}(\Lambda^2 V, V)$ is a Lie algebra if and only if

$$\sum_{\mathrm{cyc}} \mu(\mu(x, y), z) = 0 \qquad \text{for all} \quad x, y, z \in V \tag{1.3}$$

and $\psi \in \mathrm{Hom}(\Lambda^2 V, V)$ is in $Z^2(L, L)$ if and only if

$$\sum_{\mathrm{cyc}} \psi([x, y], z) + \sum_{\mathrm{cyc}} [\psi(x, y), z] = 0 \qquad \text{for all} \quad x, y, z \in V. \tag{1.4}$$

1.2. Lemma. *Let L be a Lie algebra with Lie product $\mu \in \mathscr{L}_n$. The map $\mu + t\psi$ is a deformation of L if and only if ψ is a Lie algebra and $\psi \in Z^2(L, L)$.*

Proof. The map $\mu + t\psi$ is a Lie algebra over $F((t))$ if and only if

$$\begin{aligned} 0 &= \sum_{\mathrm{cyc}} (\mu + t\psi)((\mu + t\psi)(x, y), z) \\ &= \sum_{\mathrm{cyc}} \mu(\mu(x, y), z) + t\left[\sum_{\mathrm{cyc}} \mu(\psi(x, y), z) + \sum_{\mathrm{cyc}} \psi(\mu(x, y), z)\right] \\ &\quad + t^2 \sum_{\mathrm{cyc}} \psi(\psi(x, y), z). \end{aligned}$$

The first summand is zero because μ is a Lie algebra, the second term is zero if and only if $\psi \in Z^2(L, L)$, and the last term is zero if and only if ψ is a Lie algebra. ∎

1.3. LEMMA. *Let L be a Lie algebra with Lie product $\mu \in \mathscr{L}_n$. If $\mu + t\psi$ is a trivial deformation of L, then the bilinear map ψ is in $B^2(L, L)$.*

Proof. If $\mu + t\psi$ is a trivial deformation of μ, then there is an automorphism g_t of $V_{F((t))}$ of the form (1.2) such that $(\mu + t\psi)(x, y) = g_t(\mu(g_t^{-1}x, g_t^{-1}y))$ (or equivalently, an automorphism g_t of the form (1.2) satisfying $(\mu + t\psi)(x, y) = g_t^{-1}(\mu(g_t x, g_t y))$). Then for all $x, y \in V$,

$$\begin{aligned}(\mu + t\psi)(x, y) &= g_t^{-1}(\mu(g_t x, g_t y)) \\ &= g_t^{-1}(\mu(x, y) + t[\mu(x, g_1 y) + \mu(g_1 x, y)] \\ &\quad + \text{higher order terms} \\ &= \mu(x, y) + t[\mu(x, g_1 y) + \mu(g_1 x, y) - g_1(\mu(x, y))] \\ &\quad + \text{higher order terms.}\end{aligned}$$

Comparing the terms of degree 1, we see that ψ is equal to the 2-coboundary dg_1. Therefore $\psi \in B^2(L, L)$. ∎

We introduce the following notation: for $\psi \in \mathscr{L}_n$ and $x \in V$, ψ_x is the endomorphism of V given by $\psi_x(v) = \psi(x, v)$.

1.4. LEMMA. *Let $\mu + t\psi$ be a deformation of a nilpotent Lie algebra μ. If the Lie algebra ψ is not nilpotent, then the deformation $\mu + t\psi$ is non-trivial.*

Proof. Suppose that $\mu + t\psi$ is a trivial deformation. From the definition, it is easy to verify that the Lie algebra $\mu + t\psi$ (over $F((t))$) is nilpotent. Because the highest order term of $[(\mu + t\psi)_x]^q(y)$ is $t^q\psi_x^q(y)$, it follows that ψ_x is a nilpotent endomorphism of V for every $x \in V$. Therefore the Lie algebra ψ is nilpotent, contradicting the hypothesis. ∎

Let K be a codimension one ideal of a (not necessarily nilpotent) Lie algebra L. Applying Dixmier's construction in [3] to our setting, one obtains 2-cochains in $C^2(L, L)$ from 1-cochains in $C^1(K, K)$ as follows. Let $\varphi \in C^1(K, K)$ and choose $x \in L \backslash K$. Define $\bar{\varphi} \in C^2(L, L)$ by

$$\begin{aligned}\bar{\varphi}(x, k) &= \varphi(k) \qquad \text{for all} \quad k \in K \\ \bar{\varphi}(K, K) &= 0.\end{aligned} \tag{1.5}$$

If L is a solvable Lie algebra, then it has a codimension 1 ideal, and so we can always carry out this construction for solvable Lie algebras.

1.5. LEMMA. *Let K be a codimension 1 ideal of a Lie algebra L with Lie product μ. If $\varphi \in Z^1(K, K)$, then $\bar{\varphi}$ is a Lie algebra and $\bar{\varphi} \in Z^2(L, L)$. Consequently, $\mu + t\bar{\varphi}$ is a deformation of μ.*

Proof. Because $L = K \oplus Fx$ and $\bar{\varphi}(K, K) = 0$, it suffices to verify conditions (1.3) and (1.4) on $K \times K \times \{x\}$. To verify that $\bar{\varphi}$ is a Lie algebra, we consider

$$\sum_{\text{cyc}} \bar{\varphi}(\bar{\varphi}(h, k), x), \qquad h, k \in K.$$

Because $\bar{\varphi}(K, K) = 0$, each summand is zero and it follows that $\bar{\varphi}$ is a Lie algebra. To verify that $\bar{\varphi}$ is a 2-cocycle, we establish that the following sum is zero for all $k, h \in K$:

$$\sum_{\text{cyc}} [\bar{\varphi}(h, k), x] + \sum_{\text{cyc}} \bar{\varphi}([h, k], x).$$

After eliminating terms which drop out because $\bar{\varphi}(K, K) = 0$, we have:

$$\begin{aligned}&[\bar{\varphi}(k\, x), h] + [\bar{\varphi}(x, h), k] + \bar{\varphi}([h, k], x)\\ &\quad = -(\varphi(k), h] + [\varphi(h), k] - \varphi([h, k]),\end{aligned}$$

which is zero because $\varphi \in Z^1(K, K)$. Therefore $\bar{\varphi}$ is a 2-cocycle. ∎

2. LIE ALGEBRAS WITH NON-TRIVIAL DEFORMATIONS

In this section, we use the construction described by (1.5) to define deformations of Lie algebras.

2.1. THEOREM. *Let L be a nilpotent Lie algebra with Lie product μ and let K be a codimension 1 ideal of L such that some element of $Z^1(K, K) = Der(K)$ has a nonzero eigenvalue. Then L has a non-trivial deformation.*

Proof. Let $\varphi \in Z^1(K, K)$ such that φ has a nonzero eigenvalue. Construct $\bar{\varphi}$ as in (1.5). It follows from Lemma 1.5 that $\mu + t\bar{\varphi}$ is a deformation of L. From the construction of $\bar{\varphi}$ we see that an eigenvector of φ is an eigenvector of $\bar{\varphi}_x$ of the same eigenvalue. Because φ has a nonzero eigenvalue, $\bar{\varphi}_x$ is not nilpotent. Therefore the Lie algebra $\bar{\varphi}$ is not nilpotent. It follows from Lemma 1.4 that $\mu + t\bar{\varphi}$ is a non-trivial deformation of μ. ∎

In [12], Jacobson proved that, over a field of characteristic 0, a Lie algebra with a nonsingular derivation is nilpotent. If the converse were true, then Theorem 2.1 would provide non-trivial deformations for every nilpotent Lie algebra. Unfortunately, the converse is false; Dixmier and Lister have a counterexample [4]. We obtain Propositions 2.2 through 2.9 by establishing that certain types of Lie algebras have nonsingular derivations.

2.2. PROPOSITION. *Let L be a nilpotent Lie algebra with a codimension one ideal K whose center $Z(K)$ is not contained in $[K, K]$. Then L has a non-trivial deformation.*

Proof. Let $k \in Z(K) \setminus [K, K]$. Choose a linear transformation $\varphi: K \to Z(K)$ such that $\varphi(k) = k$ and $\varphi([K, K]) = 0$. When φ is regarded as a linear transformation from K into K, it is easy to verify that φ is a 1-cocycle. Since the 1-cocycle φ has a nonzero eigenvalue, it follows from Theorem 2.1 that L has a non-trivial deformation. ∎

2.3. PROPOSITION. *If a nilpotent Lie algebra L has a codimension one ideal K of nilpotency class 2, then L has a non-trivial deformation.*

Proof. Let $K = [K, K] \oplus W$ (vector space direct sum), and define $\varphi \in C^1(K, K)$ as follows:

$$\varphi(v + w) = 2v + w \qquad \text{where} \quad v \in [K, K] \text{ and } w \in W.$$

Because $[[K, K], K] = \{0\}$, we see that $\varphi \in Z^1(K, K)$:

$$\begin{aligned} &\varphi([v_1 + w_1, v_2 + w_2]) - [\varphi(v_1 + w_1), v_2 + w_2] + [\varphi(v_2 + w_2), v_1 + w_1] \\ &\quad = 2[w_1, w_2] - [w_1, w_2] + [w_2, w_1] \qquad (\text{because } v_i \in Z(K)) \\ &\quad = 0. \end{aligned}$$

Since φ is a nonsingular derivation of K, it follows from Theorem 2.1 that L has a non-trivial deformation. ∎

2.4. COROLLARY. *If L is a Lie algebra of nilpotency class* 1 *or* 2, *then L has a non-trivial deformation.*

Proof. If L is abelian (nilpotency class 1), then the conclusion follows from Proposition 2.2. If L is of nilpotency class 2, then L has a codimension 1 ideal K of nilpotency class 2 or of nilpotency class 1. In the first case, the conclusion follows from Proposition 2.3. In the second case, the conclusion follows from Proposition 2.2. ▮

A Lie algebra L is *graded* if there is a vector space decomposition $L = \bigoplus_{i=1}^{s} V_i$ such that, for $e_i \in V_i$ and $e_j \in V_j$, we have $[e_i, e_j] \in V_{i+j}$.

2.5. Proposition. *Every nilpotent Lie algebra with a graded ideal K of codimension 1 has a non-trivial deformation.*

Proof. We construct a nonsingular derivation δ of K as follows. If K is graded by $K = \bigoplus_{i=1}^{s} V_i$, let $\delta: K \to K$ be given by

$$\delta(e_i) = ie_i \qquad \text{for each} \quad e_i \in V_i.$$

We see that the endomorphism δ is a derivation as follows:

$$[\delta e_i, e_j] + [e_i, \delta e_j] = i[e_i, e_j] + j[e_i, e_j] = (i+j)[e_i, e_j].$$

Because the vector $[e_i, e_j]$ is in V_{i+j}, it follows that $(i+j)[e_i, e_j] = \delta[e_i, e_j]$. Therefore δ is a nonsingular derivation of the ideal K. It follows from Theorem 2.1 that the Lie algebra L has a non-trivial deformation. ▮

Every graded solvable Lie algebra of dimension greater than 1 has a homogeneous ideal of codimension 1. Furthermore, every free nilpotent Lie algebra is graded (see [1, Chap. II.5] for definitions and results). Consequently, we have the following corollaries:

2.6. Corollary. *Every graded nilpotent Lie algebra has a non-trivial deformation.*

2.7. Corollary. *If a nilpotent Lie algebra L is free nilpotent or has a codimension 1 ideal which is free nilpotent, then L has a non-trivial deformation.*

2.8. Proposition. *Let L be a nilpotent Lie algebra with a codimension 1 ideal K and a codimension 2 ideal H such that $H \subset K$ and $\mathrm{Cent}_K(H) \not\subseteq H$. Then L has a non-trivial deformation.*

Proof. By Theorem 2.1, it suffices to construct a derivation δ of K which has a nonzero eigenvalue. Let $z \in \mathrm{Cent}_K(H) \backslash H$. Since $z \notin H$, we have $K = H \oplus Fz$. Define $\delta: K \to K$ as follows:

$$\delta(h + \lambda z) = \lambda z.$$

The endomorphism δ is a derivation:

$$\begin{aligned}[\delta(h_1+\lambda_1 z), h_2+\lambda_2 z] &+ [h_1+\lambda_1 z, \delta(h_2+\lambda_2 z)]\\ &= [\lambda_1 z, h_2+\lambda_2 z] + [h_1+\lambda_1 z, \lambda_2 z]\\ &= 0 = \delta[h_1+\lambda_1 z, h_2+\lambda_2 z].\end{aligned}$$

The vector z is an eigenvector of δ with eigenvalue 1. By Theorem 2.1, the derivation δ defines a non-trivial deformation of L. ∎

2.9. COROLLARY. *If L is a nilpotent Lie algebra, then $L \times F$ has a non-trivial deformation, where F is the 1-dimensional abelian Lie algbra.*

Proof. Let K be any codimension 1 ideal of L; then $K\times F$ is a codimension 1 ideal of $L\times F$ and $K\times\{0\}$ is a codimension 2 ideal contained in $K\times F$ such that $\mathrm{Cent}_{K\times F}(K\times\{0\}) \not\subseteq K\times\{0\}$. It follows from Proposition 2.8 that $L\times F$ has a non-trivial deformation. ∎

For any Lie algebra ψ define the multilinear map ψ^i, $i\geq 1$, as follows.

Let $\psi^1=\psi$ and let $\psi^i(v_1, ..., v_{i+1}) = \psi(\psi^{i-1}(v_1, ..., v_i), v_{i+1})$.

If ψ is a nilpotent Lie algebra of nilpotency class m, then $\psi^m=0$.

2.10. PROPOSITION. *Let L be a nilpotent Lie algebra of nilpotency class m with Lie product μ. If L has a codimension 1 ideal K and a derivation $\delta\in \mathrm{Der}(K)$ such that $\delta^m\neq 0$, then L has a non-trivial deformation.*

Proof. Let $\delta\in\mathrm{Der}(K)$, and choose $k\in K$ such that $\delta^m(k)\neq 0$ and let $\bar\delta$ denote the 2-cochain defined as in (1.5). If the deformation $\mu+t\bar\delta$ is trivial, then $(\mu+t\bar\delta)^m=0$. But the mth coefficient of $(\mu+t\bar\delta)^m$ is $\bar\delta^m$ and

$$\bar\delta^m(k, x, x, ..., x) = \delta^m(k)\neq 0,$$

contradicting the assumption that $(\mu+t\bar\delta)^m=0$. Therefore the deformation $\mu+t\bar\delta$ is non-trivial. ∎

Even though Dixmier's construction appeared in the context of nilpotent Lie algebras, it certainly applies to non-nilpotent Lie algebras. In the following, we consider solvable Lie algebras and arbitrary Lie algebras satisfying certain conditions.

For $\mu\in\mathrm{Hom}(\Lambda^2 V, V)$, define $\mu^{(i)}\in\mathrm{Hom}(\otimes^{2^i} V, V)$, $i\geq 0$, as follows:

$$\mu^{(0)}(v)=v$$

$$\mu^{(i+1)}(v_1, ..., v_{2^{i+1}}) = \mu(\mu^{(i)}(v_1, ..., v_{2^i}), \mu^{(i)}(v_{2^i+1}, ..., v_{2^{i+1}})).$$

If μ is a solvable Lie algebra, then $\mu^{(i)}=0$ for some i; let

$$\mathrm{sol}(\mu)=\max\{i\colon \mu^{(i)}\neq 0\}.$$

2.11. PROPOSITION. *Let L be a solvable Lie algebra with Lie product μ. If L has a codimension 1 ideal K and a derivation $\delta\in \mathrm{Der}(K)$ such that $\mu^{(s)}(\mathrm{im}\,\delta, ..., \mathrm{im}\,\delta)\neq 0$, where $s=\mathrm{sol}(\mu)$, then L has a non-trivial deformation.*

Proof. Let δ be a derivation of K such that $\mu^{(s)}(\mathrm{im}\,\delta, ..., \mathrm{im}\,\delta)\neq 0$. We prove that $\mu+t\bar{\delta}$ is a non-trivial deformation by showing that $\mathrm{sol}(\mu+t\bar{\delta})$ is strictly greater than $\mathrm{sol}(\mu)$.

For $i\geqslant 0$, we have

$$(\mu+t\bar{\delta})^{(i)}=\sum_{j=1}^{2^i-1} t^j\varphi_j^{(i)},$$

where $\varphi_j^{(i)}\in \mathrm{Hom}(\bigoplus^{2^i} L, L)$. Since every codimension 1 ideal of a solvable Lie algebra contains the derived ideal of the Lie algebra, we have $\mu(L, L)\subseteq K$. Also $\mathrm{im}\,\bar{\delta}=\mathrm{im}\,\delta\subseteq K$; therefore $\mathrm{im}\,\varphi_j^{(i)}\subseteq K$ for all $i\geqslant 1$ and for all j. Because $\bar{\delta}(K, K)=0$, we see that for $i\geqslant 1$ we have

$$\begin{aligned}(\mu+t\bar{\delta})^{(i+1)}&(v_1, ..., v_{2^{i+1}})\\ &=(\mu+t\bar{\delta})((\mu+t\bar{\delta})^{(i)}(v_1, ..., v_{2^i}), (\mu+t\bar{\delta})^{(i)}(v_{2^i+1}, ..., v_{2^{i+1}}))\\ &=\mu((\mu+t\bar{\delta})^{(i)}(v_1, ..., v_{2^i}), (\mu+t\bar{\delta})^{(i)}(v_{2^i+1}, ..., v_{2^{i+1}})).\end{aligned}\tag{2.1}$$

Thus $\deg(\mu+\bar{\delta})^{(i+1)}\leqslant 2\deg(\mu+t\bar{\delta})^{(i)}$. Since $\deg(\mu+t\bar{\delta})^{(1)}=1$, it follows that $\deg(\mu+t\bar{\delta})^{(i+1)}\leqslant 2^i$. From (2.1) we see that

$$\varphi_{2^i}^{(i+1)}=\mu(\varphi_{2^{i-1}}^{(i)}, \varphi_{2^{i-1}}^{(i)}),\qquad i\geqslant 1.$$

Arguing inductively we see that $\mathrm{im}\,\varphi_{2^i}^{(i+1)}=\mu^{(i)}(\mathrm{im}\,\delta, ..., \mathrm{im}\,\delta)$:
In the case $i=0$ we have $\mathrm{im}\,\varphi_1^{(1)}=\mathrm{im}\,\bar{\delta}=\mathrm{im}\,\delta$.
For $i>0$, we have

$$\begin{aligned}\mathrm{im}\,\varphi_{2^i}^{(i+1)}&=\mu(\mathrm{im}\,\varphi_{2^{i-1}}^{(i)}, \mathrm{im}\,\varphi_{2^{i-1}}^{(i)})\\ &=\mu(\mu^{(i-1)}(\mathrm{im}\,\delta, ..., \mathrm{im}\,\delta), \mu^{(i-1)}(\mathrm{im}\,\delta, ..., \mathrm{im}\,\delta))\\ &\qquad\text{(by induction)}\\ &=\mu^{(i)}(\mathrm{im}\,\delta, ..., \mathrm{im}\,\delta).\end{aligned}$$

Because $\mu^{(s)}(\mathrm{im}\,\delta, ..., \mathrm{im}\,\delta)\neq 0$, it follows that $\varphi_{2^s}^{(s+1)}\neq 0$ and hence

$$\mathrm{sol}(\mu+t\bar{\delta})\geqslant s+1>\mathrm{sol}(\mu),$$

contradicting the assumption that $\mu+t\bar{\delta}$ is a trivial deformation. ∎

2.12. PROPOSITION. *Let L be a (not necessarily nilpotent) Lie algebra with Lie product μ such that L has a codimension one ideal K. If $Z(K)$ is not contained in $[L, L]$ and $Z(L) \cap K$ is not contained in $[K, K]$, then L has a non-trivial deformation.*

Proof. Choose $x \in L \backslash K$, $k \in (Z(L) \cap K) \backslash [K, K]$, and $y \in Z(K) \backslash [L, L]$. Write K as a direct sum $H \oplus Fk$ where $[K, K] \subseteq H$. Choose $\varphi \in C^1(K, K)$ such that $\varphi(H) = 0$ and $\varphi(k) = y$. Because im $\varphi \subseteq Z(K)$ and $\varphi([K, K]) = 0$, it follows that $\varphi \in Z^1(K, K)$. Then by Lemma 1.5, $\mu + t\bar{\varphi}$ is a deformation of μ.

To establish that this deformation is non-trivial, we show that $\bar{\varphi}$ is not a 2-coboundary. If there is a 1-cochain $\sigma \in C^1(L, L)$ with $d\sigma = \bar{\varphi}$, then

$$y = \bar{\varphi}(x, k_1) = d\sigma(x, k_1) = [x, \sigma(k_1)] - [k_1, \sigma(x)] - \sigma([x, k_1]).$$

Because $k_1 \in Z(L)$, the last two terms are zero and it follows that $y \in [L, L]$, a contradiction.

Therefore, by Lemma 1.3, $\mu + t\bar{\varphi}$ is a non-trivial deformation of μ. ∎

We have established that many classes of solvable Lie algebras have non-trivial deformations. We see in the following propositions that the Lie algebra μ shares certain properties with the deformation $\mu + t\bar{\delta}$ we have constructed.

2.13. PROPOSITION. *Let L be a solvable Lie algebra with Lie product μ and let $\delta \in \mathrm{Der}(K)$, where K is a codimension 1 ideal of L. Then $\mu + t\bar{\delta}$ is a solvable Lie algebra and* $\mathrm{sol}(\mu) \leqslant \mathrm{sol}(\mu + t\bar{\delta}) \leqslant 1 + \mathrm{sol}(\mu)$.

Proof. In the proof of Proposition 2.11, we established that $\deg(\mu + t\bar{\delta})^{(i)} \leqslant 2^{i-1}$. Let $(\mu + t\bar{\delta})^{(i)} = \sum_{j=1}^{2^{i-1}} t^j \varphi_j^{(i)}$. Using induction on i, we prove that im $\varphi_j^{(i)} \subseteq \mathrm{im}\, \mu^{(i-1)}$ for all i, j. The case $i = 1$ is trivial. From (2.1), we have

$$\begin{aligned}
&(\mu + t\bar{\delta})^{(i+1)}(v_1, ..., v_{2^{i+1}}) \\
&= \mu((\mu + t\bar{\delta})^{(i)}(v_1, ..., v_{2^i}), (\mu + t\bar{\delta})^{(i)}(v_{2^i+1}, ..., v_{2^{i+1}})) \\
&= \mu\left(\sum_{j=1}^{2^{i-1}} t^j \varphi_j^{(i)}(v_1, ..., v_{2^i}), \sum_{j=1}^{2^{i-1}} t^j \varphi_j^{(i)}(v_{2^i+1}, ..., v_{2^{i+1}})\right) \\
&= \sum_{j=1}^{2^i} t^j \left[\sum_{p+q=j} \mu(\varphi_p^{(i)}(v_1, ..., v_{2^i}), \varphi_q^{(i)}(v_{2^i+1}, ..., v_{2^{i+1}}))\right].
\end{aligned} \tag{2.2}$$

From the induction hypothesis we have

$$\operatorname{im} \varphi_p^{(i)} \subseteq \operatorname{im} \mu^{(i-1)} \qquad \text{and} \qquad \operatorname{im} \varphi_q^{(i)} \subseteq \operatorname{im} \mu^{(i-1)}.$$

It follows from (2.2) that

$$\begin{aligned} \operatorname{im} \varphi_j^{(i+1)} &= \sum_{p+q=j} \mu(\operatorname{im} \varphi_p^{(i)}, \operatorname{im} \varphi_q^{(i)}) \\ &\subseteq \mu(\operatorname{im} \mu^{(i-1)}, \operatorname{im} \mu^{(i-1)}) \\ &= \operatorname{im} \mu^{(i)}. \end{aligned} \tag{2.3}$$

Let $s = \operatorname{sol}(\mu)$. Because $\mu^{(s+1)} = 0$, we see that $\mu + t\bar{\delta}$ is solvable and $\operatorname{sol}(\mu + t\bar{\delta}) \leqslant 1 + \operatorname{sol}(\mu)$. Note that $\varphi_0^{(i)} = \mu^{(i)}$ for all i; since $\mu^{(s)} \neq 0$, it follows that $\operatorname{sol}(\mu + t\bar{\delta}) \geqslant \operatorname{sol}(\mu)$. ∎

2.14. PROPOSITION. *Let L be a nilpotent Lie algebra with Lie product μ and let $\delta \in \operatorname{Der}(K)$, where K is a codimension 1 ideal of L. Then $\mu + t\bar{\delta}$ is a solvable Lie algebra which is a nilpotent Lie algebra if and only if the derivation δ is a nilpotent endomorphism.*

Proof. From Proposition 2.13, we know that $\mu + t\bar{\delta}$ is solvable. In the proof of Theorem 2.1, we established that the deformation $\mu + t\bar{\delta}$ is not a nilpotent Lie algebra if the endomorphism δ is not nilpotent.

Suppose the endomorphism δ is nilpotent. Reversing the argument in the proof of Theorem 2.1, we see that the Lie algebra $\bar{\delta}$ is nilpotent. For any nilpotent Lie algebra ψ on a finite-dimensional vector space V, if W is a nonzero subspace of V, then $\dim \psi(V, W) < \dim W$ (otherwise ψ would not be nilpotent). Since the coefficients of $(\mu + t\bar{\delta})^i$ are sums of i successive products, where each product is either μ or $\bar{\delta}$ (both nilpotent Lie algebras), we conclude that the images of the coefficients of $(\mu + t\bar{\delta})^i$ have dimension less than or equal to $\dim L - i$. It follows that $(\mu + t\bar{\delta})^{\dim L} = 0$. ∎

Even if a nilpotent Lie algebra has a non-trivial deformation (and we conjecture that all nilpotent Lie algebras have non-trivial deformations), it does not necessarily have a non-trivial *nilpotent* deformation. The 3-dimensional Heisenberg Lie algebra has no non-trivial nilpotent deformations.

3. Deformations and Degenerations

Consider the 3-dimensional solvable Lie algebra μ given by

$$[e_1, e_2] = e_2$$
$$[e_i, e_j] = 0 \qquad \text{otherwise.}$$

Let $K = \langle e_2, e_3 \rangle$ and let $\delta: K \to K$ be given by

$$\delta(e_2) = 0$$
$$\delta(e_3) = e_3.$$

The resulting deformation $\mu + t\bar{\delta}$ is given by

$$(\mu + t\bar{\delta})(e_1, e_2) = e_2$$
$$(\mu + t\bar{\delta})(e_1, e_3) = te_3$$
$$(\mu + t\bar{\delta})(e_i, e_j) = 0 \qquad \text{otherwise.}$$

From [13], we know that $\mu + \alpha\bar{\delta}$ and $\mu + \beta\bar{\delta}$ are isomorphic if and only if $\alpha\beta = 1$; i.e., $\mu + \alpha\bar{\delta}$ and $\mu + \beta\bar{\delta}$ are in different orbits unless $\alpha\beta = 1$. Therefore the deformation $\mu + t\bar{\delta}$ does not produce a degeneration.

On the other hand, the example (0.3) in the Introduction is a deformation which *does* realize a degeneration. Moreover, every degeneration of a nilpotent Lie algebra of dimension less than 6 can be realized by a linear deformation (simply examine every case presented in [9]). Which of the deformations constructed in Section 2 represent degenerations? Can all degenerations be represented by linear deformations? In this section we present examples which demonstrate various aspects of these questions.

A *filtration* on a Lie algebra L is a nested sequence of subspaces of L

$$\cdots \supseteq V_{-2} \supseteq V_{-1} \supseteq V_0 \supseteq V_1 \supseteq \cdots$$

such that $\mu(V_i, V_j) \subseteq V_{i+j}$. For each filtration on L there is an associated graded Lie algebra W (of the same dimension as L) defined as follows. Let $W = \bigoplus_{j \in \mathbb{Z}} V_j/V_{j+1}$ and define a Lie product φ on W by

$$\varphi(\bar{x}, \bar{y}) = \overline{[x, y]} \in V_{s+t}/V_{s+t+1} \qquad \text{for} \quad x \in V_s \text{ and } y \in V_t.$$

In [9], we observed that every filtration of a Lie algebra μ produces a degeneration from μ to the associated graded Lie algebra φ. All of the degenerations considered in [9] are given by filtrations. It is unknown whether or not every degeneration arises from a filtration. All of the examples of degeneration of Lie algebras of which we are aware are given by filtrations and can be realized by linear deformations. We conjecture that every degeneration given by a filtration can be realized by a linear deformation. The following proposition lends support to this conjecture:

3.1. PROPOSITION. *Let L be a Lie algebra with Lie product μ which is filtered in the following manner:*

$$L = V_0 \supseteq V_1 \supseteq V_2 = 0 \qquad [V_i, V_j] \subseteq V_{i+j}.$$

Then the degeneration to the associated graded Lie algebra can be realized by a linear deformation.

Proof. Choose a complementary subspace Y to V_1 in V_0 so that $V_0 = V_1 \oplus Y$. Let π_i be the projection map onto the ith summand, $i = 1, 2$. Via the natural correspondence between Y and V_0/V_1, we may realize the associated graded Lie algebra product μ' as a product on V_0 as follows:

$$\begin{aligned} \mu'(y_1, y_2) &= \pi_2(\mu(y_1, y_2)) && \text{for } y_1, y_2 \in Y \\ \mu'(y, w) &= \mu(y, w) && \text{for } y \in Y, w \in V_1 \\ \mu'(w_1, w_2) &= \mu(w_1, w_2) = 0 && \text{for } w_1, w_2 \in V_1. \end{aligned}$$

Then $\mu = \mu' + \psi$, where

$$\begin{aligned} \psi(y_1, y_2) &= \pi_1(\mu(y_1, y_2)) && \text{for } y_1, y_2 \in Y \\ \psi(y, w_1) &= \psi(w_1, w_2) = 0 && \text{for } y \in Y, w_1, w_2 \in V_1. \end{aligned}$$

The map ψ is a Lie algebra because $\psi(\psi(V_0, V_0), V_0) = 0$. The fact that $\psi \in Z^2(\mu, \mu)$ follows from straightforward computation using the fact that μ is a Lie algebra. It follows from Lemma 1.2 that $\mu' + t\psi$ is a deformation of μ'. ∎

By constructing non-trivial deformations for large classes of nilpotent Lie algebras, we have produced evidence to support the conjecture that every nilpotent Lie algebra has a non-trivial deformation. If so, is every nilpotent Lie algebra the degeneration of some other Lie algebra? We invite the reader to explore the open quetions we have presented here.

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