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Note(s)

# Asymptotic distributions of maxima of complete and incomplete samples from multivariate stationary Gaussian sequences

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ABSTRACT

Let $(\mathbf{X}_n)$ be a sequence of $d$-dimensional stationary Gaussian vectors, and let $\mathbf{M}_n$ denote the partial maxima of $\{\mathbf{X}_k, 1 \le k \le n\}$. Suppose that there are missing data in each component of $\mathbf{X}_k$ and let $\widetilde{\mathbf{M}}_n$ denote the partial maxima of the observed variables. In this note, we study two kinds of asymptotic distributions of the random vector $(\widetilde{\mathbf{M}}_n, \mathbf{M}_n)$ where the correlation and cross-correlation satisfy some dependence conditions.



## 1. Introduction

Let $(X_n)$ be a sequence of standard stationary Gaussian random variables. Define $M_n = \max_{1\le k\le n} X_k$ and $r(n) = \mathrm{E}X_1X_{n+1}$, the partial maximum and the correlation, respectively. It is well known that the limiting distribution of the normalized maxima differs according to the rate of convergence of $r(n)$. Berman [1] proved that the limiting distribution of $M_n$ is similar to that for an independent and identically distributed Gaussian sequence if $r(n)\log n \to 0$, i.e.

$$\lim_{n\to\infty} \mathrm{P}\left(a_n^{-1}(M_n - b_n) \le x\right) = \exp\{-\exp(-x)\},$$

where

$$a_n = (2\log n)^{-1/2}, \qquad b_n = a_n^{-1} + a_n(\log\log n + \log 4\pi)/2. \tag{1.1}$$

For $r(n)\log n \to \gamma \in (0,\infty)$, Mittal and Ylvisaker [8] proved that

$$\lim_{n\to\infty} \mathrm{P}\left(a_n^{-1}(M_n - b_n) \le x\right) = \int_{-\infty}^{\infty} \exp\left\{-\exp\left(-x-\gamma+\sqrt{2\gamma}z\right)\right\}\phi(z)\mathrm{d}z,$$

where $\phi(x)$ is the probability density function (pdf) of a standard Gaussian random variable. For $r(n)\log n \to \infty$ with some regular conditions for $r(n)$, the strongly dependent case, McCormick and Mittal [6] proved that

$$\lim_{n\to\infty} \mathrm{P}\left(r^{-1/2}(n)\left(M_n - (1-r(n))^{1/2}b_n\right) \le x\right) = \Phi(x),$$

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where $\Phi(x)$ is the cumulative distribution function (cdf) of a standard Gaussian random variable. For more details, see Sections 4.3, 6.5 and 6.6 of Leadbetter et al. [4]. McCormick [5] introduced the following condition:

$$\frac{\log n}{n}\sum_{k=1}^{n}|r(k)-r(n)|=o(1) \tag{1.2}$$

as $n\to\infty$ and considered the maximum centered at the sample mean

$$\overline{X}_n=\frac{1}{n}\sum_{i=1}^{n}X_i.$$

For $r(k)<1$ for some $k$ and (1.2) holding, McCormick [5] showed that

$$\lim_{n\to\infty}\mathrm{P}\left(a_n^{-1}\left(\frac{M_n-\overline{X}_n}{\sqrt{1-r(n)}}-b_n\right)\le x\right)=\exp\left\{-\exp(-x)\right\},\quad x\in R.$$

McCormick and Qi [7] and Peng and Nadarajah [11] considered the joint limiting behavior of the partial sum and maximum. Hu et al. [2] and Peng [10] studied the joint limiting distribution of the partial sum and point process of exceedances.

For the limiting distribution of the extremes of vector Gaussian sequences, see [3,12–14]. The aim of this note is to establish the joint limiting distribution of the maxima of complete and incomplete samples of stationary Gaussian vector sequences under some weakly and strongly dependent conditions similar to those of [9,8], where the univariate case is considered.

Let $\{\mathbf{X}_n=(X_{n1},X_{n2},\ldots,Z_{nd}),\ n\ge 1\}$ be a sequence of $d$-dimensional stationary Gaussian random vectors, i.e.

$$\mathrm{E}X_{ni}=0,\qquad \mathrm{Var}\,(X_{ni})=1 \tag{1.3}$$

for $n\ge 1$ and $1\le i\le d$ and

$$\mathrm{Cov}\left(X_{li},X_{kj}\right)=r_{ij}(|l-k|) \tag{1.4}$$

for $1\le i,j\le d,\ k,l\ge 1$. Let $\mathbf{M}_n^{(s)}$ denote the $s$th order statistic (componentwise) and $\mathbf{M}_n^{(1)}=\mathbf{M}_n$. Define the norming constants

$$\mathbf{a}_n=(a_n,\ldots,a_n),\qquad \mathbf{b}_n=(b_n,\ldots,b_n) \tag{1.5}$$

with $a_n$ and $b_n$ defined by (1.1). Wiśniewski [13] proved the following main result:

**Theorem 1.1.** *Let* $(\mathbf{X}_n)$ *satisfy* $r_{ij}(n)\log n\to\rho_{ij}$ *for* $1\le i,j\le d$ *and* $\max_{1\le i\ne j\le d,n\ge 0}|r_{ij}|<1$. *Then*

$$\mathbf{a}_n^{-1}\left(\mathbf{M}_n^{(s)}-\mathbf{b}_n\right)\xrightarrow{\mathrm{d}}\mathbf{M}^{(s)}(\boldsymbol{\rho})+\mathbf{R}_{\rho}\mathbf{Z}$$

*for* $s\ge 1$, *where* $\boldsymbol{\rho}=(\rho_{ii},\ 1\le i\le d)$ *and* $\mathbf{R}_{\rho}=\left(\sqrt{2\rho_{ii}},\ 1\le i\le d\right)$, $\mathbf{Z}$ *is a standard Gaussian vector with variance–covariance matrix* $\left(\rho_{ij}/\sqrt{\rho_{ii}\rho_{jj}}\right)_{d\times d}$ *and*

$$\mathrm{P}\left(\mathbf{M}^{(s)}(\boldsymbol{\rho})\le x\right)=\prod_{i=1}^{d}\exp\left\{-\exp\left(-x_i-\rho_{ii}\right)\right\}\sum_{j=0}^{s-1}\left\{\exp\left(-x_i-\rho_{ii}\right)\right\}^j/j!.$$

*Furthermore,* $\mathbf{M}^{(s)}(\boldsymbol{\rho})$ *and* $\mathbf{Z}$ *are independent of each other.*

Now suppose that some of the variables of $\mathbf{X}_k$ can be observed. Let $\varepsilon_{ki}$ denote the indicator of the event that $X_{ki}$ is observed. Then $S_{ni}=\varepsilon_{1i}+\varepsilon_{2i}+\cdots+\varepsilon_{ni}$ is the number of observed random variables from the set $\{X_{1i},X_{2i},\ldots,X_{ni}\}$, where $1\le i\le d$. In order to prove the main results, the following conditions are needed:

C$_1$. The sequence $(S_{ni})$ satisfies

$$\frac{S_{ni}}{n}\xrightarrow{\mathrm{P}}p_i\in(0,1]$$

as $n\to\infty$ for $1\le i\le d$.

C$_2$. The indicator random variables $\{\varepsilon_{ki},\ 1\le k\le n,\ 1\le i\le d\}$ are independent, and also independent of $\{\mathbf{X}_k\}$.

For simplicity, we will use the following notation throughout this note:

$$\mathbf{u}_n=(u_{n1},\ldots,u_{nd}),\qquad \mathbf{v}_n=(v_{n1},\ldots,v_{nd}), \tag{1.6}$$

where $u_{ni}=a_nx_i+b_n$, $v_{ni}=a_ny_i+b_n$ and $x_i<y_i$ for $1\le i\le d$, and

$$M_{ni}=\max\{X_{1i},\ldots,X_{ni}\},\qquad \widetilde{M}_{ni}=\begin{cases}\max\{X_{ki},\ 1\le k\le n,\ \varepsilon_{ki}=1\}, & \text{if } S_{ni}\ge 1,\\ -\infty, & \text{if } S_{ni}=0.\end{cases}$$

## 2. Main results

In this section, we consider the asymptotic distribution of the maxima of complete and incomplete samples from multivariate weakly dependent and strongly dependent stationary Gaussian sequences. We obtain two theorems. Theorem 2.1 is for the weakly dependent case and Theorem 2.2 is for the strongly dependent case. Corollaries 2.1 and 2.2 are particular cases of these theorems.

**Theorem 2.1.** *Let the d-dimensional stationary Gaussian vector sequence* $\{\mathbf{X}_n\}$ *satisfy the conditions* (1.3) *and* (1.4)*. Suppose that the conditions* $\mathrm{C}_1$ *and* $\mathrm{C}_2$ *hold. Assume further that*

$$r_{ij}(n)\log n \to 0, \quad 1 \le i,\ j \le d \tag{2.1}$$

*as* $n \to \infty$*. Then, for* $\mathbf{u}_n$ *and* $\mathbf{v}_n$ *defined by* (1.6)*, we have*

$$\mathrm{P}\left\{\widetilde{\mathbf{M}}_n \le \mathbf{u}_n, \mathbf{M}_n \le \mathbf{v}_n\right\} \xrightarrow{\mathrm{d}} \prod_{i}^{d} \exp\{-p_i \exp(-x_i)\}\exp\{-(1-p_i)\exp(-y_i)\} \tag{2.2}$$

*as* $n \to \infty$.

**Corollary 2.1.** *Under the conditions of* Theorem 2.1*, for* $\mathbf{a}_n$ *and* $\mathbf{b}_n$ *defined by* (1.5)*, we have*

$$\mathrm{P}\left\{\widetilde{\mathbf{M}}_n \le \mathbf{a}_n\mathbf{x} + \mathbf{b}_n\right\} \xrightarrow{\mathrm{d}} \prod_{i}^{d} \exp\{-p_i \exp(-x_i)\}$$

*as* $n \to \infty$*, where* $\mathbf{x} = (x_1, x_2, \ldots, x_d)$.

**Theorem 2.2.** *Let the d-dimensional stationary Gaussian vector sequence* $\{\mathbf{X}_n\}$ *satisfy the conditions* (1.3) *and* (1.4)*. Suppose that the conditions* $\mathrm{C}_1$ *and* $\mathrm{C}_2$ *hold. Assume further that*

$$r_{ij}(n)\log n \to \rho_{ij} \in (0, \infty), \quad 1 \le i,\ j \le d \tag{2.3}$$

*as* $n \to \infty$ *and*

$$\sup_{\substack{1\le i\ne j\le d\\ n\ge 0}} \left|r_{ij}(n)\right| < 1. \tag{2.4}$$

*Then, for* $\mathbf{u}_n$ *and* $\mathbf{v}_n$ *defined by* (1.6)*, we have*

$$\lim_{n\to\infty} \mathrm{P}\left\{\widetilde{\mathbf{M}}_n \le \mathbf{u}_n, \mathbf{M}_n \le \mathbf{v}_n\right\} = \int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} \prod_{i}^{d} \exp\left\{-p_i \exp\left(-x_i - \rho_{ii} + \sqrt{2\rho_{ii}}z_i\right)\right\}$$
$$\times \exp\left\{-(1-p_i)\exp\left(-y_i - \rho_{ii} + \sqrt{2\rho_{ii}}z_i\right)\right\}\phi(z_1, z_2, \ldots, z_d)\,\mathrm{d}z_1\cdots\mathrm{d}z_d,$$

*where* $\phi(z_1, z_2, \ldots, z_d)$ *is the joint pdf of a d-variate Gaussian vector* $\mathbf{X}_0$ *with zero mean and variance–covariance matrix* $\left(\rho_{ij}/\sqrt{\rho_{ii}\rho_{jj}}\right)_{d\times d}$.

**Corollary 2.2.** *Under the conditions of* Theorem 2.2*, for* $\mathbf{a}_n$ *and* $\mathbf{b}_n$ *defined by* (1.5)*, we have*

$$\lim_{n\to\infty} \mathrm{P}\left\{\widetilde{\mathbf{M}}_n \le \mathbf{a}_n\mathbf{x} + \mathbf{b}_n\right\} = \int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} \prod_{i}^{d} \exp\left\{-p_i \exp\left(-x_i - \rho_{ii} + \sqrt{2\rho_{ii}}z_i\right)\right\}\phi(z_1, z_2, \ldots, z_d)\,\mathrm{d}z_1\cdots\mathrm{d}z_d$$

*as* $n \to \infty$*, where* $\mathbf{x} = (x_1, x_2, \ldots, x_d)$.

## 3. Proofs

In this section, we shall prove the main results. Firstly, let $\{\mathbf{X}_k^*, k \ge 1\}$ denote a sequence of independent $d$-dimensional Gaussian vectors with $\mathrm{E}X_{ki}^* = 0$, $\mathrm{Var}(X_{ki}^*) = 1$, $1 \le i \le d$, $\mathrm{Cov}(X_{ki}^*, X_{kj}^*) = 0$ for $i \ne j$, $0 \le k \le n$ and $\mathrm{Cov}(X_{ki}^*, X_{lj}^*) = 0$ for $0 \le k \ne l \le n$. Define $\mathbf{M}_n^*$ to be the partial maxima of the sequence $\{\mathbf{X}_k^*, 1 \le k \le n\}$, i.e. $M_{ni}^* = \max\{X_{1i}^*, \ldots, X_{ni}^*\}$ for $1 \le i \le d$. Also let $\widetilde{\mathbf{M}}_n^*$ denote the partial maxima of the observed variables, i.e.

$$\widetilde{M}_{ni}^* = \begin{cases} \max\left\{X_{ki}^*, 1 \le k \le n, \varepsilon_{ki} = 1\right\}, & \text{if } S_{ni} \ge 1,\\ -\infty, & \text{if } S_{ni} = 0.\end{cases}$$

For $\mathbf{v} = (v_1, \ldots, v_d)$, $v_i \in (0, \infty)$, define

$$\mathbf{A}(\mathbf{v}) = \begin{pmatrix} {v_1}^{1/2} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & {v_d}^{1/2} \end{pmatrix}, \qquad \mathbf{B}(\mathbf{v}) = \begin{pmatrix} (1-v_1)^{1/2} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & (1-v_d)^{1/2} \end{pmatrix}.$$

For the strongly dependent case, we need the following construction. Let $\mathbf{X}_0$ be a $d$-variate Gaussian vector with zero mean and variance–covariance matrix $\left(\rho_{ij}/\sqrt{\rho_{ii}\rho_{jj}}\right)_{d\times d}$. Suppose that $\mathbf{X}_0$ is independent of $\{\mathbf{X}_k^*, k \geq 1\}$. For $\boldsymbol{\rho}(n) = (\rho_{11}(n), \ldots, \rho_{dd}(n))$ and $\rho_{ii}(n) = \rho_{ii}/(\log n)$, define the $d$-variate Gaussian variables

$$\mathbf{Y}_k = \mathbf{X}_0^*\mathbf{A}(\boldsymbol{\rho}(n)) + \mathbf{X}_k^*\mathbf{B}(\boldsymbol{\rho}(n)) \tag{3.1}$$

for $1 \leq k \leq n$, i.e.

$$Y_{ki} = \rho_{ii}^{1/2}(n)X_{0i}^* + (1-\rho_{ii}(n))^{1/2} X_{ki}^* \tag{3.2}$$

for $1 \leq k \leq n$ and $1 \leq i \leq d$. It is easy to check that $\mathrm{E}Y_{ki} = 0$, $\mathrm{Var}(Y_{ki}) = 1$ and $\mathrm{Cov}(Y_{ki}, Y_{lj}) = \rho_{ij}(n)$ for $1 \leq k \neq l \leq n$, where $\rho_{ij}(n) = \rho_{ij}/(\log n)$. Let $\mathbf{M}_n^{**}$ and $\widetilde{\mathbf{M}}_n^{**}$ denote, respectively, the partial maxima of $\{\mathbf{Y}_k, 1 \leq k \leq n\}$ and that of its observed variables, i.e.

$$M_{ni}^{**} = \max\{Y_{1i}, \ldots, Y_{ni}\}, \qquad \widetilde{M}_{ni}^{**} = \begin{cases} \max\{Y_{ki}, 1 \leq k \leq n, \varepsilon_{ki} = 1\}, & \text{if } S_{ni} \geq 1, \\ -\infty, & \text{if } S_{ni} = 0 \end{cases}$$

for $1 \leq i \leq d$. Noting (3.1) and (3.2), we have

$$\mathbf{M}_n^{**} = \mathbf{X}_0^*\mathbf{A}(\boldsymbol{\rho}(n)) + \mathbf{M}_n^*\mathbf{B}(\boldsymbol{\rho}(n)), \qquad \widetilde{\mathbf{M}}_n^{**} = \mathbf{X}_0^*\mathbf{A}(\boldsymbol{\rho}(n)) + \widetilde{\mathbf{M}}_n^*\mathbf{B}(\boldsymbol{\rho}(n)), \tag{3.3}$$

i.e.

$$M_{ni} = \rho_{ii}^{1/2}(n)X_{0i}^* + (1-\rho_{ii}(n))^{1/2} M_{ni}^*, \qquad \widetilde{M}_{ni} = \rho_{ii}^{1/2}(n)X_{0i}^* + (1-\rho_{ii}(n))^{1/2} \widetilde{M}_{ni}^*$$

for $1 \leq i \leq d$. Here, we use the convention $a - \infty = -\infty$.

To prove our main results, we need some lemmas. The first one will be used to prove Theorem 2.1.

**Lemma 3.1.** *Let the $d$-dimensional stationary Gaussian vector sequence $\{\mathbf{X}_k\}$ satisfy the conditions (1.3), (1.4) and (2.1). Suppose that the conditions $C_1$ and $C_2$ hold. With $\{\mathbf{X}_k^*\}$ defined as above, we have*

$$\lim_{n\to\infty} \left| \mathrm{P}\left\{\widetilde{\mathbf{M}}_n \leq \mathbf{u}_n, \mathbf{M}_n \leq \mathbf{v}_n\right\} - \mathrm{P}\left\{\widetilde{\mathbf{M}}_n^* \leq \mathbf{u}_n, \mathbf{M}_n^* \leq \mathbf{v}_n\right\}\right| = 0.$$

**Proof.** For fixed $i \in (1, \ldots, d)$ and component sequence $(X_{ki})$, firstly suppose that just $\{X_{i_1,i}, \ldots, X_{i_{k_i},i}\}$ have been observed from the set $\{X_{1i}, \ldots, X_{ni}\}$, which is one case of $\{S_{ni} = k_i\}$. Let $\mathcal{N} = \{1, 2, \ldots, n\}$, $I_i = \{i_1, \ldots, i_{k_i}\}$, $M(I_i) = \max\{X_{li}, l \in I_i\}$ and $M^*(I_i) = \max\{X_{li}^*, l \in I_i\}$. Then, by the Normal Comparison lemma given in [4] and noting that $u_{ni} \leq v_{ni}$ for $1 \leq i \leq d$, we have

$$\begin{aligned} &\left|\mathrm{P}\{M(I_i) \leq u_{ni}, M(\mathcal{N}/I_i) \leq v_{ni}, 1 \leq i \leq d\} - \mathrm{P}\left\{M^*(I_i) \leq u_{ni}, M^*(\mathcal{N}/I_i) \leq v_{ni}, 1 \leq i \leq d\right\}\right| \\ &\quad \leq K_1 \sum_{i=1}^{d} n \sum_{k=1}^{n} |r_{ii}(k)| \exp\left\{-\frac{u_{ni}^2}{1+|r_{ii}(k)|}\right\} + K_2 \sum_{1\leq i\neq j\leq d} n \sum_{k=0}^{n} \left|r_{ij}(k)\right| \exp\left\{-\frac{u_{ni}^2 + u_{nj}^2}{2\left(1+\left|r_{ij}(k)\right|\right)}\right\} \end{aligned}$$

for some absolute constants $K_1 > 0$ and $K_2 > 0$. So, by (2.1) and by arguments similar to those of the proof of Lemma 4.3.2 in [4], we obtain

$$n \sum_{k=0}^{n} \left|r_{ij}(k)\right| \exp\left\{-\frac{u_{ni}^2 + u_{nj}^2}{2\left(1+\left|r_{ij}(k)\right|\right)}\right\} \to 0 \tag{3.4}$$

and

$$n \sum_{k=1}^{n} |r_{ii}(k)| \exp\left\{-\frac{u_{ni}^2}{1+|r_{ii}(k)|}\right\} \to 0 \tag{3.5}$$

uniformly for all $k_i$, $1 \leq i \leq d$ as $n \to \infty$. The result follows by the condition $C_2$, the total probability formula and the uniform convergence of (3.4) and (3.5). □

The following result is useful for proving Theorem 2.2.

**Lemma 3.2.** *Let the d-dimensional stationary Gaussian vector sequence $\{\mathbf{X}_k\}$ satisfy the conditions (1.3) and (1.4). Let $\{\mathbf{Y}_k\}$ be the Gaussian vector sequence defined above with equal correlations. Suppose that the conditions $C_1$ and $C_2$ hold. Assume further that both (2.3) and (2.4) hold. Then*

$$\lim_{n\to\infty}\left|\mathrm{P}\left\{\widetilde{\mathbf{M}}_n\le\mathbf{u}_n,\mathbf{M}_n\le\mathbf{v}_n\right\}-\mathrm{P}\left\{\widetilde{\mathbf{M}}_n^{**}\le\mathbf{u}_n,\mathbf{M}_n^{**}\le\mathbf{v}_n\right\}\right|=0.$$

**Proof.** For fixed $i\in(1,\dots,d)$ and component sequence $(X_{ki})$, firstly suppose that just $\{X_{i_1,i},\dots,X_{i_{k_i},i}\}$ have been observed from the set $\{X_{1i},\dots,X_{ni}\}$, which is one case of $\{S_{ni}=k_i\}$. Define $\mathcal{N}=\{1,2,\dots,n\}$, $I_i=\{i_1,\dots,i_{k_i}\}$, $M(I_i)=\max\{X_{li},l\in I_i\}$ and $M^{**}(I_i)=\max\{X_{li}^{**},l\in I_i\}$. Then, by the Normal Comparison lemma of [4] and noting that $u_{ni}\le v_{ni}$ for $1\le i\le d$, we have

$$\begin{aligned}&\left|\mathrm{P}\{M(I_i)\le u_{ni},M(\mathcal{N}/I_i)\le v_{ni},1\le i\le d\}-\mathrm{P}\left\{M^{**}(I_i)\le u_{ni},M^{**}(\mathcal{N}/I_i)\le v_{ni},1\le i\le d\right\}\right|\\&\le K_3\sum_{i=1}^{d}n\sum_{k=1}^{n}|r_{ii}(k)-\rho_{ii}(n)|\exp\left\{-\frac{u_{ni}^2}{1+w_{ii}(k)}\right\}+K_4\sum_{1\le i\ne j\le d}n\sum_{k=0}^{n}\left|r_{ij}(k)-\rho_{ij}(n)\right|\exp\left\{-\frac{u_{ni}^2+u_{nj}^2}{2\left(1+w_{ij}(k)\right)}\right\},\end{aligned}$$

where $w_{ij}(k)=\max\{|r_{ij}(k)|,\rho_{ij}(n)\}$, and $K_3$ and $K_4$ are absolute constants. So, by arguments similar to those of the proofs of Lemmas 4.3.2 and 6.4.1 in [4], we obtain

$$n\sum_{k=1}^{n}\left|r_{ij}(k)-\rho_{ij}(n)\right|\exp\left\{-\frac{u_{ni}^2+u_{nj}^2}{2\left(1+w_{ii}(k)\right)}\right\}\to 0$$

and

$$n\sum_{k=1}^{n}|r_{ii}(k)-\rho_{ii}(n)|\exp\left\{-\frac{u_{ni}^2}{1+w_{ii}(k)}\right\}\to 0$$

uniformly for all $k_i$, $1\le i\le d$, as $n\to\infty$. So, the result follows by the condition $C_2$ and the total probability formula. □

**Proof of Theorem 2.1.** By Lemma 3.1, we only need to prove

$$\lim_{n\to\infty}\mathrm{P}\left\{\widetilde{\mathbf{M}}_n^{*}\le\mathbf{u}_n,\mathbf{M}_n^{*}\le\mathbf{v}_n\right\}=\prod_{i}^{d}\exp\{-p_i\exp(-x_i)\}\exp\{-(1-p_i)\exp(-y_i)\}. \tag{3.6}$$

By using the total probability formula, we obtain

$$\mathrm{P}\left\{\widetilde{\mathbf{M}}_n^{*}\le\mathbf{u}_n,\mathbf{M}_n^{*}\le\mathbf{v}_n\right\}=\sum_{i=1}^{d}\sum_{k_i=1}^{n}P\{S_{n1}=k_1,\dots,S_{nd}=k_d\}\prod_{i=1}^{d}\{\Phi(u_{ni})\}^{k_i}\{\Phi(v_{ni})\}^{n-k_i}.$$

Define

$$\Sigma_1=\sum_{i=1}^{d}\sum_{\exists i:\left|\frac{k_i}{n}-p_i\right|>\varepsilon}\mathrm{P}\{S_{n1}=k_1,\dots,S_{nd}=k_d\}\prod_{i=1}^{d}\{\Phi(u_{ni})\}^{k_i}\{\Phi(v_{ni})\}^{n-k_i} \tag{3.7}$$

and

$$\Sigma_2=\sum_{i=1}^{d}\sum_{\forall i:\left|\frac{k_i}{n}-p_i\right|\le\varepsilon}\mathrm{P}\{S_{n1}=k_1,\dots,S_{nd}=k_d\}\prod_{i=1}^{d}\{\Phi(u_{ni})\}^{k_i}\{\Phi(v_{ni})\}^{n-k_i}. \tag{3.8}$$

By the condition $C_1$, we have

$$\Sigma_1\le\sum_{i=1}^{d}\mathrm{P}(|S_{ni}/n-p_i|\ge\varepsilon)\to 0 \tag{3.9}$$

as $n\to\infty$. By (3.8), the inequalities

$$\Sigma_2\le\prod_{i=1}^{d}\{\Phi(u_{ni})\}^{n(p_i-\varepsilon)}\{\Phi(v_{ni})\}^{n(1-p_i-\varepsilon)}\sum_{i=1}^{d}\sum_{\forall i:\left|\frac{k_i}{n}-p_i\right|\le\varepsilon}\mathrm{P}(S_{n1}=k_1,\dots,S_{nd}=k_d) \tag{3.10}$$

and

$$\Sigma_2\ge\prod_{i=1}^{d}\{\Phi(u_{ni})\}^{n(p_i+\varepsilon)}\{\Phi(v_{ni})\}^{n(1-p_i+\varepsilon)}\sum_{i=1}^{d}\sum_{\forall i:\left|\frac{k_i}{n}-p_i\right|\le\varepsilon}\mathrm{P}(S_{n1}=k_1,\dots,S_{nd}=k_d) \tag{3.11}$$

hold. By (3.7)–(3.11) and the condition $C_1$, we have

$$\limsup_{n\to\infty} \mathrm{P}\left(\widetilde{\mathbf{M}}_n^* \le \mathbf{u}_n, \mathbf{M}_n^* \le \mathbf{v}_n\right) \le \prod_{i=1}^{d} \exp\{-(p_i-\varepsilon)\exp(-x_i)\}\exp\{-(1-p_i-\varepsilon)\exp(-y_i)\} \quad (3.12)$$

and

$$\liminf_{n\to\infty} \mathrm{P}\left(\widetilde{\mathbf{M}}_n^* \le \mathbf{u}_n, \mathbf{M}_n^* \le \mathbf{v}_n\right) \ge \prod_{i=1}^{d} \exp\{-(p_i+\varepsilon)\exp(-x_i)\}\exp\{-(1-p_i+\varepsilon)\exp(-y_i)\} \quad (3.13)$$

for every $\varepsilon \in (0, \min\{p_i, 1 \le i \le d\})$. So, by letting $\varepsilon \downarrow 0$ in (3.12) and (3.13), we obtain (3.6). The proof is complete. □

**Proof of Theorem 2.2.** By Lemma 3.2, in order to prove (2.2), it sufficient to show that

$$\lim_{n\to\infty} \mathrm{P}\left(\widetilde{\mathbf{M}}_n^{**} \le \mathbf{u}_n, \mathbf{M}_n^{**} \le \mathbf{v}_n\right) = \int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} \prod_{i}^{d} \exp\left\{-p_i \exp\left(-x_i-\rho_{ii}+\sqrt{2\rho_{ii}}z_i\right)\right\}$$
$$\times \exp\left\{-(1-p_i)\exp\left(-y_i-\rho_{ii}+\sqrt{2\rho_{ii}}z_i\right)\right\}\phi(z_1,z_2,\ldots,z_d)\,\mathrm{d}z_1\cdots\mathrm{d}z_d. \quad (3.14)$$

By (3.3) and the total probability formula, we obtain

$$\begin{aligned}
\mathrm{P}\left(\widetilde{\mathbf{M}}_n^{**} \le \mathbf{u}_n, \mathbf{M}_n^{**} \le \mathbf{v}_n\right) &= \mathrm{P}\left(\mathbf{X}_0^*\mathbf{A}(\boldsymbol{\rho}(n)) + \widetilde{\mathbf{M}}_n^*\mathbf{B}(\boldsymbol{\rho}(n)) \le \mathbf{u}_n, \mathbf{X}_0^*\mathbf{A}(\boldsymbol{\rho}(n)) + \mathbf{M}_n^*\mathbf{B}(\boldsymbol{\rho}(n)) \le \mathbf{v}_n\right)\\
&= \int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\left[\mathrm{P}\left(\widetilde{\mathbf{M}}_n^* \le (\mathbf{u}_n - \mathbf{ZA}(\boldsymbol{\rho}(n)))\mathbf{B}^{-1}(\boldsymbol{\rho}(n)), \mathbf{M}_n^* \le (\mathbf{v}_n - \mathbf{ZA}(\boldsymbol{\rho}(n)))\mathbf{B}^{-1}(\boldsymbol{\rho}(n))\right)\right]\\
&\quad \times \varphi(z_1,z_2,\ldots,z_d)\,\mathrm{d}z_1\mathrm{d}z_2\cdots\mathrm{d}z_d\\
&= \int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\sum_{\substack{k_i=1\\1\le i\le d}}^{n}\mathrm{P}(S_{n1}=k_1,\ldots,S_{nd}=k_d)\prod_{i}^{d}\{\Phi(q_{ni})\}^{k_i}\{\Phi(t_{ni})\}^{n-k_i}\varphi(z_1,z_2,\ldots,z_d)\,\mathrm{d}z_1\mathrm{d}z_2\cdots\mathrm{d}z_d,
\end{aligned} \quad (3.15)$$

where $\mathbf{Z} = (z_1, z_2, \ldots, z_d)$ and

$$q_{ni} = \frac{u_{ni} - z_i\sqrt{\rho_{ii}(n)}}{\sqrt{1-\rho_{ii}(n)}}, \qquad t_{ni} = \frac{v_{ni} - z_i\sqrt{\rho_{ii}(n)}}{\sqrt{1-\rho_{ii}(n)}}, \quad 1 \le i \le d.$$

Note that $q_{ni} = a_n\left(x_i + \rho_{ii} - \sqrt{2\rho_{ii}}z_i\right) + b_n + o(a_n)$ and $t_{ni} = a_n\left(y_i + \rho_{ii} - \sqrt{2\rho_{ii}}z_i\right) + b_n + o(a_n)$ from the proof of Theorem 6.5.1 in [4]. So, one can check that

$$\lim_{n\to\infty}\{\Phi(q_{ni})\}^n = \exp\left\{-\exp\left(-x_i-\rho_{ii}+\sqrt{2\rho_{ii}}z_i\right)\right\}$$

and

$$\lim_{n\to\infty}\{\Phi(t_{ni})\}^n = \exp\left\{-\exp\left(-y_i-\rho_{ii}+\sqrt{2\rho_{ii}}z_i\right)\right\}.$$

By arguments similar to those of (3.6), we have

$$\sum_{\substack{k_i=1\\1\le i\le d}}^{n}\mathrm{P}(S_{n1}=k_1,\ldots,S_{nd}=k_d)\prod_{i}^{d}\{\Phi(q_{ni})\}^{k_i}\{\Phi(t_{ni})\}^{n-k_i}$$
$$\to \prod_{i}^{d}\exp\left\{-p_i\exp\left(-x_i-\rho_{ii}+\sqrt{2\rho_{ii}}z_i\right)\right\}\exp\left\{-(1-p_i)\exp\left(-y_i-\rho_{ii}+\sqrt{2\rho_{ii}}z_i\right)\right\}$$

as $n \to \infty$. Combining this with (3.15), we obtain (3.14) by the dominated convergence theorem. The proof is complete. □

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