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Decomposition into pairs-of-pants for complex algebraic hypersurfaces

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Abstract

It is well-known that a Riemann surface can be decomposed into the so-called *pairs-of-pants*. Each pair-of-pants is diffeomorphic to a Riemann sphere minus 3 points. We show that a smooth complex projective hypersurface of arbitrary dimension admits a similar decomposition. The n-dimensional pair-of-pants is diffeomorphic to $\mathbb{CP}^n$ minus $n+2$ hyperplanes.

Alternatively, these decompositions can be treated as certain fibrations on the hypersurfaces. We show that there exists a singular fibration on the hypersurface with an n-dimensional polyhedral complex as its base and a real n-torus as its fiber. The base accommodates the geometric genus of a hypersurface V. Its homotopy type is a wedge of $h^{n,0}(V)$ spheres S^n.

MSC: 14J70

Keywords: Topology of projective hypersurfaces; Pairs of pants decomposition; Singular torus fibrations

1. Introduction

1.1. Main question

In this paper, we study non-singular algebraic hypersurfaces in $\mathbb{CP}^{n+1}$ and other toric varieties. Let V be such a hypersurface. Naturally, V is a complex variety and thus has the underlying structure of a smooth manifold. Furthermore, V is a symplectic manifold. The symplectic structure is induced by the embedding to $\mathbb{CP}^{n+1}$.

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Since V is non-singular, its diffeomorphism and symplectomorphism types depend only on its degree, i.e. the degree of the defining polynomial f. All smooth hypersurfaces of the same degree are isotopic in the ambient $\mathbb{CP}^{n+1}$ even though the complex structure of V varies with the coefficients of f.

Thus, from the point of view of differential topology or symplectic topology a smooth projective hypersurface V is given by two numbers: its dimension n and its degree d.

Question. *Given n and d, describe a non-singular hypersurface $V \subset \mathbb{CP}^{n+1}$ of degree d as a smooth manifold and as a symplectic manifold.*

More generally, one can ask a similar question where $\mathbb{CP}^{n+1}$ is replaced by an arbitrary toric variety. The degree d would then be replaced with a convex lattice polygon $\Delta \subset \mathbb{R}^{n+1}$.

1.2. State of knowledge for small values of n and d

1.2.1. Case $n = 1$

The answer to this question is well-known if $n=1$. Then V is a Riemann surface. Topologically it is a sphere with g handles, where the genus g can be computed from the degree d by the adjunction formula $g = ((d-1)(d-2))/2$.

Recall that one way to understand Riemann surfaces is via their decomposition to primitive pieces each diffeomorphic to a sphere with 3 holes. These primitive pieces are called *pairs-of-pants* and such a decomposition can be thought of as some (singular) fibration of the Riemann surface over a 3-valent graph, see Fig. 5. Note that the first Betti number of the base graph coincides with the genus of the Riemann surface.

1.2.2. Case $n = 2$

In this case V is a smooth 4-manifold. If d is 1, 2 or 3 then V is diffeomorphic to $\mathbb{CP}^2$, $\mathbb{CP}^1 \times \mathbb{CP}^1$ or $\mathbb{CP}^2 \# 6\overline{\mathbb{CP}}^2$, the connected sum of $\mathbb{CP}^2$ and 6 copies of $\mathbb{CP}^2$ with the inverse orientation. In these cases the geometric genus $p_g = h^{2,0}(V)$ vanishes.

If $d=4$ then $p_g=1$ and V is the celebrated K3 surface (named so, according to A. Weil, in honor of Kähler, Kodaira, Kummer and the K2-mountain in Pakistan). This manifold is primitive, it does not decompose as a connected sum. One way to understand its topology is via a singular fibration $\lambda : V \to S^2$. A generic fiber of λ is a torus while 24 fibers are special and are homeomorphic to a torus with its meridian collapsed to a point (so-called fishtail fibers). The fibration λ can be chosen so that all generic fibers are Lagrangian submanifolds, i.e. so that the symplectic 2-form restricted to these fibers vanishes.

For any value of d V is simply connected and if $d \geqslant 4$ it does not decompose into a connected sum. We have $p_g(V) = (d-1)(d-2)(d-3)/6$ (cf., e.g. a more general Khovanskii's formula [8]). A simply connected smooth 4-manifolds is determined up to a homeomorphism once we know its Euler characteristic χ, its signature σ and whether it is spin or not. Our manifold V is spin iff d is even, $\chi = d^3 - 4d^2 + 6d$ and $\sigma = 2(2p_g + 1) - (\chi - 2) = 4(p_g + 1) - \chi = (4d - d^3)/3$.

The diffeomorphism (and symplectomorphism) type of V is, however, more mysterious as it is not determined by purely homological data. E.g. the surface of degree 5 is a non-spin manifold with $\chi = 55$ and $\sigma = -35$, but there might be many non-diffeomorphic manifolds with these data.

1.2.3. Case $d = n + 2$

In this case the canonical class of V is trivial and there exists a nowhere-degenerate holomorphic n-form Ω on V. Such V is called a *Calabi–Yau* manifold. Here we have $p_g = 1$. According to the Strominger–Yau–Zaslow conjecture [15] there is supposed to exist a singular special Lagrangian fibration of V over the sphere S^n. This means that a generic fiber should be Lagrangian and such that the imaginary part of Ω restricted to the fiber is zero as a real 3-form at every point.

It was verified in [19,14] that such fibrations exist in this case at least if we relax a special Lagrangian condition to simply Lagrangian. Note that special Lagrangian condition makes use of the non-degenerate holomorphic n-form from a Calabi–Yau manifold. Thus, at least literally, the Strominger–Yau–Zaslow conjecture only makes sense if $d = n + 2$ in our setup. However, a relaxed version of this conjecture makes sense for all values of d and n.

1.3. Results of the paper

Here we state the main results of the paper informally. See Section 3 for precise statements.

1.3.1. Torus fibration and pairs-of-pants decomposition

Theorem 1 asserts that for any value of n and d the hypersurface V admits a singular fibration λ over an n-dimensional polyhedral complex $\bar{\Pi}$. A generic fiber of λ is diffeomorphic to a smooth torus T^n. The base $\bar{\Pi}$ here is homotopy equivalent to the bouquet of p_g copies of S^n, thus this theorem can be interpreted as a geometric interpretation of the geometric genus p_g.

Furthermore, the local topological structure of the polyhedral complex $\Pi \subset \mathbb{R}^{n+1}$ is known in differential topology as the local structure of so-called *special spines*. In particular, there is a natural stratification of Π and regular neighborhoods of the vertices essentially exhaust the complex Π.

It turns out that the stratification of the base Π determines a decomposition of the hypersurface V into d^{n+1} copies of $\mathscr{P}_n$, where $\mathscr{P}_n$ is diffeomorphic to $\mathbb{CP}^n$ minus $(n+2)$ hyperplanes in general position. This decomposition can be considered as a higher-dimensional analogue of the pair-of-pants decomposition of Riemann surfaces. In particular $\mathscr{P}_1$ is the classical pair-of-pants $\hat{\mathbb{C}} \setminus \{0, 1, \infty\}$.

1.3.2. A projective hypersurface as a piecewise-linear object

The base Π of the fibration λ is a piecewise-linear n-dimensional complex in $\mathbb{R}^{n+1}$. The dimension over $\mathbb{R}$ of the hypersurface V is $2n$. Yet the hypersurface V can be reconstructed (as a smooth manifold) from $\Pi \subset \mathbb{R}^{n+1}$. It turns out that Π (together with its PL-embedding to $\mathbb{R}^{n+1}$) encodes the combinatorics of gluing of d^{n+1} copies of $\mathscr{P}_n$ needed to obtain V. Theorem 4 is the corresponding reconstruction theorem.

1.3.3. Lagrangian submanifolds in projective hypersurfaces

It turns out that the fibration λ produces a number of Lagrangian submanifolds in V. Different fibers of λ are not necessarily homologous and $p_g = h^{n,0}$ disjoint embedded Lagrangian tori come as fibers of λ. This tori are linearly independent in $H_n(V)$. In addition we have $h^{n,0}$ linearly independent embedded Lagrangian spheres coming as partial sections of λ. In particular, we have Corollary 3.1.

2. Preliminaries

2.1. Balanced polyhedra

Definition 1. A subset $\Pi \subset \mathbb{R}^{n+1}$ is called a *proper rational polyhedral complex* (or just a *polyhedral complex* in this paper) if it can be presented as a finite union of closed sets in $\mathbb{R}^{n+1}$ called *cells* with the following properties:

- Each cell is a closed convex (possibly semi-infinite) polyhedron. The dimension of the cell is, by definition, the dimension of its affine span, the smallest affine subspace of $\mathbb{R}^{n+1}$ which contains it. We call a cell of dimension k a k-cell.
- The slope of the affine span of each cell is rational, i.e. the linear subspace of $\mathbb{R}^{n+1}$ parallel to the affine span is defined over $\mathbb{Q}$.
- The boundary (i.e. the boundary in the corresponding affine span) of a k-cell is a union of $(k-1)$-cells.
- Different open cells (i.e. the interiors of the cells in the corresponding affine spans) do not intersect.

Informally speaking, a proper polyhedral complex in $\mathbb{R}^{n+1}$ is a cellular space where each cell is a convex polyhedron with a rational slope and where some cells are allowed to go to infinity.

As usual, the dimension of Π is the maximal dimension of its cells.

Definition 2. A polyhedral n-complex is called *weighted* if there is a natural number $w(F)$, called *weight*, prescribed to each of its n-cell F. (Of course, any polyhedral complex can be considered as a weighted polyhedral complex by prescribing 1 to each n-cell.)

Let $\Pi \subset \mathbb{R}^{n+1}$ be a weighted polyhedral n-complex. Note that its complement $\mathbb{R}^{n+1} \setminus \Pi$ consists of a finite union of connected components. Let $F \subset \Pi$ be an n-cell.

Recall that by Definition 1 the n-cell F has a rational slope in $\mathbb{R}^{n+1}$. Therefore, it defines an integer covector

$$\pm c_F : \mathbb{Z}^{n+1} \to \mathbb{Z}$$

up to its sign. Here are the characteristic properties of c_F:

- the kernel of c_F is parallel to F and
- $(1/w(F))c_F$ is a primitive (i.e. non-divisible) integer covector $\mathbb{Z}^{n+1} \to \mathbb{Z}$.

Furthermore, even the sign of c_F becomes well-defined once we co-orient $F \subset \mathbb{R}^{n+1}$.

Polyhedral complexes that appear in this paper have the following additional property.

Definition 3. A weighted polyhedral n-complex $\Pi \subset \mathbb{R}^{n+1}$ is called *balanced* if for every $(n-1)$-cell $G \subset \Pi$ the following condition holds (Fig. 1). Let $F_1, \ldots, F_k$ be the n-cells adjacent to G. A choice of a rotational direction about G defines a coherent co-orientation on these n-cells. The balancing

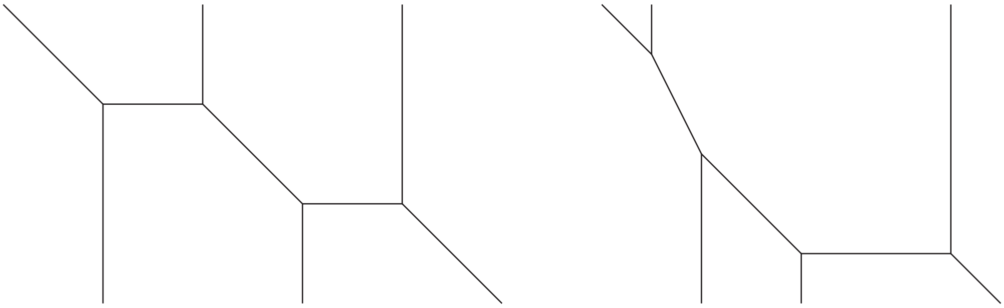

Fig. 1. Balanced graphs in $\mathbb{R}^2$.

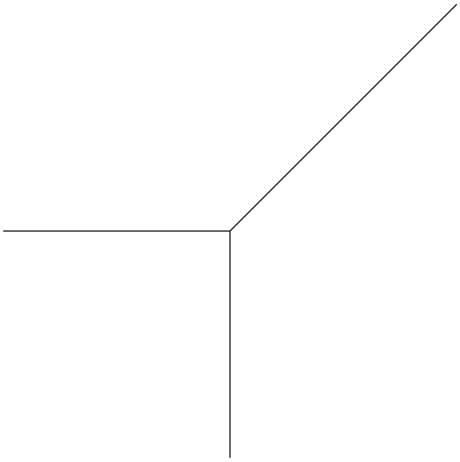

Fig. 2. Primitive complex Σ_n.

condition is

$$\sum_{j=1}^{k} c_{F_j} = 0.$$

Example 1. Consider the function

$$H(x_1,\dots,x_{n+1}) = \max\{0, x_1,\dots,x_{n+1}\}.$$

This is a convex piecewise-linear function $\mathbb{R}^{n+1} \to \mathbb{R}$. We define the *primitive complex* $\Sigma_n \subset \mathbb{R}^{n+1}$ as the *corner locus* of H, i.e. the set of points where H is not smooth.

Note that Σ_n is a balanced proper polyhedral complex in $\mathbb{R}^{n+1}$ (Fig. 2). Its k-cells are formed by the points where at least $n+2-k$ of the functions $0, x_1,\dots,x_{n+1}$ achieve the value of H. In fact, it is easy to see that topologically Σ_n is the cone over the $(n-1)$-skeleton of the $(n+1)$-simplex. The fact that Σ is balanced follows from Proposition 2.2.

The following example is a generalization of the previous one. As the following propositions show, it is the fundamental example of balanced polyhedra.

Example 2. Let $A \subset \mathbb{Z}^{n+1}$ be a finite set and let $v : A \to \mathbb{R}$ be any function. Let $\Delta \subset \mathbb{R}^{n+1}$ be the convex hull of A. We associate the following polyhedral complex Π_v to v.

Take the Legendre transform of v, $L_v : \mathbb{R}^{n+1} \to \mathbb{R}$,

$$L_v(y) = \max_{x \in A} (xy - v(x)).$$

Here $x, y \in \mathbb{R}^{n+1}$ and xy is their scalar product. Since the maximum is taken over a finite set, the result L_v is a convex piecewise-linear function. We define Π_v as the corner locus of L_v (recall that this is the set of points where L_v is not smooth).

To present Example 1 as a special case of Example 2 we take the vertices of the standard simplex

$$\Delta_1 \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \,|\, x_j \geqslant 0, x_1 + \cdots + x_{n+1} \leqslant 1\} \tag{1}$$

for A and set $v \equiv 0$.

Recall that a polyhedron in $\mathbb{R}^{n+1}$ is called *a lattice polyhedron* if all its vertices belong to $\mathbb{Z}^{n+1}$. A subdivision of a polyhedron into smaller polyhedra is called *a lattice subdivision* if all its subpolyhedra are lattice.

Proposition 2.1. *The set Π_v from Example* 2 *is a proper rational polyhedral complex dual to a certain lattice subdivision of Δ.*

Proof. We start by associating to v a certain lattice subdivision $\mathscr{D}_v$ of Δ. Let $O\Gamma(v)$ be the overgraph of v, i.e. the set of vertical rays upwards in $\mathbb{R}^{n+1} \times \mathbb{R}$ starting at the points of the graph of v. The convex hull of $O\Gamma(v)$ is a semi-infinite closed polyhedral domain. The projections of its finite faces to $\mathbb{R}^{n+1}$ form the subdivision $\mathscr{D}_v$.

We claim that Π_v is a polyhedral complex dual to $\mathscr{D}_v$. Namely, a k-dimensional polyhedron Δ' in $\mathscr{D}_v$, $k > 0$, gives a $(n+1-k)$-cell of Π_v. This cell is compact iff $\Delta' \subset \Delta$.

This claim follows from the duality property of the Legendre transform. Consider the function $\underline{v}$ whose graph is given by the lower boundary of the convex hull of $O\Gamma(v)$. If v is convex then the function $\underline{v}$ extends v and is defined on the whole polyhedron Δ, not just on its lattice points. It is a convex piecewise-linear function. The Legendre transform of v coincides with the Legendre transform of $\underline{v}$. (In fact the function $\underline{v}$ can be defined by applying the Legendre transform to v twice.) By duality, the graph of $L_{\underline{v}}$ has the facets en lieu of the vertices of the graph of $\underline{v}$ and so on. $\square$

Note that Π_v is naturally weighted. Indeed, an n-cell $F \subset \Pi_v$ comes as a corner between the graphs of two integer linear functions. The difference between these functions is an integer covector c_F. We define $w(F) \in \mathbb{N}$ as the maximum integer divisor of c_F.

Proposition 2.2. *The weighted polyhedral complex Π_v is balanced.*

Proof. The proposition easily follows from the definition of the covectors c_{F_j} for the n-cells F_j adjacent to an $(n-1)$-cell $G \subset \Pi$. $\square$

Remark 2.3. Note that several different functions v define the same complex Π_v by the construction of Example 2. Here is the list of ambiguities.

(1) Let $v' = v + \text{const} : A \to \mathbb{R}$ be a function different with v by a constant. Then $\Pi_v = \Pi_{v'}$.
(2) Let $A' = A + c$, where $c \in \mathbb{Z}^{n+1}$ and $v' : A' \to \mathbb{R}$ is defined by $v'(z + c) = v(z)$. Then $\Pi_v = \Pi_{v'}$.
(3) Let A' be such that its convex hull Δ' coincides with Δ, the convex hull of A. Let v (resp. v') be the maximal convex function such that $v \leqslant v$ (resp. $v' \leqslant v'$). Suppose that $v = v'$. Then $\Pi_v = \Pi_{v'}$.

The following proposition shows that Example 2 is fundamental.

Proposition 2.4. *Suppose that $\Pi \subset \mathbb{R}^{n+1}$ is a weighted balanced proper rational polyhedral complex. Then there exists a finite set $A \subset \mathbb{Z}^{n+1}$ and a function $v : A \to \mathbb{Z}$ such that $\Pi = \Pi_v$ (see Example 2). The convex hull $\Delta \subset \mathbb{R}^{n+1}$ of A is unique up to a translation in $\mathbb{Z}^{n+1}$. The choice of the function v is unique up to the ambiguity of Remark 2.3.*

Proof. First we define a convex piecewise-linear function H whose corner locus is Π and then choose a function v such that H is the Legendre transform L_v of v. Note that the finiteness condition in Definition 1 implies that there are finitely many connected components in $\mathbb{R}^{n+1} \setminus \Pi$.

We define the function H inductively. Choose any connected component D_0 of $\mathbb{R}^{n+1} \setminus \Pi$ as a "reference component". Define $H|_{D_0} \equiv 0$. Suppose that D' is a component of $\mathbb{R}^{n+1} \setminus \Pi$ such that there exists an adjacent component D where H is already defined.

Let F be the n-cell of Π separating D from D'. Let c_F be the covector associated to F (recall that the weight of F is incorporated into c_F) with the co-orientation directed from D to D'. Let $l_D : \mathbb{R}^{n+1} \to \mathbb{R}$ be the affine-linear function extending $H|_D$. We define $H|_{D'} = l_D + c_F + c$, where the constant c is chosen so that $H|_D$ and $H|_{D'}$ agree on F. By the balancing condition the result does not depend on the choice of the adjacent component D where H is already defined.

To define v we take the Legendre transform of H. This amounts to associating to each component D a point $z \in \mathbb{Z}^{n+1}$ equal to the gradient of $H|_D$ and setting $v(z) = -l_D(0)$. Thus, the number of elements of the set A is equal to the number of components of $\mathbb{R}^{n+1} \setminus \Pi$.

The ambiguity Remark 2.3.3 comes from taking the Legendre transform of non-convex functions v. It coincides with the Legendre transform of the underlying convex function v. (In fact, nothing changes if we assume that v is defined on the whole $\mathbb{Z}^{n+1}$ by letting $v(z) = +\infty$ for $z \notin A$.) The ambiguities Remark 2.3.1 and 2.3.2 come from the ambiguity in assigning a linear function for $H|_{D_0}$. □

Corollary 2.5. *Any n-dimensional balanced polyhedral complex $\Pi \subset \mathbb{R}^{n+1}$ determines a convex lattice polyhedron $\Delta \subset \mathbb{R}^{n+1}$ (defined up to translation) and a lattice subdivision of Δ* (Fig. 3).

This corollary follows from Propositions 2.4 and 2.1.

The next corollary illustrates the strength of the balancing condition that we require just at the n-cells. We do not use this corollary elsewhere in the paper.

Let B be a vertex of Π and let $E_1, \ldots, E_k$ be the edges adjacent to B. Let $v_j \in \mathbb{Z}^{n+1}$, $j = 1, \ldots, k$, be the primitive integer vectors parallel to E_j and directed outwards from B. Suppose that each E_j

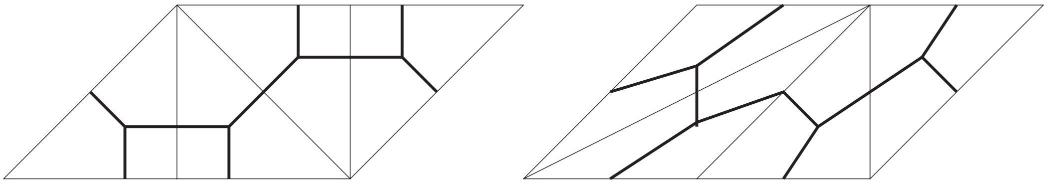

Fig. 3. The lattice polyhedron subdivisions dual to the balanced graphs from Fig. 1.

is adjacent to exactly $n+2$ connected components of $\mathbb{R}^{n+1} \setminus \Pi$ (note that this is a general position situation).

Corollary 2.6. *If $\Pi \subset \mathbb{R}^{n+1}$ is a balanced n-complex then there exists a weight $w_j \subset \mathbb{N}$ for E_j, $j = 1, \dots k$, such that*

$$\sum_{j=1}^{k} w_j v_j = 0.$$

Proof. By Proposition 2.4 Π comes as a corner locus of a convex piecewise-linear function F on $\mathbb{R}^{n+1}$. Let $y = a_{j,1}x_1 + \cdots + a_{j,n+1}x_{n+1}$, $j = 1, \dots, n+2$ be the equations of the linear functions on the adjacent components of $\mathbb{R}^{n+1} \setminus \Pi$. Then $u_j = (a_{j,1}, \dots, a_{j,n+1}, -1)$ are the vectors in $\mathbb{R}^{n+2} = \mathbb{R}^{n+1} \times \mathbb{R}$ normal to the linear portions of the graph of F adjacent to B.

The $\mathbb{R}^{n+2}$-version of the vector product associates a normal vector to $(n+1)$ other vectors in $\mathbb{R}^{n+2} = \mathbb{R}^{n+1} \times \mathbb{R}$. We take all possible such products among u_j and project them to $\mathbb{R}^{n+1}$. The result is the vectors which are multiples of v_j. By linear algebra the sum of these vectors is zero. □

2.2. *Maximal polyhedral complexes and their decomposition into primitive pieces*

Definition 4. We call Π a *dual Δ-complex* if it corresponds to the convex polyhedron $\Delta \subset \mathbb{R}^{n+1}$ by Proposition 2.4. We call Π a *maximal* polyhedral complex if the elements of the corresponding subdivision from Corollary 2.5 are simplices of volume $1/(n+1)!$ (a so-called *unimodular* lattice triangulation).

Proposition 2.7. *The minimal positive volume of a lattice polyhedron in $\mathbb{R}^{n+1}$ is $1/(n+1)!$. Any lattice polyhedron of volume $1/(n+1)!$ can be identified with the standard simplex Δ_1 (see (1)) by an element of $ASL_{n+1}(\mathbb{Z})$.*

Here $ASL_{n+1}(\mathbb{Z})$ stands for the group of affine-linear transformations of $\mathbb{R}^{n+1}$ whose rotation part belongs to $SL_{n+1}(\mathbb{Z})$.

Proof. We may assume that our lattice polyhedron is a simplex, since otherwise we can triangulate it to smaller polyhedra. Fix one of its vertice and consider the $(n+1)$ integer vectors connecting it to other vertices. The volume of the simplex is equal to the determinant of the sublattice generated by these vectors divided by $(n+1)!$ □

Example 3. Clearly, a dual Δ_1-complex (see (1)) is necessarily maximal. The complexes from Fig. 1 are maximal dual Δ-complexes for the polyhedra Δ pictured in Fig. 3.

Proposition 2.8. *Any dual Δ_1-complex is the result of a translation of Σ_n in $\mathbb{R}^{n+1}$.*

Proof. Such a complex Π is determined by a function $v : \Delta_1 \cap \mathbb{Z}^{n+1} \to \mathbb{R}$, i.e. by $n+2$ numbers $a_1, \dots, a_{n+1}, b \in \mathbb{R}$. Recall (see Example 2) that Π is the corner locus $L_v(x_1, \dots, x_{n+1}) = \max\{x_j - a_j, -b\}$. If $a_j = b = 0$ for all j than $\Pi = \Sigma_1$. Adding the same real number to all numbers does not change Π. Changing a_j by t results in a translation by t in the direction of x_j. □

Remark 2.9. Not for every Δ there exists maximal dual Δ-complex. E.g. a lattice simplex in $\mathbb{R}^3$, whose vertices are $(1,0,0)$, $(0,1,0)$, $(1,1,0)$ and $(0,0,n)$, cannot be further subdivided. On the other hand, a maximal dual Δ-complex is, of course, not unique.

Proposition 2.10. *If Π is a maximal dual Δ-complex then Π is homotopy equivalent to the bouquet of* $\#(\operatorname{Int} \Delta \cap \mathbb{Z}^{n+1})$ *copies of S^n.*

Proof. Because of its maximality, the polyhedron Π is dual to a unimodular triangulation of Π. Such a triangulation cannot be further subdivided and therefore its vertices are all the lattice points of Δ. Therefore, Π is homotopy equivalent to $\operatorname{Int} \Delta \setminus \mathbb{Z}^{n+1}$. □

Here is a way to canonically cut a maximal complex $\Pi \subset \mathbb{R}^{n+1}$ into standard-looking subsets U_j. We define the cutting locus Ξ as the following simplicial complex that is partially dual to Π. The vertices of Ξ are the baricenters of all bounded k-cells, $k > 0$ from Π. The simplices of Ξ have the baricenters of positive-dimensional cells $F_k \subset \Pi$ in the embedded towers $F_1 \subset \cdots \subset F_l$ as its vertices. Note that $\Xi \subset \Pi$ is a finite simplicial $(n-1)$-complex.

Definition 5. The connected components of $\Pi \setminus \Xi$ are called *the primitive pieces* of Π. We denote them with U_j. These open sets are parametrized by the vertices of Π or, equivalently, by the $(n+1)$-simplices of the triangulation of Δ.

Proposition 2.11. *For each U_j there exists $M_j \in ASL_{n+1}(\mathbb{Z})$ such that $M_j(U_j) \subset \Sigma_n$ is an open set in the primitive complex Σ_n from Example* 1.

Proof. This proposition also follows from the duality with a unimodular triangulation $\mathscr{D}$ of Δ. Let U_j be a primitive piece. It corresponds to a simplex of volume $1/(n+1)!$ in $\mathscr{D}$. There is an element of $SL_{n+1}(\mathbb{Z})$ which takes this simplex to the standard simplex Δ_1^{n+1} (see (1)). Then the image of U_j by the adjoint to the inverse of this element is contained in a dual Δ_1-complex. Such a complex is the result of a translation of Σ_n by Proposition 2.8. □

Recall that a polyhedral complex Π is called *generic* at a point $x \in \Pi$ of an open k-cell if x has a neighborhood homeomorphic to $\mathbb{R}^k \times \Sigma_{n-k}$.

Thus, Proposition 2.10 implies that a maximal dual Δ-complex is a generic polyhedron. In topology such polyhedra often appear as the so-called *special spines* of smooth manifolds. In the next section, we see that Π can be compactified so as to become a spine of the polyhedron Δ after puncturing it in the interior lattice points.

2.3. Toric varieties and compactification of balanced polyhedra

Consider the complex algebraic torus $(\mathbb{C}^*)^{n+1}$, where $\mathbb{C}^* = \mathbb{C} \setminus 0$. It is a commutative Lie group under multiplication. The 2-form

$$\frac{1}{2i}\sum_{j=1}^{n+1} \frac{\mathrm{d}z}{z} \wedge \frac{\mathrm{d}\bar{z}}{\bar{z}} \tag{2}$$

is an invariant symplectic form on $(\mathbb{C}^*)^{n+1}$. There is an action of the real torus $T^{n+1} = S^1 \times \cdots \times S^1$ on $(\mathbb{C}^*)^{n+1}$ by coordinatewise multiplication (we treat $S^1 \subset \mathbb{C}^*$ as the unit circle). The action of T^{n+1} is Hamiltonian and thus we have a well-defined *moment map* (we refer to [1] for the general definition or to a textbook, e.g. [2]) $\mathrm{Log} : (\mathbb{C}^*)^{n+1} \to \mathbb{R}^{n+1}$

$$\mathrm{Log}\,(z_1, \ldots, z_{n+1}) = (\log|z_1|, \ldots, \log|z_{n+1}|). \tag{3}$$

Let $\Delta \subset \mathbb{R}^{n+1}$ be a convex polyhedron with integer (from $\mathbb{Z}^{n+1}$) vertices. Recall (see e.g. [4]) that there is a complex toric variety $\mathbb{C}T_\Delta \supset (\mathbb{C}^*)^{n+1}$. One way to construct it is to consider *the Veronese embedding* $(\mathbb{C}^*)^{n+1} \to \mathbb{CP}^{\#(\Delta\cap\mathbb{Z}^{n+1})-1}$ defined by the linear system of monomials associated to $\Delta \cap \mathbb{Z}^{n+1}$. Here we associate to a point $(p_1, \ldots, p_{n+1})$ a monomial $z^{p_1} \ldots z_{n+1}^{p_{n+1}}$. We define $\mathbb{C}T_\Delta$ as the closure of the image of the Veronese embedding. Note that the standard, *Fubini-Study*, symplectic form on the ambient space $\mathbb{CP}^{\#(\Delta\cap\mathbb{Z}^{n+1})-1}$ defines a symplectic form on $\mathbb{C}T_\Delta$ (as long as the variety $\mathbb{C}T_\Delta$ is non-singular). In particular, it gives a symplectic form ω_Δ on $(\mathbb{C}^*)^{n+1}$ that is invariant with respect to the action of T_Δ. This gives us a moment map with respect to ω_Δ

$$\mu_\Delta : (\mathbb{C}^*)^{n+1} \to \Delta, \quad \mu_\Delta(z) = \frac{1}{\sum_{j\in\Delta\cap\mathbb{Z}^{n+1}} |z^{2j}|} \sum_{j\in\Delta\cap\mathbb{Z}^{n+1}} j|z^{2j}|.$$

The image of this embedding is the interior $\mathrm{Int}\,\Delta$. The map $\mu\Delta$ can be compactified to the moment map $\bar{\mu}_\Delta : \mathbb{C}T_\Delta \to \Delta$.

The maps $\mathrm{Log} : (\mathbb{C}^*)^{n+1} \to \mathbb{R}^{n+1}$ and $\mu_\Delta : (\mathbb{C}^*)^{n+1} \to \mathrm{Int}\,\Delta$ both have the orbits of T^{n+1} as their fibers. Thus, they define a natural reparametrization

$$\Phi_\Delta : \mathbb{R}^{n+1} \to \mathrm{Int}\,\Delta.$$

Definition 6. Let $\Pi \subset \mathbb{R}^{n+1}$ be an n-dimensional balanced polyhedral complex. By Proposition 2.4 there is a convex lattice polyhedron Δ dual to Π. We define $\bar{\Pi} \subset \Delta$, *the compactification of* Π, by taking the closure of $\Phi_\Delta(\Pi)$ in Δ. We call $\bar{\Pi} \setminus \Phi_\Delta(\Pi)$ *the boundary* of $\bar{\Pi}$. For convenience from now on we identify Π and $\Phi_\Delta(\Pi)$.

Proposition 2.12. *Let Π be a dual Δ-complex and let $\Delta' \subset \Delta$ be a $(k+1)$-dimensional face. Then the intersection $\bar{\Pi} \cap \Delta'$ is a compactification of a dual Δ'-complex Π'. If Π is maximal then Π' is also maximal.*

We prove this proposition simultaneously with the following proposition describing the behavior of Π near infinity. Recall that a *supporting vector* $\vec{v}$ at a face $\Delta' \subset \Delta$ is a vector such that $p_{\vec{v}}|_{\Delta}$ reaches its maximum precisely over Δ', where $p_{\vec{v}}$ is the orthogonal projection in the direction of $\vec{v}$.

Proposition 2.13. *The complex Π' from Proposition* 2.12 *can be obtained in the following way. Let $L \subset \mathbb{R}^{n+1}$ be the linear $(k+1)$-subspace parallel to the face Δ'. Let $\vec{v}$ be a supporting vector at Δ'. For a sufficiently large $R > 0$ we have $\Pi' = (\Pi - R\vec{v}) \cap L$.*

Proof. From the finiteness condition in Definition 1 we have that the complex $\Pi' = (\Pi - R\vec{v}) \cap L \subset L$ does not depend on the choice of $R > 0$ and $\vec{v}$ as long as $\vec{v}$ is supporting and R is sufficiently large. The proof of Proposition 2.4 ensures that Π' is a dual Δ'-complex. If Π is maximal then it is dual to a triangulation of Δ into simplices of minimal volume. Such a triangulation induces a triangulation into simplices of minimal volume on the faces Δ' and thus Π' is also maximal. □

If Π is a maximal dual Δ-complex then it is generic everywhere except at the points of its boundary $\partial\Pi$. The following proposition describes the local topology of $\bar{\Pi}$ near the boundary. It is a corollary of Proposition 2.12.

Proposition 2.14. *Suppose that Π is a maximal dual Δ-complex. A point x in $\bar{\Pi}$ has a neighborhood of one of the following $((n+1)(n+2))/2$ types*: $\mathbb{R}^k \times \Sigma_{l-k} \times [0, +\infty)^{n-l}$, *where $k \leqslant l \leqslant n$. Here k is the dimension of the open cell of $\bar{\Pi}$ which contains x while $l+1$ is the dimension of the open face of Δ which contains x.*

We call a point with such a neighborhood *a (k,l)-point* of $\bar{\Pi}$.

Remark 2.15. The concept of generic polyhedron is closely related to that of special spine in topology. We remind its definition. Let M be a compact $(n+1)$-manifold with boundary and $\bar{\Pi} \subset M$ be an n-dimensional CW-complex such that every open cell is smoothly embedded to M. The complex $\bar{\Pi}$ is called *a spine* of M if $\bar{\Pi}$ is a deformational retract of M. The spine $\bar{\Pi}$ is called *special* if for any point $x \in \bar{\Pi} \setminus \partial M$ from an open k-cell there exists a neighborhood isomorphic to $\mathbb{R}^k \times \Sigma^{n-k}$.

Note that if $\operatorname{Int} \Delta \cap \mathbb{Z}^{n+1} = \emptyset$ then all the triangulation vertices of a dual Δ-polyhedron Π are from $\partial\Delta$ then $\bar{\Pi}$ is a spine of Δ. In general, $\bar{\Pi}$ is a spine of the polyhedron Δ minus a small neighborhood of the interior lattice points. Note that $\bar{\Pi}$ can be treated as a special spine of Δ if we treat Δ as a manifold with corners.

2.4. Stratified fibrations

Let V and F be smooth manifolds, $\Delta \subset \mathbb{R}^{n+1}$ be a lattice polyhedron of full dimension and Π be a maximal dual Δ-complex.

Definition 7. A smooth map $\lambda : V \to \bar{\Pi}$ is called *a stratified F-fibration* if

- the restriction of λ to any open n-cell $e \subset \bar{\Pi}$ is a trivial fibration with the fiber F;
- for each integer pair (l,k), $0 \leqslant k \leqslant l \leqslant n$ there exists a smooth "model" map $\lambda_{l,k} : V_{l,k} \to \Pi_{l,k}$, where $\Pi_{l,k} \approx \mathbb{R}^k \times \Sigma_{l-k} \times [0, +\infty)^{n-l}$, such that any (l,k)-point of $\bar{\Pi}$ has a neighborhood $U \supset x$

such that

$$\lambda|_U : \lambda^{-1}(U) \to U$$

is diffeomorphic to the model map. The model map depends only on l and k.

The map $\lambda_{l,k}$ is called *the* (l,k)*-fiber degeneration*; the fiber $F_{l,k} = \lambda_{l,k}^{-1}(x)$ is called *the* (l,k)*-fiber* of λ.

The following proposition is a direct corollary of Definition 7.

Proposition 2.16. *Let $\lambda : V \to \bar{\Pi}$ be a stratified fibration over the compactification of a maximal dual Δ-complex Π. For any open (l,k)-cell e of $\bar{\Pi}$ the restriction of λ to e is a trivial fibration over e with the fiber $F_{l,k}$.*

Remark 2.17. Definition 7 can be generalized in a straightforward way to the case when the base is any space with a prescribed stratification. Here we used the stratification given by Proposition 2.14.

2.5. Hypersurfaces in toric varieties

Let $f : (\mathbb{C}^*)^{n+1} \to \mathbb{C}$ be a Laurent polynomial

$$f(z) = \sum_j a_j z^j,$$

where $z \in (\mathbb{C}^*)^{n+1}$ and $j \in \mathbb{Z}^{n+1}$ is a multi-index.

We recall that *the Newton polyhedron Δ of f* is the convex hull in $\mathbb{R}^{n+1}$ of the set of all indices $j \in \mathbb{Z}^{n+1}$ such that $a_j \neq 0$. Since by assumption f is a polynomial this set is finite and Δ is a bounded convex lattice polyhedron. We also call Δ the Newton polyhedron of the hypersurface $V^\circ = \{z \in (\mathbb{C}^*)^{n+1} \mid f(z) = 0\}$. According to Gelfand et al. [4] we call the image $\mathrm{Log}(V^\circ) \subset \mathbb{R}^{n+1}$ *the amoeba* of V°.

For the rest of the paper we assume that Δ has a non-empty interior in $\mathbb{R}^{n+1}$. Otherwise after a suitable (multiplicative) change of coordinates the polynomial f can be transformed to a polynomial in a smaller number of variables.

Let $\mathbb{C}T_\Delta$ be the complex toric variety (see e.g. [4]) associated to Δ. We define V as the closure of the hypersurface $V^\circ = \{z \in (\mathbb{C}^*)^{n+1} \mid f(z) = 0\}$ in $\mathbb{C}T_\Delta$. Taking the Newton polyhedron for Δ is a canonical choice. Of course, we can take such compactification for any convex lattice $(n+1)$-polyhedron Δ, even if it was not the Newton polyhedron of V°. However the choice of the Newton polyhedron of V° as Δ produces the best results as the next proposition shows. Recall that in the toric construction there is a k-dimensional complex toric subvariety $\mathbb{C}T_{\Delta'}$ associated to any k-dimensional face $\Delta' \subset \Delta$.

Proposition 2.18. *The hypersurface V is disjoint from the points* (*i.e. the* 0*-dimensional toric varieties*) *corresponding to the vertices of Δ, but intersects all the tori corresponding to any positive-dimensional face of Δ.*

Furthermore, this property characterizes $\mathbb{C}T_\Delta$ in the following sense. Let $\bar{\Delta}$ be a convex lattice polyhedron in $\mathbb{R}^{n+1}$ with a non-empty interior and $\bar{V}$ be the closure of V° in $\mathbb{C}T_{\bar{\Delta}} \supset (\mathbb{C}^*)^{n+1}$. If a *hypersurface* $\bar{V}$ *is disjoint from the points corresponding to the vertices of* $\bar{\Delta}$ *but intersects all the tori corresponding to positive-dimensional faces of* $\bar{\Delta}$ *then* $\mathbb{C}T_{\bar{\Delta}} = \mathbb{C}T_{\Delta}$.

Remark 2.19. Note that even though $\mathbb{C}T_{\Delta}$ is unique by this proposition, the polyhedron $\bar{\Delta}$ itself is not unique even up to a translation. The image of Δ by a homothety with an integer coefficient for $\bar{\Delta}$ corresponds to the same toric variety.

Proof. Proposition 2.18 follows from the following Lemma. Note that a vertex is a 0-face of Δ. □

Lemma 2.20. *Let* $\Delta' \subset \Delta$ *be a face. The intersection* $V \cap \mathbb{C}T_{\Delta'}$ *coincides with the hypersurface cut on* $\mathbb{C}T_{\Delta'}$ *by the closure of the zero set of the following* Δ'*-truncation of the polynomial* f:

$$f_{\Delta'}(z) = \sum_{j \in \Delta'} a_j z^j.$$

Proof. To prove the lemma it suffices to note that the monomials from $\mathbb{Z}^{n+1} \cap \Delta'$ have higher order of vanishing when $z \to \mathbb{C}\,\Delta'$. □

Remark 2.21. The property of V from Proposition 2.18 can be alternatively reformulated in terms of the moment map $\bar{\mu}_{\Delta} : \mathbb{C}T_{\Delta} \to \Delta$, see Section 2.3. *The image* $\mu(V)$ *is disjoint from the vertices of* Δ *but intersects every positive-dimensional face of* Δ. According to Gelfand et al. [4] the image $\mu(V)$ is called *the compactified amoeba* of V°. This restatement is equivalent to the property from Proposition 2.18, since for any face $\Delta' \subset \Delta$ we have $\mu(\mathbb{C}T_{\Delta'}) = \Delta'$.

Example 4. Let $f(z,w) = zw + z + w - 1$. Then $V^\circ \subset (\mathbb{C}^*)^2$ is a hyperbola. The Newton polygon Δ is a square $\{(x,y) \in \mathbb{R}^2 \mid 0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 1\}$ and the corresponding toric surface $\mathbb{C}T_{\Delta}$ is the hyperboloid $\mathbb{CP}^1 \times \mathbb{CP}^1$.

Take now $\bar{\Delta} = \{(x,y) \in \mathbb{R}^2 \mid 0 \leqslant x, 0 \leqslant y, x + y \leqslant 1\}$. The corresponding toric surface is $\mathbb{CP}^2 \supset (\mathbb{C}^*)^2$. The images of V° under the associated moment maps are sketched in Fig. 4.

The following example treats projective hypersurfaces.

Example 5. Let $V \subset \mathbb{CP}^{n+1} \supset (\mathbb{C}^*)^{n+1}$ be a projective hypersurface of degree d not passing through the points $[1 : 0 : \cdots : 0], \ldots, [0 : \cdots : 0 : 1]$. Then $V^\circ = V \cap (\mathbb{C}^*)^{n+1}$ is given by a polynomial f whose Newton polyhedron is

$$\Delta_d = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid 0 \leqslant x_j, \sum_j x_j \leqslant d\}.$$

Vice versa, $\mathbb{C}T_{\Delta} = \mathbb{CP}^{n+1}$ and the closure of V° in $\mathbb{CP}^{n+1}$ is V.

2.6. Pairs-of-pants in higher dimensions

Definition 8. Let $\mathscr{H} \subset \mathbb{CP}^n$ be the union of $n+2$ generic hyperplanes in $\mathbb{CP}^n$. Let $\mathscr{U} \subset \mathbb{CP}^n$ be the union of their ε-neighborhoods for a very small $\varepsilon > 0$.

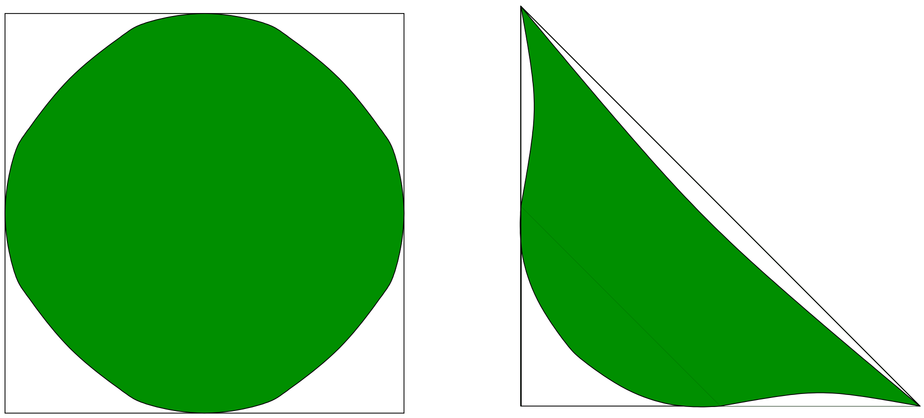

Fig. 4. Images of the hyperbola $zw + z + w - 1 = 0$ under the moment maps corresponding to its Newton polygon and another polygon.

The complement $\bar{\mathscr{P}}_n = \mathbb{CP}^n \setminus \mathscr{U}$ is a manifold with corners. We call $\bar{\mathscr{P}}_n$ *the n-dimensional pair-of-pants*. We call $\mathscr{P}_n = \mathbb{CP}^n \setminus \mathscr{H}$ *the n-dimensional open pair-of-pants*.

Immediately we have the following proposition.

Proposition 2.22. *A pair-of-pants is a compact manifold with boundary. An open pair-of-pants is diffeomorphic to the pair-of-pants minus its boundary.*

Remark 2.23. Note that the choice of $n+2$ generic hyperplane in $\mathbb{CP}^n$ is unique up to the action of $PSL_{n+1}(\mathbb{C})$. Thus $\mathscr{P}_n$ can be given a canonical complex structure.

Note that $\mathscr{P}_1$ is diffeomorphic to the Riemann sphere punctured 3 times, while $\bar{\mathscr{P}}_1$ is diffeomorphic to a closed disk with 2 holes. Thus Definition 8 agrees with the classical, one-dimensional, pair-of-pants definition.

The following proposition describes a natural stratification of the boundary $\partial\bar{\mathscr{P}}$.

Proposition 2.24. *We have the following canonical decomposition of the boundary* $\partial\overline{\mathscr{P}_n} = \bigcup_{j=0}^{n-1} \partial_j\overline{\mathscr{P}_n}$, *where* $\partial_j\overline{\mathscr{P}_n}$ *is a* $(2n-j)$*-dimensional smooth manifold such that each one of its connected components is a trivial* T^j*-fibration over* $\mathscr{P}_{n-j}$ *(recall that* T^j *is a j-dimensional torus* $S^1 \times \cdots \times S^1$*). Different parts do not intersect:* $\partial_j\overline{\mathscr{P}_n} \cap \partial_k\overline{\mathscr{P}_n} = \emptyset$, *if* $j \neq k$, *but the closure of* $\partial_j\overline{\mathscr{P}_n}$ *contains* $\partial_k\overline{\mathscr{P}_n}$ *for all* $k \leqslant j$. *The number of connected components of* $\partial\overline{\mathscr{P}_n}$ *is* $\binom{n+2}{j+2}$.

Proof. Connected components of the manifold $\partial_j\overline{\mathscr{P}_n}$ can be obtained as the intersections of the boundaries of the ε-neighborhoods of j different hyperplanes from $\mathscr{H}$. $\square$

3. Statement of the results

Let $V \subset \mathbb{CP}^{n+1}$ be a smooth hypersurface of degree d. We choose homogeneous coordinates $[Z_0 : \cdots : Z_{n+1}]$ so that V is transverse to coordinate hyperplanes $Z_j = 0$ and all their intersections. The complement of the coordinate hyperplanes in $\mathbb{CP}^{n+1}$ is $(\mathbb{C}^*)^{n+1}$. Denote $V^\circ = V \cap (\mathbb{C}^*)^{n+1}$. Then the hypersurface $V^\circ \in (\mathbb{C}^*)^{n+1}$ is given by the equation $f(z) = 0$, where

$$z = (z_1, \ldots, z_{n+1}) = (Z_1/Z_0, \ldots, Z_{n+1}/Z_0)$$

stands for affine coordinates in $(\mathbb{C}^*)^{n+1}$ and f is a polynomial with the Newton polyhedron Δ_d from Example 5. Recall that we denote the real n-dimensional torus with $T^n = S^1 \times \cdots \times S^1$.

Theorem 1. *For every maximal dual Δ_d-complex Π there exists a stratified T^n-fibration $\lambda : V \to \bar{\Pi}$. This fibration satisfies to the following properties*:

- *the induced map $\lambda^* : H^n(\bar{\Pi}; \mathbb{Z}) \to H^n(V; \mathbb{Z})$ is injective, where $H^n(\bar{\Pi}; \mathbb{Z}) \approx \mathbb{Z}^{p_g}$, $p_g = h^{n,0}$ is the geometric genus of V;*
- *for each primitive piece U_j of Π (see Definition 5) the inverse image $\lambda^{-1}(U_j)$ is an open pair-of-pants $\mathscr{P}_n$;*
- *for each n-cell e of $\bar{\Pi}$ there exists a point $x \in e$ such that the fiber $\lambda^{-1}(x)$ is a Lagrangian n-torus $T^n \subset V$;*
- *there exist Lagrangian embedding $\phi_k : S^n \to V$, $k = 1, \ldots, p_g$ such that the cycles $\lambda(\phi_k(S^n))$ form a basis of $H_n(\bar{\Pi})$.*

Maximal dual Δ_d-complexes exist for every degree d and every dimension n.

Corollary 3.1. *A $2h^{n,0}$-dimensional subspace of $H_n(V)$ has a basis represented by embedded Lagrangian tori and spheres.*

Theorem 1 admits a straightforward generalization to toric varieties other than $\mathbb{CP}^{n+1}$. Let Δ be a bounded convex lattice polyhedron such that all singularities of the toric variety $\mathbb{C}T_\Delta$ are isolated. Note that the isolated singular points of $\mathbb{C}T_\Delta$ necessarily correspond to some vertices of Δ. Consider the space $(\mathbb{C}^*)^{\#(\Delta \cap \mathbb{Z}^{n+1})}$ of all polynomials of the type $f(z) = \sum_{j \in \Delta} a_j z^j$ such that $a_j \neq 0$. Then for a generic choice of a polynomial f from this space the closure V in $\mathbb{C}T_\Delta$ of the zero set of f is a smooth hypersurface transverse to all toric subvarieties $\mathbb{C}T_{\Delta'}$ corresponding to the faces $\Delta' \subset \Delta$. All such V are diffeomorphic and, if we equip them with the symplectic form from $\mathbb{C}T_\Delta$, are symplectomorphic varieties.

Theorem 1′. *For every maximal dual Δ-complex Π there exists a stratified T^n-fibration $\lambda : V \to \bar{\Pi}$. This fibration satisfies to the following properties*

- *the induced map $\lambda^* : H^n(\bar{\Pi}; \mathbb{Z}) \to H^n(V; \mathbb{Z})$ is injective, where $H^n(\bar{\Pi}; \mathbb{Z}) \approx \mathbb{Z}^{p_g}$, $p_g = h^{n,0}$ is the geometric genus of V;*
- *for each primitive piece U_j of Π (see Definition 5) the inverse image $\lambda^{-1}(U_j)$ is an open pair-of-pants $\mathscr{P}_n$;*

- *for each n-cell e of $\bar{\Pi}$ there exists a point $x \in e$ such that the fiber $\lambda^{-1}(x)$ is a Lagrangian n-torus $T^n \subset V$;*
- *there exist Lagrangian embeddings $\phi_k : S^n \to V$, $k = 1, \dots, p_g$ such that the cycles $\lambda(\phi_k(S^n))$ form a basis of $H_n(\bar{\Pi})$.*

By Remark 2.9 not all convex lattice polyhedra have maximal dual complexes. However, in the case of Δ_d (the polyhedra corresponding to the projective space), such subdivisions exists for any d. Maximal subdivisions also exist for products of different Δ_d (this corresponds to hypersurfaces in the product of projective spaces). It is conjectured that for any lattice polyhedron Δ there exists a sufficiently large integer N that $N\Delta$ (the result of scaling of Δ by N) has a maximal subdivision.

The next theorem describes the behavior of the fibration λ with respect to a complex structure on V. Recall that, unlike the smooth and symplectic structures, the complex structure on V depends on the polynomial f and not just on Δ.

Recall that a map $\lambda : V \to \bar{\Pi}$ is called *a totally real fibration* if for any $z \in V$ the tangent space to the fiber through z is totally real i.e. contains no positive-dimensional complex subspaces (as long as the fiber is smooth near z). We say that a hypersurface $V \subset \mathbb{C}T_\Delta$ is *defined over* $\mathbb{R}$ if it can be obtained as the closure of the zero set of a polynomial $f : (\mathbb{C}^*)^{n+1} \to \mathbb{C}$ whose coefficients are real.

Theorem 2. *For every maximal dual Δ-complex Π there exists a smooth hypersurface $V \subset \mathbb{C}T_\Delta$ defined over $\mathbb{R}$ such that the map λ from Theorem $1'$ preserves the real structure of V, i.e. $\lambda \circ \mathrm{conj} = \lambda$, where $\mathrm{conj} : V \to V$ is the involution of complex conjugation. Furthermore, λ is a totally real fibration.*

Theorems $1'$ and 2 can be extended further to polyhedra Δ corresponding to toric varieties with non-isolated singularities. However, in order to do that, one has to modify the definition of stratified fibrations to include singular total spaces V. We do not do that. In the next theorem we no longer have any restrictions on the convex lattice polyhedron Δ, but its statement concerns only the toric, non-singular, part $V^\circ \subset (\mathbb{C}^*)^{n+1}$ of the hypersurface V.

Theorem 3. *For every maximal dual Δ-complex Π there exists a stratified T^n-fibration λ°: $V^\circ \to \Pi$. This fibration satisfies to the following properties*

- *the induced map $(\lambda^\circ)^* : H^n(\Pi; \mathbb{Z}) \to H^n(V^\circ; \mathbb{Z})$ is injective, where $H^n(\Pi; \mathbb{Z}) \approx \mathbb{Z}^{p_g}$, $p_g = h^{n,0}$ is the geometric genus of V;*
- *for each primitive piece U_j of Π (see Definition 5) the inverse image $(\lambda^\circ)^{-1}(U_j)$ is an open pair-of-pants $\mathcal{P}_n$;*
- *for each n-cell e of Π there exists a point $x \in e$ such that the fiber $(\lambda^\circ)^{-1}(x)$ is a Lagrangian n-torus $T^n \subset V$;*
- *there exist Lagrangian embeddings $\phi_k : S^n \to V^\circ$, $k = 1, \dots, p_g$ such that the cycles $\lambda^\circ(\phi_k(S^n))$ form a basis of $H_n(\Pi)$.*

Remark 3.2. These theorems generalize to complete intersections. The base of the fibration in this case is the intersection of the maximal dual balanced polyhedra for the corresponding hypersurfaces (we have to choose them in a mutually general position).

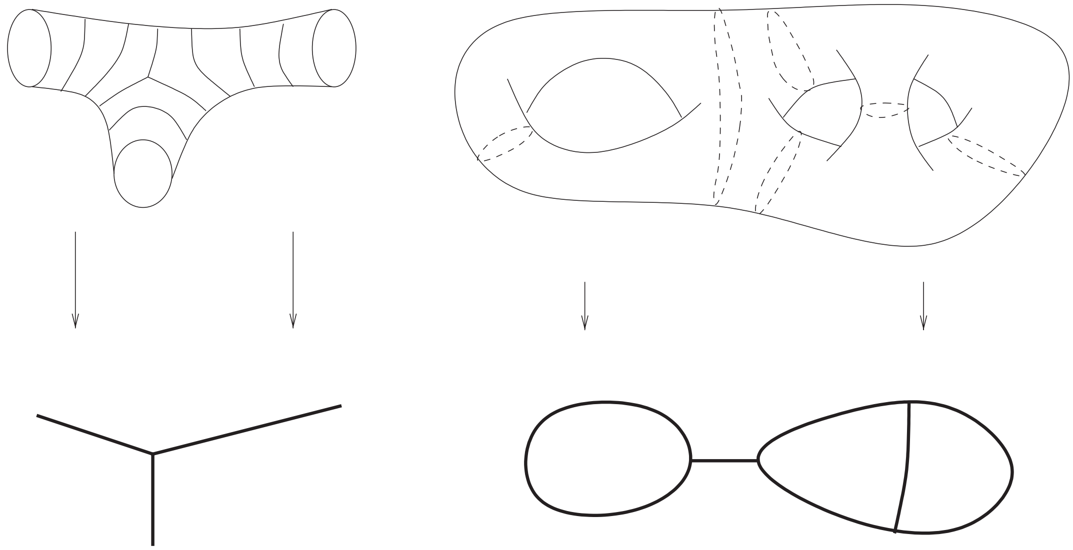

Fig. 5. Circle fibrations on a pair-of-pants and on a surface with a pair-of-pants decomposition.

From a different point of view the base is dual to a maximal mixed lattice subdivision of the Newton polyhedra of the participating equations. The primitive pieces for complete intersections are products of the primitive pieces for hypersurfaces. Sturmfels' generalization [16] of the patchworking technique allows to produce in this case the Lagrangian lifts of the base cycles.

This generalization will be the subject of a future paper.

4. Some examples

4.1. Riemann surfaces

Let S be a closed Riemann surface of genus $g > 1$. It is well-known that S admits a decomposition into pairs-of-pants. Namely, there exist $3g-3$ disjoint embedded circles $C_j \subset S$ such that $S \setminus \bigcup_{j=1}^{3g-3} C_j$ is a disjoint union of $2g-2$ copies of the pair-of-pants P. The pair-of-pants surface P is homeomorphic to the Riemann sphere $\mathbb{CP}^1$ punctured in three points.

To such a decomposition we associate a graph Γ. The vertices of Γ correspond to the pairs-of-pants while the edges correspond to the circles C_j. Each edge joins the vertices corresponding to the adjacent pairs-of-pants.

There exists a fibration $\pi : S \to \Gamma$ such that the circles C_j are inverse images of the midpoints of the edges of Γ. Such fibration is canonically associated to our decomposition into pairs-of-pants. To construct it we fiber each individual pair-of-pants over a tripod graph as pictured on the left-hand-side of Fig. 5. Corresponding diagrams in the Newton polygon of a polynomial were explored in [13].

4.2. The elliptic curve and the K3-surface

Here we consider the well-known fibrations of the elliptic curve and the K3-surface.

Let $\mathbb{C}E$ be an elliptic curve, i.e. a Riemann surface of genus 1. Since $\mathbb{C}E$ is topologically a torus, there is a trivial S^1-fibration $\lambda_E : \mathbb{C}E \to S^1$.

Suppose that the elliptic curve $\mathbb{C}E \subset \mathbb{C}T_\Delta$ is presented as a curve in a toric surface $\mathbb{C}T_\Delta$, where Δ is the Newton polygon of a polynomial defining $\mathbb{C}E$. By the genus formula (see [8]), Int Δ contains a unique lattice point. By Proposition 2.10 a dual Δ-complex is homotopy equivalent to a circle. It is easy to see that the fibration from Theorem 1′ coincides up to homotopy with the trivial S^1-fibration $\mathbb{C}E \approx S^1 \times S^1 \to S^1$.

Another famous fibration $\lambda_K : \mathbb{C}K \to S^2$ has the K3-surface $\mathbb{C}K$ as its total space. All its fibers, except for 24 of them are Lagrangian tori.

Suppose that the polyhedron Δ has exactly one interior lattice point. Then, by Khovanskii's formula [8], the zero locus $\mathbb{C}K$ of a generic polynomial with the Newton polyhedron Δ is a K3-surface. A dual Δ-complex is homotopy equivalent to a sphere S^2 by Proposition 2.10.

Again, the fibration λ can be deformed to a fibration like λ_K by so-called shelling of $\bar{\Pi}$.[1]

In higher dimensions, if Δ is a non-singular polyhedron with a unique interior lattice point, then the corresponding hypersurface $V \subset \mathbb{C}T_\Delta$ is a smooth Calabi–Yau manifold. Singular torus fibrations $V \to S^n$ were constructed by Zharkov [19]. Ruan [14] noted that such fibrations can be made Lagrangian.

Theorem 1′ constructs in this case a stratified torus fibration over a polyhedral complex homotopy equivalent to S^n.

4.3. Hyperplanes in the projective space

This is a fundamental example for the main theorems. Let $H = \{z_1 + \cdots + z_{n+1} + 1 = 0\} \subset \mathbb{CP}^{n+1}$ be a hyperplane. Its toric part $H^\circ = H \cap (\mathbb{C}^*)^{n+1}$ is an open pair-of-pants.

Let Log be the moment map for $(\mathbb{C}^*)^{n+1}$ (see (3)).

Lemma 4.1. $\Sigma_n \subset \mathrm{Log}(H^\circ)$.

Proof. By Passare [12] Σ_n is a spine of the amoeba $\mathrm{Log}(H^\circ)$ and, therefore, its subset. The lemma can alternatively be verified by writing explicit inequalities defining $\mathrm{Log}(H^\circ)$. □

The complement $\mathbb{R}^{n+1} \setminus \Sigma_n$ consists of $n+2$ components. Each component is the region where one of the functions $0, x_1, \dots, x_{n+1}$ is maximal. In the component corresponding to x_j we consider the foliation into straight lines parallel to the gradient of x_j (the jth basis vector). In the component corresponding to 0 we consider the foliation into straight lines parallel to $(1, \dots, 1)$. These foliations glue to a singular foliation $\mathscr{F}'$ which has singularities at Σ_n.

It is easy to smooth out $\mathscr{F}'$ (in a symmetric way with respect to the homogeneous coordinates permutations) at the open n-cells of Σ_n (see Fig. 6). However, the singularities at the

[1] A higher-dimensional version of such deformation will be the subject for a future paper.

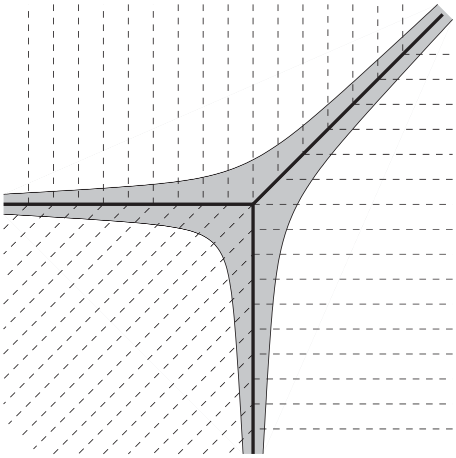

Fig. 6. The amoeba $\mathrm{Log}(H^\circ)$ together with the foliation $\mathscr{F}'$ and its deformation $\mathscr{F}$.

smaller-dimensional cells are essential. The leaves passing through an open $(n-k)$-cell are homeomorphic to the cone over $k+2$ points.

We denote the resulting foliation with $\mathscr{F}$. The foliation $\mathscr{F}$ is a singular fibration and defines the projection $\pi_{\mathscr{F}} : \mathbb{R}^{n+1} \to \Sigma_n$.

The following statement is a key lemma in the proof of the main theorems of this paper.

Lemma 4.2. *The composition*

$$\lambda_H = \pi_{\mathscr{F}} \circ \mathrm{Log} : H^\circ \to \Sigma_n$$

is a stratified T^n-fibration in the sense of Definition 7. *It satisfies to all conclusions of Theorem* 3 *except for the third one. The fibration λ_H can be deformed so that the third condition will also hold.*

The proof of this lemma occupies the rest of this subsection.

To figure out the fibers of λ_H we need to understand the critical points of $\mathrm{Log}|_{H^\circ}$. Following [6,11] for a hypersurface $V^\circ \subset (\mathbb{C}^*)^{n+1}$ we define *the logarithmic Gauss map*

$$\gamma : V^\circ \to \mathbb{CP}^n$$

by taking the composition of a branch of a holomorphic logarithm of each coordinate with the conventional Gauss map. This produces the following formula

$$\gamma(z_1, \ldots, z_{n+1}) = \left[z_1 \frac{\partial f}{\partial z_1} : \cdots : z_{n+1} \frac{\partial f}{\partial z_{n+1}} \right],$$

where f is the polynomial defining V°.

Note that the Newton polyhedron of $z_j(\partial f/\partial z_j)$ coincides with the Newton polyhedron Δ of f. Therefore, by Kouchnirenko's formula [9], $\deg \gamma = (n+1)!\mathrm{Vol}\,\Delta$. In particular, if $V^\circ = H^\circ$ then $\deg \gamma = 1$.

Lemma 4.3 (cf. Lemma 3 of Mikhalkin [11]). *The set of critical points of* $\mathrm{Log}|_{V^\circ}$ *coincides with* $\gamma^{-1}(\mathbb{R}P^n)$.

Proof. Let $z \in V^\circ$ and let $\mathscr{Log}$ be a branch of a holomorphic logarithm $(z_1, \dots, z_{n+1}) \mapsto (\log(z_1), \dots, \log(z_{n+1}))$ defined in a neighborhood of z. The point z is critical for $\mathrm{Log}\,|_{V^\circ}$ iff V° and the orbit of the real torus T^n are not transversal at z. But $\mathscr{Log}$ takes the tangent space to an orbit of T^n to a translate of $i\mathbb{R}^{n+1}$ in $\mathbb{C}^{n+1}$.

Therefore, z is critical iff $\mathscr{Log}(T_z V^\circ)$ contains at least n purely imaginary vectors which is, in turn, equivalent to $\gamma(z) \in \mathbb{R}P^n$. □

Corollary 4.4. *The set of critical points of* $\mathrm{Log}\,|_{H^\circ}$ *coincides with the real locus* $\mathbb{R}H^\circ$ *of* H° (*i.e. with the set of real solutions of* $z_1 + \cdots + z_{n+1} + 1 = 0$).

Proof. Note that, since H° is defined over $\mathbb{R}$, we have $\gamma(\mathbb{R}H^\circ) \subset \mathbb{R}\mathbb{P}^n$. Note that γ extends to a map $H \to \mathbb{C}\mathbb{P}^n$ which is an isomorphism, since $\deg \gamma = 1$. □

Corollary 4.5. *The locus* $\mathscr{D} \subset \mathrm{Log}(H^\circ)$ *of critical values of* $\mathrm{Log}|_{H^\circ}$ *is an immersed manifold transverse to the foliation* $\mathscr{F}$.

Proof. The map $\mathrm{Log}\,|_{\mathbb{R}H^\circ} : \mathbb{R}H^\circ \to \mathscr{D} \subset \mathbb{R}^{n+1}$ is an immersion since the map $\mathrm{Log}\,|_{(\mathbb{R}^*)^{n+1}} : (\mathbb{R}^*)^{n+1} \to \mathbb{R}^{n+1}$ is an immersion (it is a trivial 2^{n+1}-covering of $\mathbb{R}^{n+1}$).

To see the transversality we recall the definition of the foliation $\mathscr{F}'$. For each component of $\mathbb{R}^{n+1} \setminus \Sigma_n$ the foliation $\mathscr{F}'$ is parallel to a vector $\vec{v}$ normal to a facet Δ' of the Newton polyhedron of H°. Therefore, any hyperplane tangent to $\gamma(\mathbb{R}H^\circ)$ is transverse to $\vec{v}$ (hyperplanes parallel to $\vec{v}$ correspond to the intersection of $\mathbb{R}H$ with the divisor corresponding to Δ'). Furthermore, hyperplanes close to being parallel to $\vec{v}$ are close to the hyperplane in $\mathbb{C}\mathbb{P}^{n+1}$ corresponding to this facet and therefore are far from the given component of $\mathbb{R}^{n+1} \setminus \Sigma_n$. Thus the result $\mathscr{F}$ of smoothing is also transverse to $\mathscr{D}$ and the angle between them in $\mathbb{R}^{n+1}$ is separated from 0. □

Note that $\pi_{\mathscr{F}}$ is a stratified $[-1, 1]$-fibration. Thus, the transversality of $\mathscr{D}$ and $\mathscr{F}$ implies that λ_H is a stratified fibration for Σ_n. We need to show that the restriction of π_F to open n-cells of Σ is a torus fibration.

Consider a point $x = (-t, \dots, -t, 0)$ for a large $t > 0$. Note that $\mathscr{D}$ is almost horizontal near x. Thus the fiber of λ_H over x is diffeomorphic to the fiber F of a composition of $\mathrm{Log}\,|_{H^\circ}$ and the linear projection onto the first n coordinates. Note that the map $F \to T^n$ obtained by taking the arguments of the first n coordinates is a diffeomorphism. Recall that H° is given by the equation $z_1 + \cdots + z_{n+1} + 1 = 0$. The absolute values of the coordinates $z_1, \dots, z_n$ are fixed. For any value of their argument we take $z_{n+1} = 1 - z_1 - \cdots - z_n$ to get the unique point from F corresponding to this choice of the arguments. Since $|z_1|, \dots, |z_n|$ are small $z_{n+1} \neq 0$.

We verify the conclusions of Theorem 3 item-by-item. The first and the last conclusions are vacuous in this case, since Σ_n (and, therefore, $\bar{\Sigma}_n$ as well) is contractible. The second one holds since H° is itself an open pair-of-pants.

To make the third conclusion true we have to modify λ_H a little. The fiber F is not Lagrangian, but it is close to a Lagrangian torus $\Lambda = \{|z_j| = \mathrm{const},\ j = 1, \dots, n,\ z_{n+1} = -1\}$. We can deform

H° a little in a neighborhood of $\mathrm{Log}^{-1}(x)$ to make it intersect the fiber of $\pi_{\mathscr{F}} \circ \mathrm{Log}$ along Λ. Therefore, F is Lagrangian for a nearby symplectic structure. By Moser's trick (see e.g. [2]) there exists a self-diffeomorphism h of H° constant outside of a neighborhood of $\mathrm{Log}^{-1}(x)$ and taking one symplectic structure to another. We redefine λ as $\lambda \circ h$. This ensures a Lagrangian fiber over one of the $\binom{n+2}{2}$ open n-cells of Σ_n. We do the same for all other n-cells.

4.4. A localization $Q^n \subset (\mathbb{C}^*)^{n+1}$ of the standard hyperplane

The toric part $H^\circ \subset (\mathbb{C}^*)^{n+1}$ of a hyperplane from 4.3 is a nice embedding of $\mathscr{P}_n$ to $(\mathbb{C}^*)^{n+1}$. However for our purposes it is convenient to modify it in a neighborhood of infinity to get a different submanifold Q^n which is better suited for gluing.

Note that the symmetric group S_{n+2} acts on $\mathbb{CP}^{n+1}$ by interchanging the $n+2$ homogeneous coordinates. This action leaves $(\mathbb{C}^*)^{n+1}$ and $H^\circ \subset (\mathbb{C}^*)^{n+1}$ invariant.

Proposition 4.6. *There exists a proper submanifold $Q^n \subset (\mathbb{C}^*)^{n+1}$ such that*

- *Q^n is embedded in $(\mathbb{C}^*)^{n+1}$ symplectically, i.e. so that the restriction of the form (2) to Q^n is a symplectic form.*
- *Q^n is isotopic to H° in $(\mathbb{C}^*)^{n+1}$.*
- *The composition $\pi_{\mathscr{F}} \circ \mathrm{Log}_t|_{Q^n}$ is a stratified T^n-fibration that satisfies to all hypotheses of Theorem 3.*
- *the closure $\overline{Q^n}$ of Q^n in $\mathbb{CP}^{n+1} \supset (\mathbb{C}^*)^{n+1}$ is a smooth manifold isotopic to H.*
- *Q^n is invariant with respect to the action of the symmetric group S_{n+2} on $(\mathbb{C}^*)^{n+1}$ (see above).*
- *For a sufficiently large $M > 0$*

  $$Q^n \cap (\mathbb{C}^*)^{n+1}_{-M} = Q^{n-1} \times \mathbb{C}^*_{-M},$$

  where $(\mathbb{C}^*)^{n+1}_{-M} = \{(z_1, \dots, z_{n+1}) \in (\mathbb{C}^*)^{n+1} \mid \log|z_{n+1}| < -M\}$ and $\mathbb{C}^*_{-M} = \{z \in \mathbb{C}^* \mid \log|z| < -M\}$. In particular, the intersection $Q^n \cap (\mathbb{C}^*)^{n+1}_{-M}$ is invariant under a translation $z_{n+1} \mapsto cz_{n+1}$, $0 < c < 1$.

Proof. We construct Q^n inductively by dimension n. If $n = 0$ then H° is a point and $Q^0 = H^\circ$.

Assume that Q^k, $k < n$ is already constructed. Consider the simplex

$$\Delta_n(R) = \left\{ x \in \mathbb{R}^{n+1} \,\middle|\, -x_j \leqslant R, \sum_j x_j \leqslant R \right\}.$$

Each its k-dimensional face is dual to a $(n+1-k)$-cell of Σ_n. Fix a sufficiently large number $R_n > 0$ (Fig. 7).

First we define $Q^n \cap \mathrm{Log}^{-1}(\partial\Delta(R_n))$. Each k-face of $\Delta(R_n)$ is contained in a unique affine k-space A in $\mathbb{R}^{n+1}$. Furthermore, the adjoint faces cut the polyhedron $\Delta_{k-1}(R_n) \subset A$. Thus we may identify A with $\mathbb{R}^k$ and, therefore, $\mathrm{Log}^{-1}(A)$ with $(\mathbb{C}^*)^k$. By the induction assumption we already have $Q^{k-1} \subset (\mathbb{C}^*)^k \to \mathbb{R}^k$. We define $Q^n \cap \mathrm{Log}^{-1}(\partial\Delta(R_n))$ to be equal to the union of these Q^k for all faces of $\partial\Delta(R_n)$. By the induction hypothesis (and since R_n was large enough) the choices over different faces agree.

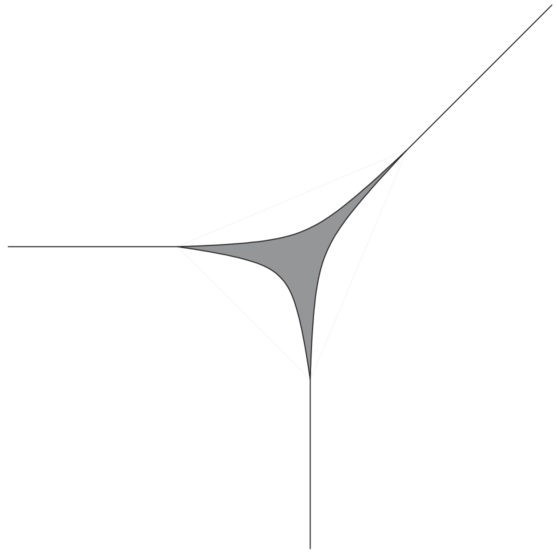

Fig. 7. The amoeba of the localization Q^n of a hyperplane.

Our next step is to extend Q^n to the complement of $\mathrm{Log}^{-1}(\Delta(R_n))$. For each face Δ' of $\partial\Delta(R_n)$ consider its outer normal cone $C_{\Delta'} \subset \mathbb{R}^{n+1}$ (e.g. if Δ' is a facet then $C_{\Delta'}$ is a ray). We define

$$Q^n \cap \mathrm{Log}^{-1}(\Delta' + C_{\Delta'}) = \bigcup_{\vec{v}\in C_{\Delta'}} e^{\vec{v}} Q^n \cap \mathrm{Log}^{-1}(\Delta').$$

In other words, we span the region above the normal cone of a k-face Δ' by the translates of the manifold Q^k.

We set $Q^n \cap \mathrm{Log}^{-1}(\Delta(R_n - 1)) = H^\circ \cap \mathrm{Log}^{-1}(\Delta(R_n - 1))$. By now we have defined Q^n everywhere, but $\mathrm{Log}^{-1}(\Delta(R_n) \setminus \Delta(R_n - 1))$.

Consider a facet Δ' of $\partial\Delta(R_n - 1)$, e.g. the one sitting in the hyperplane $A = \{x_{n+1} = R_n - 1\}$. Since R_n is large enough, $z_{n+1}^{R_n - 1}$ is small enough and the intersection $H^\circ \cap \mathrm{Log}^{-1}(A)$ is close enough to the zero set of $z_1 + \cdots + z_n + 1 = 0$. By the induction hypothesis this zero set can be deformed to Q^{n-1}. We define $Q^n \cap \{\mathrm{Log}|z_{n+1}| = t\}$, $-R_n \leqslant t \leqslant -R_n + 1$ using this deformation. We repeat the same procedure for all other facets of $\Delta(R_n - 1)$. $\square$

Denote $\bar{Q}^n = Q^n \cap \mathrm{Log}^{-1}(\Delta(R_n + 1))$. This is the core part of Q^n and is diffeomorphic to a closed pair-of-pants $\overline{\mathscr{P}_n}$ (as a manifold with corners).

5. Reconstruction of the complex hypersurface from a balanced polyhedron Π

Theorem 1′ can be treated as a pair-of-pants decomposition for V. We can use this presentation to reconstruct V from Π. This allows to interpret a maximal balanced polyhedral complex Π as the complex encoding the gluing pattern of pairs-of-pants in order to get V. Here is the way to reconstruct V from Π.

For each vertex v_j of Π take a copy $\bar{Q}_j$ of $\bar{\mathscr{P}}_n$. This copy can be identified with the localized hyperplane $\bar{Q}^n \subset (\mathbb{C}^*)^{n+1}$. Recall that by Proposition 2.24 $\partial_1(\bar{Q}_j)$ consists of $n+2$ components. Each such component corresponds to a 1-cell of Π adjacent to v_j.

Let e_{jk} be a 1-cell of Π connecting the vertices v_j and v_k. For each such 1-cell we identify the closures F_j and F_k of the corresponding components of $\partial_1(\bar{Q}_j)$ and $\partial_1(\bar{Q}_k)$ in the following way.

Without the loss of generality we may assume that in both copies $\bar{Q}_j, \bar{Q}_k$ of $\bar{Q}^n$ the edge e_{jk} corresponds to the facet $x_{n+1} = -R_n$ of $\Delta(R_n)$ (see 4.4). (Note that such correspondence is given by matrices M_j, M_k from Proposition 2.11.) We attach F_j to F_k by the map

$$(z_1,\dots,z_n,z_{n+1}) \mapsto (z_1,\dots,z_n,\bar{z}_{n+1}), \log|z_{n+1}| = -R_n,$$

where $\bar{z}_{n+1}$ is the complex conjugate to z_{n+1}.

The result U of this gluing is a manifold with boundary. The boundary comes from the unbounded cells of Π. Denote $W^\circ = U \setminus \partial U$. The boundary is formed by the closures F of the components of $\partial_1(Q_j)$ that correspond to unbounded 1-cells in Π. By Proposition 2.24 each such F is a circle fibration over a union of lower-dimensional pairs-of-pants $\mathscr{P}_{n-1}$. Let W be the result of collapsing all fibers of these fibrations on ∂U. Note that W is canonically a smooth manifold since this procedure locally coincides with collapsing the boundary on $\bar{\mathscr{P}}_n$ which results in $\mathbb{CP}^n$.

Theorem 4. *The manifold W is diffeomorphic to V. The manifold W° is diffeomorphic to V°.*

Corollary 5.1. *The manifolds W and W° depend only on the lattice polyhedron Δ associated to Π, not on Π itself.*

Remark 5.2. With a little more care this reconstruction process can be made in the symplectic category, i.e. the result W of gluing can be given a natural symplectic structure. This is due to the following two reasons. The first one is that the pair-of-pants possesses a natural symplectic structure (the one which gives the standard symplectic $\mathbb{CP}^n$ after the symplectic reduction of the boundary). The second one is that two pairs-of-pants get identified along a part F of their boundary which is a symplectically flat hypersurface, it has a neighborhood $F \times [0,1]$ symplectically isomorphic to $Q_{n-1} \times A$, where $A \subset \mathbb{C}^*$ is an annulus. This product is consistent with the S^1-fibration $F \to Q_{n-1}$ from Proposition 2.24.

6. Proof of the main theorems

We are free to choose any smooth hypersurface V with the Newton polyhedron Δ to construct the stratified fibration λ, since all such hypersurfaces are isotopic. We use Viro's patchworking construction [17] to choose a convenient V. Recall that the Newton polyhedron $\Delta \subset \mathbb{R}^{n+1}$ of V is a convex polyhedron whose vertices are lattice points.

6.1. Viro's patchworking

Let $v : \Delta \cap \mathbb{Z}^{n+1} \to \mathbb{R}$ be any function and $a(z) = \sum_{j \in \Delta \cap \mathbb{Z}^{n+1}} a_j z^j$ be any polynomial. Following [17] we define the *patchworking polynomial* for any $t > 0$ by

$$f_t^v(z) = \sum_{j \in \Delta \cap \mathbb{Z}^{n+1}} a_j t^{-v(j)} z^j,$$

where $a_j \neq 0$ for any $j \in \Delta \cap \mathbb{Z}^{n+1}$. Note that if v is integer-valued then f_t^v makes sense also for any $t \in \mathbb{C}^*$.

Remark 6.1. In [17] the patchworking polynomial was used for construction of real algebraic hypersurfaces with controlled topology. The topology of the zero set of a real patchworking polynomial for $t \gg 0$ depends only on the function v and on the signs of the coefficients a_j.

6.2. Non-Archimedian amoebas

If $V \subset (\mathbb{C}^*)^{n+1}$ be an algebraic variety. The image $\mathrm{Log}(V) \subset (\mathbb{C}^*)^{n+1}$ is called the *amoeba* of V, see [4]. Note that amoebas make sense also for varieties over other fields K as long as we have a norm $K^* = K \setminus \{0\} \to \mathbb{R}_+$. The map $\mathrm{Log}_K : (K^*)^{n+1} \to \mathbb{R}^{n+1}$ is defined by $\mathrm{Log}_K(z_1, \ldots, z_{n+1}) = (\log \|z_1\|_K, \ldots, \log \|z_{n+1}\|)$ and the amoeba of $V_K \subset (K^*)^{n+1}$ is defined to be $\mathrm{Log}_K(V_K)$.

A particularly useful case is when K is an algebraically closed field with a non-Archimedian valuation.[2] Recall that a non-Archimedian valuation[2] is a function $\mathrm{val} : K^* \to \mathbb{R}$ such that $\mathrm{val}(a+b) \leqslant \max\{\mathrm{val}(a), \mathrm{val}(b)\}$ and $\mathrm{val}(ab) = \mathrm{val}(a) + \mathrm{val}(b)$. Note that e^{val} gives a norm on K and Log_K is nothing but taking the coordinatewise valuation.

Non-Archimedian amoebas of hypersurfaces were completely described in [7]. An example of such field is the field K of the Puiseux series with complex coefficients in t. Namely an element of K is a formal series $b(t) = \sum_{k \in J} b_k t^k$, $b_k \in \mathbb{C}^*$ where $J \subset \mathbb{R}$ is any bounded from below set contained in a finite union of arithmetic progressions. The valuation is defined by $\mathrm{val}\, \|b(t)\| = -\min J$. Note that we used irrational as well as rational powers in the Puiseux series to make the valuation surjective.

Theorem (Kapranov [7]). *If $V_K \subset (K^*)^{n+1}$ is a hypersurface given by a polynomial $f = \sum a_j z^j$, $a_j \in K^*$ then the (non-Archimedian) amoeba of V_K is the balanced polyhedral complex corresponding to the function $v(j) = \mathrm{val}(a_j)$ defined on the lattice points of the Newton polyhedron Δ of V_K as in Example* 2.

6.3. Lifts of non-Archimedian amoebas to $(\mathbb{C}^*)^{n+1}$

Consider the map $u : K^* \to S^1$ defined by $u(b) = \arg(b_{-\mathrm{val}(b)})$, $b = \sum_{k \in J} b_k t^k$. In other words, u takes the argument of the coefficient at the lowest power of t. This is a homomorphism from the multiplication group K^*. Together with val it gives a homomorphism $w = (\mathrm{val}, u) : K^* \to \mathbb{C}^* \approx \mathbb{R} \times S^1$ and thus a homomorphism $W : (K^*)^{n+1} \to (\mathbb{C}^*)^{n+1}$.

[2] Sometimes a valuation is defined as minus such a function.

Lemma 6.2. *If $V \subset (K^*)^{n+1}$ is a hypersurface given by a polynomial $f = \sum a_j z^j$, $a_j \in K^*$ then $W(V_K) \subset (\mathbb{C}^*)^{n+1}$ depends only on the values $w(a_j) \in \mathbb{C}^*$ of the coefficients.*

Proof. Kapranov's theorem takes care of $\mathrm{Log}(w(V_K)) = \mathrm{Log}_K(V_K)$. We need to prove that the values $u(a_j)$ determine the arguments of $W(V_K)$. Let $x \in \mathrm{Log}_K(V_K)$. By Kapranov's theorem it means that there is a set of indices $j_1, \ldots, j_l$, $l \geqslant 2$, such that $\mathrm{val}(a_{j_1}) = \cdots = \mathrm{val}(a_{j_l}) \geqslant \mathrm{val}(a_j)$ for any other index j. Let $z \in (K^*)^{n+1}$ be a point such that $\mathrm{Log}_K(z) = x$. The lowest powers of t in the Puiseux series $f(z)$ are contributed by the monomials $a_{j_1} z^{j_1}, \ldots, a_{j_l} z^{j_l}$. If $f(z) = 0$ then the coefficients at these lowest powers are such that their sum is zero. Conversely, the higher powers of t can be arranged to make $f(z) = 0$ without the change of $W(z)$ as in the proof of Kapranov's theorem. □

6.4. Maslov's dequantization

Consider the following family of binary operations on $\mathbb{R} \ni x, y$:

$$x \oplus_t y = \log_t(t^x + t^y)$$

for $t > 1$ and

$$x \oplus_\infty y = \lim_{t \to 0} x \oplus_t y = \max\{x, y\}.$$

This is a commutative semigroup operation (no inverse elements and no zero) for each t. The set $\mathbb{R}$ equipped with this operation for addition and with $x \odot y = x + y$ for multiplication is a semiring $\mathbb{R}_t$. Indeed, for any $x, y, z \in \mathbb{R}$ we have $x \odot (y \oplus_t z) = (x \odot z) \oplus_t (y \odot_t z)$.

Passing from a finite t to infinity in this family of semirings is called *Maslov's dequantization*, cf. [10]. Note that for all finite values of t the semiring is isomorphic to the semiring of real positive numbers equipped with the usual addition and multiplication. But the behavior at $t = \infty$ is qualitatively different, the addition becomes idempotent, $x \oplus_\infty x = x$. The prefix "de" reflects the fact that in this deformation the classical calculus operations appear on the quantum side.

There is a universal bound for the convergence of the operations $\oplus_t$ to $\oplus_\infty = \max$. Namely, we have

$$\max\{x_1, \ldots, x_N\} \leqslant x_1 \oplus_t \cdots \oplus_t x_N \leqslant \max\{x_1, \ldots, x_N\} + \log_t N \tag{4}$$

The dequantization point of view can be used to reinterpret Viro's patchworking, see [18]. Instead of deforming the coefficients of the polynomial we may keep them constant, but deform the addition operation instead. This point of view yields some useful estimates on the zero set of the patchworking polynomial as shown below.

One way to think of a polynomial is to think of it as a collection of coefficients at its monomials. Fix a polynomial $p(x) = \sum_j c_j x^j$ in $n+1$ variables, where the arithmetic operations are taken from the semiring $\mathbb{R}_t$. Depending on t this polynomial defines different functions $p_t : \mathbb{R}^{n+1} \to \mathbb{R}$. Note that the function

$$f_t(z) = t^{p_t(\mathrm{Log}_t(z))}$$

coincides with the patchworking polynomials where all $a_j = 1$ and $v(j) = c_j$. Here $\mathrm{Log}_t(z_1, \ldots, z_{n+1}) = (\log_t(z_1), \ldots, \log_t(z_{n+1}))$.

Lemma 6.3. *If a point $x \in \mathbb{R}^{n+1}$ belongs to the amoeba*

$$\mathrm{Log}_t(\{z \in (\mathbb{C}^*)^{n+1} \mid f_t(z) = 0\})$$

then the monomials $c_j x^j$ from p_t satisfy the generalized triangle inequality in $\mathbb{R}_t$, i.e. for each index k we have

$$c_k \odot x^k \leqslant \bigoplus_{j \neq k} c_j \odot x^j.$$

Proof. If $x = \mathrm{Log}_t(z)$ with $f_t(z) = 0$ then the sum of the monomials $t^{c_j} z^j$ is zero and thus their norms must satisfy the triangle inequality. □

Let $f_t = \sum_{j \in \Delta \cap \mathbb{Z}^{n+1}} a_j t^{-v(j)} z^j$ now be a general patchworking polynomial. Denote $V_t^\circ = \{f_t = 0\} \subset (\mathbb{C}^*)^{n+1}$. The family f_t can be treated as a single polynomial in $(K^*)^{n+1}$ (see 6.2). It defines a hypersurface $V_K^\circ \subset (K^*)^{n+1}$. Recall that *the Hausdorff distance* between two closed subsets $A, B \subset \mathbb{R}^{n+1}$ is the number

$$\max\left\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\right\},$$

where $d(a, B)$ is the Euclidean distance between a point a and a set B in $\mathbb{R}^{n+1}$. Denote $\mathscr{A}_t = \mathrm{Log}_t(V_t^\circ)$ and $\mathscr{A}_K = \mathrm{Log}_K(V_K^\circ)$.

Corollary 6.4. *The amoebas $\mathscr{A}_t$ converge in the Hausdorff metric to the non-Archimedean amoeba $\mathscr{A}_K$ when $t \to \infty$.*

Proof. Lemma 6.3 and inequality (4) imply that $\mathscr{A}_t$ converge to a subset of $\mathscr{A}_K$. Indeed, for each t we can rewrite $|a_j t^{v(j)} z^j|$ as $|t^{c_j} z^j|$, $c_j = v(j) + \log_t |a_j|$. Such a monomial induces a linear function $c_j + jx$ in $\mathbb{R}^{n+1}$. The inequalities

$$c_k + kx \leqslant \max_{j \neq k} (c_j + jx) + \log_t(N), \tag{5}$$

where $N + 1$ is the number of monomials in f_t, cut out a uniformly bounded neighborhood of $\mathscr{A}_K$ which contains $\mathscr{A}_K$.

The limit of $\mathscr{A}_t$ cannot be any smaller than $\mathscr{A}_K$ by the following topological reason. A component of the complement of the set described by inequalities (5) is given by the inequality $c_k + kx > \max_{j \neq k} (c_j + jx) + \log_t(N)$. By Forsberg et al. [3] this component is contained in the component of $\mathbb{R}^{n+1} \setminus \mathscr{A}_t$ corresponding to the index k. Thus, different components of the set described by (5) must be contained in different components of $\mathbb{R}^{n+1} \setminus \mathscr{A}_t$. □

This corollary can be strengthened to describe the limits of the varieties $V_t^\circ \subset (\mathbb{C}^*)^{n+1}$ under the corresponding renormalization of the norms of their points. The description is in terms of the lifts of non-Archimedean amoebas, see 6.3. Let $H_t : (\mathbb{C}^*)^{n+1} \to (\mathbb{C}^*)^{n+1}$ be the transformation defined by

$$H_t(z_1,\dots,z_{n+1}) = \left(t^{-|z_1|}\frac{z_1}{|z_1|},\dots,t^{-|z_{n+1}|}\frac{z_{n+1}}{|z_{n+1}|}\right).$$

We have $\mathrm{Log}_t = \mathrm{Log} \circ H_t$.

Theorem 5. *The sets $H_t(V_t^\circ)$ converge in the Hausdorff metric to $W(V_K^\circ)$ when $t \to \infty$.*

The proof is the same as the proof of Corollary 6.4. The only difference we have to make is to incorporate the arguments of the monomials to inequalities (5).

6.5. *Construction of the fibration $\lambda_t : V_t^\circ \to \Pi$*

Let Π be a maximal dual Δ-complex and $v : \Delta \cap \mathbb{Z}^{n+1} \to \mathbb{R}$ be the function such that $\Pi = \Pi_v$ as in Proposition 2.4. It gives us a patchworking polynomial $f_t = \sum_{j\in\Delta\cap\mathbb{Z}^{n+1}} t^{-v(j)}z^j$. As before we denote with $V_t^\circ \subset (\mathbb{C}^*)^{n+1}$ the zero set of this polynomial.

We construct $\lambda_t : V_t^\circ \to \Pi$ for a sufficiently large t by gluing the fibrations λ_H from 4.3.

To do it we construct a singular foliation $\mathscr{F}_\Pi$ in a neighborhood $\mathscr{N} \supset \Pi$. By Proposition 2.11 Π can be locally identified with Σ_n by elements of $ASL_{n+1}(\mathbb{Z})$. Recall that an element $M \in ASL_{n+1}(\mathbb{Z})$ is a rotation defined by a unimodular integer $(n+1)\times(n+1)$-matrix $(m_{j,k})$ followed by a translation by $m = (m_1,\dots,m_{n+1})$ in $\mathbb{R}^{n+1}$. This transformation of $\mathbb{R}^{n+1}$ lifts to $(\mathbb{C}^*)^{n+1}$ as

$$H_M : z_j \mapsto e^{m_j} z_1^{m_{j,1}} \dots z_{n+1}^{m_{j,n+1}}.$$

We patch the foliations $\mathscr{F}$ constructed in 4.3 for the primitive n-complex Σ_n. Let $v_j \in \Pi$ be a vertex. By Proposition 2.11 there exists a neighborhood $U_j \ni v_j$ in Π and $M_j \in ASL_{n+1}(\mathbb{Z})$ such that $M_j(U_j)$ is a neighborhood of 0 in Σ_n. Let N_j be a small neighborhood of the closure of $M_j(U_j)$.

Consider the pull-back under M_j of the foliation $\mathscr{F}$ constructed in 4.3 restricted to N_j. Note that $M_j^{-1}(N_j)$ cover Π. The pull-back foliations at the overlaps $M_j^{-1}(N_j)\cap M_k^{-1}(N_k)$ do agree in general. Nevertheless, they have the same type of singularities at the same points and their non-singular leaves are transverse to Π. A partition of unity gives a foliation $\mathscr{F}_\Pi$ in a neighborhood $\mathscr{N}$ of Π. Note that we can ensure that $\mathscr{N}$ contains an ε-neighborhood of Π for some $\varepsilon > 0$. Following 4.3 we denote $\pi_{\mathscr{F}_\Pi} : \mathscr{N} \to \Pi$ the projection along the leaves of $\mathscr{F}_\Pi$.

By Corollary 6.4 for a sufficiently large $t > 0$ we have $\mathrm{Log}_t(V_t) \subset \mathscr{N}$, and we define

$$\lambda_t = \pi_{\mathscr{F}_\Pi} \circ \mathrm{Log}_t : V_t \to \Pi.$$

6.6. *Proof of Theorems 2 and 4*

Here we prove that V_t is non-singular and that λ_t satisfies to all hypotheses of Theorem 3 for a large $t > 0$.

Note that if $\mathrm{Log}_t z = x$ then $\|t^{-v(j)}z^j\| = t^{jx-v(j)}$, where $jx \in \mathbb{R}$ stands for the scalar product. Let $F \subset \Pi$ be an open $(n+2-k)$-cell.

Lemma 6.5. *There exists k monomials $t^{-v(j_1)}z^{j_1},\dots,t^{-v(j_k)}z^{j_k}$ that dominate f_t in a neighborhood of F. Namely, any other monomial evaluated at a point near F has a smaller order by t. Furthermore, the hypersurface*

$$\sum_{m=1}^{k} t^{-v(j_m)}z^{j_m} = 0$$

is isomorphic to the hyperplane $z_1+\cdots+z_{k-1}+1=0$ under the multiplicative change of coordinates by an element of $ASL_{n+1}(\mathbb{Z})$.

Proof. This follows from the maximality of Π. By Proposition 2.1 F is dual to a k-dimensional polyhedron from a subdivision of Δ. Since Π is maximal, this polyhedron is the standard $(k-1)$-simplex up to action of $ASL_{n+1}(\mathbb{Z})$. □

This lemma implies that V_t° is non-singular for large $t>0$. Indeed, it is covered by a finite number of open sets and in each set it is a small perturbation of the image of a hyperplane. Furthermore, its compactification $V_t\subset\mathbb{C}T_\Delta$ is smooth and transverse to the coordinate hyperplanes as the same reasoning with the terms of smaller order applies to the affine charts of $\mathbb{C}T_\Delta$.

Our next step is to isotop V_t over N_j as in 4.4. Recall that N_j was defined in 6.5 as a small neighborhood of $\bar{U}_j\subset\Pi$ in $\mathbb{R}^{n+1}$. Denote

$$Q_j^n = M_j^{-1}(H_t(Q^n))\cap \mathrm{Log}_t^{-1}(N_j).$$

By the last conclusion of Proposition 4.6 these manifolds coincide over $N_j\cap N_k$ for $t\gg 0$. We set

$$Q_\Pi=\bigcup_j Q_j^n.$$

Note that for $t\gg 0$ V_t° is isotopic to Q_Π by the same isotopy as in the proof of Proposition 4.6 since all other monomials of f_t have smaller order in t. This proves Theorem 4. As in Proposition 4.6 the closure $\bar{Q}_\Pi\subset\mathbb{C}T_\Delta$ is a smooth manifold. Similarly, V_t is isotopic to $\bar{Q}_\Pi$ in $\mathbb{C}T_\Delta$.

In the proof of Theorem 3 we may assume that $V^\circ=V_t^\circ$ since its closure $V_t\subset\mathbb{C}T_\Delta$ is smooth and transverse to the coordinate hyperplanes. Similarly, in the proof of Theorems 1, 1′ and 2 we may assume that $V=V_t$. We define $\lambda^\circ: V^\circ\to\Pi$ as a composition of the isotopy $V\approx Q_\Pi$, the map $\mathrm{Log}_t:(\mathbb{C}^*)^{n+1}\to\mathbb{R}^{n+1}$ and the projection $\pi_{\mathscr{F}_\Pi}:\mathscr{N}\to\Pi$. Note that the isotopy $V\approx Q_\Pi$ is a symplectomorphism by the Moser trick. (By the Moser trick, see e.g. [2] any smooth deformation of a symplectic structure on a simply connected manifolds is isomorphic to the original symplectic structure.) Note also that the Moser trick can be done equivariantly with respect to the complex conjugation if V is defined over $\mathbb{R}$.

To define $\lambda: V\to\bar{\Pi}$ we compactify the previous construction by using $\bar{Q}_\Pi$ and the reparametrized moment map to Δ as in 2.3 (Fig. 8).

This proves Theorem 2, since everything in our construction is equivariant with respect to complex conjugation as long as a_j in the patch-working polynomial are real. The fibration λ is totally real since it is totally real for a hyperplane.

Also, by Proposition 4.6 this proves the second and the third conclusions in Theorems 1, 1′ and 3. The homotopy type of $\bar{\Pi}$ and Π is the wedge of p_g copies of S^n, where $p_g=h^{n,0}$ by Proposition 2.10.

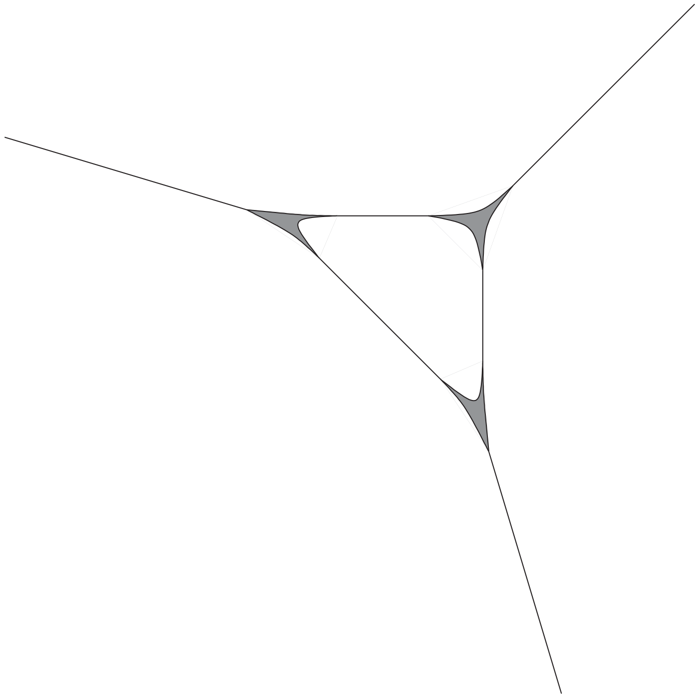

Fig. 8. The amoeba of the localization Q_Π of a hypersurface.

To finish the proof of Theorems 1, 1′ and 3 we need to prove injectivity of the induced homomorphism in cohomology and to exhibit the Lagrangian spheres lifting the cycles from Π.

6.7. Proof of Theorems 1, 1′ and 3

The Lagrangian spheres will come from components of certain real hypersurfaces whose complexification is isotopic to V.

Let j be a lattice point of Δ. We define

$$f_t^{(j)} = \sum_{k \neq j} |a_k| t^{v(k)} z^k - |a_j| t^{v(j)} z^j.$$

Denote with $V_t^{(j)} \subset (\mathbb{C}^*)^{n+1}$ the zero set of $f_t^{(j)}$ and with $\mathbb{R}V_t^{(j)} \subset (\mathbb{R}^*)^{n+1}$ its real part. The Viro patchworking theorem [17] (see also [4] for a special case of *combinatorial patchworking* and [5] for an elementary description in the case of curves) implies that $\mathbb{R}V_t^{(j)} \cap \mathbb{R}_+^{n+1}$ is diffeomorphic to a sphere S^n. This sphere $S_j^n \subset V_t^{(j)}$ is Lagrangian as a component of the real part and it maps under Log_t to $\mathcal{N} \supset \Pi$ for $t \gg 0$. Furthermore, it realizes in $H_n(\Pi)$ the class corresponding to j according to Proposition 2.10 (Fig. 9).

By 6.6 $V_t^{(j)}$ is smooth. Thus, it is isotopic to V_t and we have a diffeomorphism $h : V_t^{(j)} \to V_t$. Moreover, we can choose an isotopy among the hypersurfaces defined by the polynomials such that the norm of all monomials is constant in the course of deformation. All such hypersurfaces are

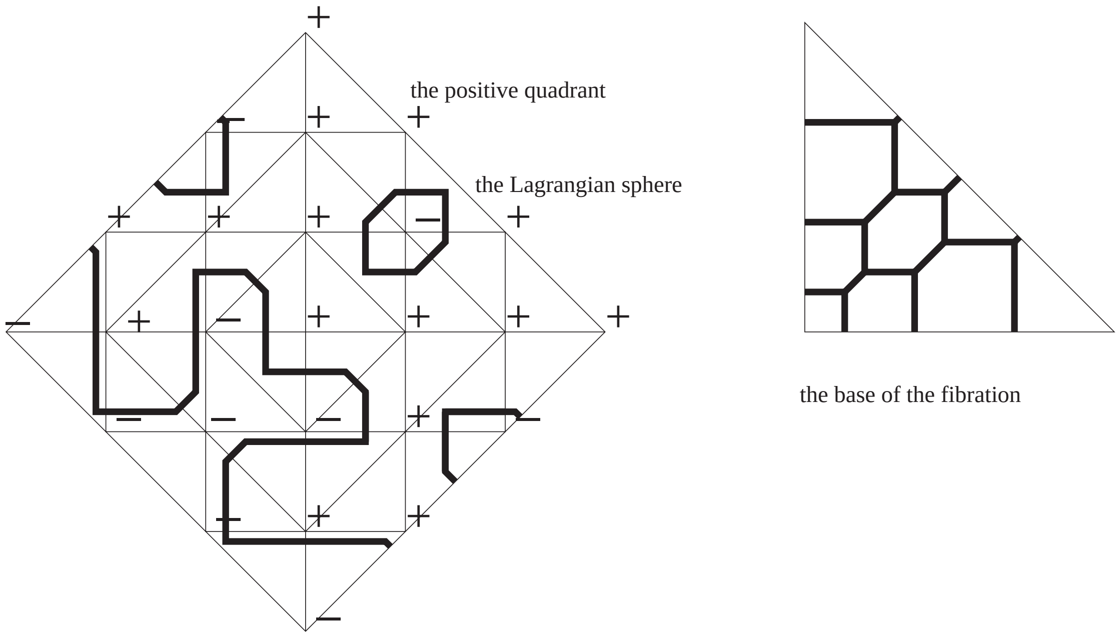

Fig. 9. Construction of the Lagrangian lift of a base cycle by the real patchworking.

smooth and their image under Log_t is contained in $\mathcal{N} \supset \Pi$ by 6.6. Therefore, the image $h(S^n_j)$ projects to the same class in $H_n(\Pi)$.

By Moser's trick, h is isotopic to a symplectomorphism. This gives a Lagrangian sphere in V_t which projects to the class in $H_n(\Pi)$ corresponding to j. Thus the last conclusion of Theorems 1 and 1′ is proved.

Existence of such spheres also implies the first conclusion of Theorems 1 and 1′. The map λ^* is injective since we can distinguish the images in $H^n(V;\mathbb{Z})$ by their evaluations on these Lagrangian spheres.

The proof of Theorem 3 is the same since these spheres belong to the toric part $\mathbb{R}V_t^\circ$ of $\mathbb{R}V$.

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