Some New Results on the Convergence of the SSOR and USSOR Methods*

M. Madalena Martins*†
Department of Mathematics
University of Coimbra
3000 Coimbra, Portugal

and

Lala B. Krishna
Department of Mathematical Sciences
The University of Akron
Akron, Ohio 44325

Submitted by Richard S. Varga

ABSTRACT

We obtain conditions for the convergence of SSOR and USSOR methods when the matrix \( A \) has a special form (so-called "red-black" ordering), using some vectorial norms. We give characterizations for the \( H \)-matrix with respect to the USSOR iteration matrix. Our results extend the work of Alefeld and Krishna.

1. INTRODUCTION

To solve the system of linear equations

\[
Ax = b,
\]

(1.1)

where \( A \in \mathbb{C}^{n \times n} \) is a nonsingular complex matrix with nonzero diagonal
elements, it is convenient to consider the splitting of $A$,

$$A = D(I - L - U),$$

(1.2)

with $D$ a diagonal matrix and $L, U$ strictly lower and upper triangular matrices, respectively.

Let us denote the iteration matrices for the Jacobi, forward successive overrelaxation (SOR), backward SOR, symmetric successive overrelaxation (SSOR), and unsymmetric successive overrelaxation (USSOR) methods, associated with the splitting (1.2), by $J, \mathcal{L}_\omega, \mathcal{U}_\omega, \mathcal{S}_\omega, \mathcal{S}_{\omega,\omega}$, respectively; they are defined by

$$J = I - D^{-1}A = L + U,$$

(1.3)

$$\mathcal{L}_\omega = (I - \omega L)^{-1}\{(1 - \omega)I + \omega U\},$$

(1.4)

$$\mathcal{U}_\omega = (I - \omega U)^{-1}\{(1 - \omega)I + \omega L\},$$

(1.5)

$$\mathcal{S}_\omega = \mathcal{U}_\omega \mathcal{L}_\omega,$$

(1.6)

$$\mathcal{S}_{\omega,\omega} = \mathcal{U}_\omega \mathcal{U}_{\omega}.$$

(1.7)

Then, the SSOR and USSOR iterative methods are defined by

$$x^{(i+1)} = \mathcal{S}_{\omega,\omega} x^{(i)} + \omega(2 - \omega)(I - \omega U)^{-1}(I - \omega L)^{-1}b, \quad i = 0, 1, \ldots,$$

(1.8)

and

$$x^{(i+1)} = \mathcal{S}_{\omega,\omega} x^{(i)} + (\omega + \omega' - \omega \omega')(I - \omega' U)^{-1}(I - \omega L)^{-1}b,$$

(1.9)

$$i = 0, 1, \ldots,$$

respectively.

In the first part of this paper, using some notions about vectorial norms, we derive intervals of convergence for the SSOR and USSOR iterative methods when the matrix $A$ has the special form (1.10). These results have some computational advantage over those established in [1] and [2]. Then, we generalize the USSOR method when $A$ is an $H$-matrix and has the form (1.10). This extends the result of Alefeld [1].
According to Young [8, pp. 463–473] we know that the SSOR method presents certain advantages over the classical SOR method. Moreover, Neithammer [3] has shown that with the exception of the first iteration, the SSOR method does not need more computation effort than the SOR method.

In this paper, we assume that $A$ has the following structure:

$$A = \begin{bmatrix} D_1 & H \\ K & D_2 \end{bmatrix}. \quad (1.10)$$

where $D_1$ and $D_2$ are nonsingular diagonal matrices. Matrices having this structure are the result of the discretization of elliptic boundary value problems if the unknowns are collected using so-called red-black ordering [8, pp. 159].

## 2. CONVERGENCE OF THE SSOR METHOD

The following theorems use vectorial norms of the matrices $L$ and $U$ generated by a vectorial norm $p$ as given in Theorem 1 of [5]. The first theorem gives sufficient conditions for the convergence of the SSOR method.

**Theorem 2.1.** Let $A \in \mathbb{C}^{n \times n}$ be a nonsingular matrix of the form (1.10). Let $p$ be a vectorial regular norm of dimension $k$ defined over $\mathbb{C}^n$, and $M$ the vectorial norm of the matrix generated by $p$. If $\rho^* = \rho(M(L) + M(U)) < 1$, then the SSOR method is convergent if $0 < \omega < 2$.

**Proof.** Because of the special form (1.10) of $A$, the SSOR iteration matrix $S_\omega$ associated with $A$ can be written as follows (see [8]):

$$S_\omega = P_\omega S_\omega = S_{\omega,0} L_{0,\omega}^0 L_{0,\omega}^0 S_{\omega,0}, \quad (2.1)$$

where

$$L_{0,\omega} = \begin{bmatrix} I_1 & 0 \\ \omega N_1 & (1 - \omega) I_2 \end{bmatrix},$$

$$S_{\omega,0} = \begin{bmatrix} (1 - \omega) I_1 & \omega M_1 \\ 0 & (1 - \omega) I_2 \end{bmatrix} \quad (2.2)$$

and $M_1 = -D_1^{-1}H$, $N_1 = -D_2^{-1}K$. 


If we interchange the matrices \( L_{0,\omega} \) and \( L_{\omega,0} \) in (2.1), we obtain a matrix \( \tilde{L}_\omega = L_{0,\omega} L_{\omega,0} \) whose spectral radius is the same as that of the matrix \( L_\omega \) (see [8, pp. 15, Theorem 1.11]). Moreover,

\[
\tilde{L}_\omega = L_{0,\tilde{\omega}} L_{\tilde{\omega},0} = L_{\tilde{\omega}},
\]

where \( \tilde{\omega} = \omega(2 - \omega) \).

From Theorem 6 of [5], it follows that \( L_{\tilde{\omega}} \) is contractive relative to the vectorial norm \( p \) if

\[
\rho^* < 1 \quad \text{and} \quad 0 < \tilde{\omega} < \frac{2}{1 + \rho^*}.
\]

Hence, from the given hypothesis and the Theorem 3 in [5], we conclude that

\[
\rho(L_{\tilde{\omega}}) < 1 \quad \text{if} \quad 0 < \omega < \frac{2}{1 + \rho^*}.
\]

For \( \rho^* < 1 \), \( 0 < \tilde{\omega} < 2/(1 + \rho^*) \) if and only if \( 0 < \omega < 2 \), so \( \rho(L_{\tilde{\omega}}) < 1 \) if \( 0 < \omega < 2/(1 + \rho^*) \). More precisely,

\[
\rho(L_{\tilde{\omega}}) < 1 \quad \text{if} \quad 0 < \omega < 2.
\]

As a consequence of this theorem, we obtain a sufficient condition for the convergence of the SSOR iterative method when the associated \( A \) matrix is an \( H \)-matrix. If we use the vectorial norm \( p \) of dimension \( n \) defined over \( \mathbb{C}^n \) such that

\[
p(x) = |x|,
\]

then we have the following result of Alefeld [1].

**Corollary 2.2.** Let \( A \in \mathbb{C}^{n \times n} \) be a nonsingular \( H \)-matrix of the form (1.10). Then the SSOR method converges if \( 0 < \omega < 2 \).

In the next section, we obtain the sufficient conditions for the convergence of the USSOR iterative methods when the matrix \( A \) has the form (1.10). We also give necessary and sufficient conditions for \( H \)-matrices.
3. USSOR METHOD

Krishna [2] has shown the following result: Let \( A = [a_{i,j}] \in \mathbb{C}^{n,n}, \ n \geq 2. \) Then the following are equivalent:

(a) \( A \) is nonsingular \( H \)-matrix.
(b) For any \( B \in \Omega(A) \) and any \( \omega \) and \( \omega' \) in the interval \( 0 < \omega, \omega' \leq 1 \), the USSOR method converges.

In the following theorem we assume that \( A \) has the form (1.10) and prove the generalization of the above theorem which will extend the result of Alefeld [1] for the SSOR method.

Given any complex matrix \( A = [a_{i,j}] \in \mathbb{C}^{n,n} \), we define its comparison matrix \( M(A) = [\alpha_{i,j}] \in \mathbb{R}^{n,n} \) by

\[
\alpha_{i,i} = |a_{i,i}|, \quad 1 \leq i \leq n;
\]

\[
\alpha_{i,j} = -|a_{i,j}|, \quad i \neq j.
\]

The collection of all matrices equimodular to \( A \) is defined by

\[
\Omega(A) = \left\{ R = [b_{i,j}] \in \mathbb{C}^{n,n}, \ |b_{i,j}| = |a_{i,j}|, 1 \leq i, j \leq n \right\}.
\]

Clearly, \( M(A) \) is an element of \( \Omega(A) \). Any real matrix \( A = [a_{i,j}] \) with \( a_{i,j} < 0, \ i \neq j \) can be written as

\[
A = \tau I - C
\]
satisfying \( \tau > 0 \) and \( C \geq 0 \). Ostrowski [4] called such a matrix a nonsingular \( M \)-matrix if \( \tau > \rho(C) \). A complex matrix \( A \) is called a nonsingular \( H \)-matrix if the associated comparison matrix \( M(A) \) is a nonsingular \( M \)-matrix.

Now we prove the following lemma and the theorem on the USSOR method when \( A \) is a nonsingular \( H \)-matrix.

**Lemma 3.1.** Let \( \omega \) and \( \omega' \) be two real numbers. Then the following are equivalent:

(a) \( 0 < \omega + \omega' - \omega \omega' \leq 1. \)

(b) Exactly one of the following holds (possibly with \( \omega \) and \( \omega' \) interchanged).

(i) \( \omega = 1 \) and \( \omega' \) is arbitrary.
(ii) $\omega < 1$ and $\omega/(\omega - 1) < \omega' < 1$;
(iii) $\omega > 1$ and $1 \leq \omega' < \omega/(\omega - 1)$.

Proof.

$$0 < \omega + \omega' - \omega' \omega \leq 1 \Rightarrow 0 \leq (1 - \omega)(1 - \omega') < 1.$$  

The above is true if and only if one of the following holds ($\omega$ and $\omega'$ are interchangeable):

(i) $1 - \omega = 0$, or $\omega = 1$ and $\omega'$ is arbitrary.
(ii) $1 - \omega > 0$ and $0 < (1 - \omega') < 1/(1 - \omega)$; i.e., $\omega < 1$ and $\omega/(\omega - 1) < \omega' < 1$.
(iii) $1 - \omega < 0$ and $1/(1 - \omega) < (1 - \omega') < 0$; i.e., $\omega > 1$ and $1 \leq \omega' < \omega/(\omega - 1)$.

THEOREM 3.2. Let $A = [a_{i,j}] \in \mathbb{C}^{n \times n}$, where $A$ has the form (1.10). Then the following are equivalent:

(a) $A$ is nonsingular H-matrix.
(b) For all $B \in \Omega(A)$ and $0 < \omega + \omega' - \omega \omega' \leq 1$, we have $\rho(\mathcal{J}_{\omega, \omega}^B) < 1$.

Proof. From (1.7),

$$\mathcal{J}_{\omega, \omega'} = \mathcal{J}_{\omega, 0} \mathcal{J}_{\omega, \omega} \mathcal{J}_{0, \omega} \mathcal{J}_{0, \omega} \mathcal{J}_{0, \omega} \mathcal{J}_{0, \omega},$$

where $\mathcal{J}_{\omega, 0}$, $\mathcal{J}_{0, \omega}$, $\mathcal{J}_{0, \omega}$, and $\mathcal{J}_{0, \omega}$ are defined as in (2.2).

Let

$$\tilde{\mathcal{J}}_{\omega, \omega'} = \mathcal{J}_{0, \omega, \omega} \mathcal{J}_{0, \omega, \omega} \mathcal{J}_{0, \omega, \omega}, \quad (3.1)$$

Then $\mathcal{J}_{\omega, \omega'}$ and $\tilde{\mathcal{J}}_{\omega, \omega'}$ have the same eigenvalues (see [8, Theorem 2.1.11]).

If $\tilde{\omega} = \omega + \omega' - \omega \omega'$, then $\mathcal{J}_{\tilde{\omega}, 0} = \mathcal{J}_{\omega', 0} \mathcal{J}_{\omega, 0}$ and $\mathcal{J}_{0, \tilde{\omega}} = \mathcal{J}_{0, \omega} \mathcal{J}_{0, \omega}$. From [8, p. 256], we have

$$\tilde{\mathcal{J}}_{\omega, \omega'} = \mathcal{J}_{0, \omega} \mathcal{J}_{\omega, 0} \mathcal{J}_{0, \omega}, \quad (3.2)$$

where

$$\mathcal{J}_{\tilde{\omega}} = [I - (\omega + \omega' - \omega \omega')L]^{-1} \{(1 - \omega)(1 - \omega')I + (\omega + \omega' - \omega \omega')U\}.$$  

The eigenvalues of $\mathcal{J}_{\omega, \omega'}$, $\tilde{\mathcal{J}}_{\omega, \omega'}$, and $\mathcal{J}_{\tilde{\omega}}$ are the same.
We write $\mathcal{J}^A_{\omega, \omega'}$ instead of $\mathcal{J}_{\omega, \omega'}$ if we want to express the dependence of $\mathcal{J}_{\omega, \omega'}$ on $A$. The same notation applies to other iterative matrices. Thus we have

(i) $\rho(\mathcal{J}^A_{\omega, \omega'}) < 1$ for some $\tilde{\omega}$, $0 < \tilde{\omega} \leq 1$, if and only if $\rho(\mathcal{J}^A_{\omega, \omega'}) < 1$ for $\omega$ and $\omega'$ satisfying $0 < \tilde{\omega} \leq 1$, where $\tilde{\omega} = \omega + \omega' - \omega'$. 

To complete the proof, we use the following additional equivalent statements given by Varga [6]:

(ii) $A$ is a nonsingular $H$-matrix;
(iii) For all $B \in \Omega(A)$, we have $\rho(J^B) \leq \rho(J^\mathcal{J}^{(A)}) < 1$;
(iv) For all $B \in \Omega(A)$ and $0 < \tilde{\omega} < 2/[1 + \rho(|J^B|)]$, we have $\rho(\mathcal{J}^B_{\tilde{\omega}}) < 1$.

Now we give the proof of the theorem.

(a) $\Rightarrow$ (b): If $A$ is nonsingular $H$-matrix, it follows from (iv) above that

$$\rho(\mathcal{J}^B_{\tilde{\omega}}) < 1 \quad \text{for all} \quad 0 < \tilde{\omega} \leq 1 \quad \text{and} \quad B \in \Omega(A).$$

Thus from (i), we have $\rho(\mathcal{J}^B_{\omega, \omega'}) < 1$ for all $\omega$ and $\omega'$ satisfying $0 < \omega + \omega' - \omega' \leq 1$ and $B \in \Omega(A)$.

(b) $\Rightarrow$ (a): Let $\rho(\mathcal{J}^B_{\omega, \omega'}) < 1$ for all $\omega$ and $\omega'$ satisfying $0 < \omega + \omega' - \omega' \leq 1$ and $B = \mathcal{J}^{(A)} \in \Omega(A)$. It is obvious that $J^{\mathcal{J}^{(A)}} \geq 0$, so by the Stein-Rosenberg theorem $\rho(\mathcal{J}^{\mathcal{J}^{(A)}}) < 1$ if and only if $\rho(\mathcal{J}^{\mathcal{J}^{(A)}}) < 1$. If $\omega = 1$ (or $\omega' = 1$), then $\tilde{\omega} = 1$ and from (i) we have $\rho(\mathcal{J}^{\mathcal{J}^{(A)}}) < 1$, which implies that $\rho(\mathcal{J}^{\mathcal{J}^{(A)}}) < 1$. Hence, condition (iii) is satisfied, and so from (ii) $A$ is a nonsingular $H$-matrix.

We remark that if we use $\omega = \omega'$, then we get the result of Alefeld [1] for the SSOR method.

The next theorem is another generalization of Theorem 2 of Alefeld [1].

**Theorem 3.3.** Let $A = [a_{i,j}]$ be an $M$-matrix of the form (1.10). Then

$$\min_{0 < \omega + \omega' - \omega \omega' \leq 1} \rho(\mathcal{J}^{(A)}_{\omega, \omega'}) = \rho(\mathcal{J}^{(A)}{1}).$$

**Proof.** Let

$$P = \frac{D}{\omega + \omega' - \omega \omega'} [I - (\omega + \omega' - \omega \omega')L]$$
and
\[
Q = \frac{D}{\omega + \omega' - \omega \omega'} \left[ I - (\omega + \omega' - \omega \omega') I + (\omega + \omega' - \omega \omega') U \right]
= \frac{(1 - \omega)(1 - \omega')}{\omega + \omega' - \omega \omega'} D + DU.
\]

Then \( A = P - Q \) and \( \mathcal{L}_\omega = P^{-1}Q \).

From Lemma 3.2, it is easy to see that
\[
\min_{0 < \omega + \omega' - \omega \omega' < 1} \frac{(1 - \omega)(1 - \omega')}{\omega + \omega' - \omega \omega'} = \min_{0 < \omega + \omega' < 1} \left[ \frac{1}{\omega + \omega' - \omega \omega'} - 1 \right]
\]
occurs when either \( \omega = 1 \) or \( \omega' = 1 \), in which case \( \bar{\omega} = 1 \). Since \( \rho(\mathcal{S}_\omega, \omega') = \rho(\mathcal{L}_\bar{\omega}) \), the result follows from Theorem 3.15 in [7].

We remark that if we let \( \omega = \omega' \) in the above theorem, we get the result obtained by Alefeld [1] for the SSOR method.

Finally, using vectorial norms, we give sufficient conditions for the convergence of the USSOR method when \( A \) has the form (1.10).

**Theorem 3.4.** Let \( A \in \mathbb{C}^{n,n} \) be a nonsingular matrix of the form (1.10). Let \( \rho \) be a vectorial regular norm of dimension \( k \) defined over \( \mathbb{C}^n \) and \( M \) the vectorial norm of a matrix generated by \( \rho \).

If
\[
\rho^* = \rho(M(L) + M(U)) < 1,
\]
then the USSOR method is convergent if exactly one of the following conditions hold (here \( \omega \) and \( \omega' \) are interchangeable):
\[
\begin{align*}
\omega = 1 & \quad \text{and} \quad \omega' \text{ is arbitrary}, \quad (3.3) \\
\omega < 1 & \quad \text{and} \quad \frac{\omega}{\omega - 1} < \omega' < 1 + \frac{\alpha}{1 - \omega}, \quad (3.4) \\
\omega > 1 & \quad \text{and} \quad 1 + \frac{\alpha}{1 - \omega} < \omega' < \frac{\omega}{\omega - 1}, \quad (3.5)
\end{align*}
\]
where \( \alpha = (1 - \rho^*)/(1 + \rho^*) \).
Proof. As given in Theorem 3.2, we know that the eigenvalues of \( L_{\omega,\omega'} \) are the same as those of \( L_{\tilde{\omega}} \), where \( \tilde{\omega} = \omega + \omega' - \omega \omega' \). The matrix \( L_{\tilde{\omega}} \) is contractive relative to the vectorial norm \( \rho \) if \( \rho^* < 1 \) and \( 0 < \tilde{\omega} < 2/(1+\rho^*) \) (see [5, Theorem 6]). Thus, from Theorem 3 in [5], it follows that \( \rho(L_{\tilde{\omega}}) < 1 \) if \( 0 < \tilde{\omega} < 2/(1+\rho^*) \).

It can be shown by a simple generalization of Lemma 3.2 that \( 0 < \tilde{\omega} < 2/(1+\rho^*) \) if and only if exactly one of the following conditions holds:

(i) \( \omega = 1 \) and \( \omega' \) is arbitrary;
(ii) \( \omega < 1 \) and \( \omega/(\omega - 1) < \omega' < 1 + \alpha(1 - \omega) \);
(iii) \( \omega > 1 \) and \( 1 + \alpha/(1 - \omega) < \omega' < \omega/(\omega - 1) \).

Hence \( \rho(L_{\omega,\omega'}) < 1 \) if one of the conditions (3.3)–(3.5) holds.

As a consequence of this theorem, we have the following corollary.

Corollary 3.5. If \( A \in C^{n \times n} \) is a nonsingular \( H \)-matrix of the form (1.10), then the USSOR method is convergent if exactly one of the conditions (3.3)–(3.5) holds.

We remark that the intervals of convergence of the previous corollary contain the intervals obtained by Krishna in [2]. However in [2], the matrix \( A \) need not necessarily be of the form (1.10).

REFERENCES


Received 19 August 1986; final manuscript accepted 8 October 1987