A formula for the braid index of links

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ABSTRACT

Morton and Franks–Williams independently gave a lower bound for the braid index $b(L)$ of a link $L$ in $S^3$ in terms of the $v$-span of the Homfly-pt polynomial $P_L(v, z)$ of $L$: $\frac{1}{2} \text{span}_v P_L(v, z) + 1 \leq b(L)$. Up to now, many classes of knots and links satisfying the equality of this Morton–Franks–Williams’s inequality have been founded. In this paper, we give a new such a class $K$ of knots and links and make an explicit formula for determining the braid index of knots and links that belong to the class $K$. This gives simultaneously a new class of knots and links satisfying the Jones conjecture which says that the algebraic crossing number in a minimal braid representation is a link invariant. We also give an algorithm to find a minimal braid representative for a given knot or link in $K$.

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1. Introduction

An $n$-string braid is a set of $n$ arcs embedded in $D^2 \times I$ such that each 2-disc $D^2 \times \{x\}$, $x \in I$, meets the $n$ arcs in exactly $n$ points, where $n \geq 1$ and $I = [0, 1]$. A closed $n$-string braid is a set of $n$ arcs embedded in $D^2 \times S^1$ such that each disk $D^2 \times \{x\}$, $x \in S^1$, meets the $n$ arcs in exactly $n$ points. The set of all $n$-string braids forms a group with concatenation product. Alexander [1] showed that every link in $S^3$ can be represented as a closed $n$-string braid. The braid index $b(L)$ of a link $L$ is the smallest positive integer $n$ such that $L$ can be represented as a closed $n$-string braid. In [26], Yamada gave an algorithm for transforming a given knot or link into a closed braid and Vogel [25] improved Yamada’s algorithm later. This algorithm gives that the minimum number of Seifert circles in any diagram of a knot or link $L$ is equal to the braid index of $L$. It is an open problem to determine the minimum number of Seifert circles among all diagrams for a given knot or link. This is equivalent to determine the braid index among all closed braid representatives of a given knot or link. One general result related to this problem is MFW inequality. More precisely, Franks and Williams [6] and Morton [16] gave independently a lower bound for the braid index $b(L)$ of a link $L$ in terms of the $v$-span of the Homfly-pt polynomial $P_L(v, z)$ of $L$ as follows

$$\frac{1}{2} \text{span}_v P_L(v, z) + 1 \leq b(L).$$

(1.1)

This inequality (1.1) is sometimes called the Morton–Franks–Williams inequality or the MFW inequality for short. Up to now, there have been founded many classes of knots and links for which the MFW inequality is sharp, that is, the MFW inequality (1.1) detects the braid index of knots and links in the class. Also, there is an obstruction of sharpness of the MFW inequality.

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In 1987, Franks and Williams [6] conjectured that for any closed positive braid, the MFW inequality is sharp and showed that the conjecture is true for torus links and closed positive braids with a full twist. Morton and Short [17] gave a counter example, a 2-cable of the trefoil knot, for this Franks–Williams conjecture. In [5], Elrifai has classified all 3-braids for which the MFW inequality is not sharp. In 1991, Murasugi [18] conjectured that for any alternating links, the MFW inequality is sharp and proved that this conjecture is true for 2-bridge links and fibered alternating links. In [19], Murasugi and Przytycki found a counter example for this conjecture. In 2004, Nakamura [21] showed that the MFW inequality is sharp for a certain family of closed positive braids and gave an infinite family of prime closed positive braids for which the MFW inequality is not sharp.

Furthermore, the MFW inequality is closely related to study the well-known conjecture given by Jones in [9], which says that the algebraic crossing number in a minimal braid representation is a link invariant. It is known that if a knot or link satisfies the equality of the MFW inequality (1.1), then the knot or link also satisfies the Jones conjecture. In 2006, Kawamuro [10] showed that there are infinitely many examples of knots and links for which the MFW inequality is not sharp but the Jones conjecture is still true and proved that if the Jones conjecture is true for K and K′, then it is also true for (p, q)-cable of K and for the connected sum of K and K′. In [22], Stoimenow discussed the MFW inequality and a minimal braid representations focused on positive knots and braids. Quit recently, Kawamuro [11] constructed knots for which the new Khovanov–Rozansky–Morton–Franks–Williams (KR-MFW) inequality gives a sharp bound for its braid index; however, the MFW inequality fails to do so.

The purpose of this paper is to introduce a new class K of knots and links in S^3 for which the MFW inequality is sharp and give an explicit formula for determining the braid index of knots and links that belong to the class K, in terms of some integers which represent knots and links in K. This gives simultaneously a new class of knots and links satisfying the Jones conjecture mentioned above. The main techniques we use here are a special representation of knots and links in S^3 which allows an integral matrix parametrization of knots and links introduced by the authors in [15] and Murasugi and Przytycki’s theory of the index of a graph with applications to knot theory developed in [19]. Murasugi and Przytycki [19] improved the Morton–Franks–Williams’s inequality (cf. Theorem 5) for the upper (resp. lower) bound for the maximal (resp. minimal) \( \nu \)-degrees of \( P_{1}(v, z) \) by using the index of the Seifert graph \( \Gamma(D) \) associated with a diagram \( D \) of a link \( L \), counting the maximal number of independent edges in \( \Gamma(D) \) with positive (resp. negative) sign. They also showed that if \( D \) is a homogeneous diagram of \( L \) and the bounds are equal to the corresponding degrees, then the braid index of \( L \) is equal to the number of Seifert circles of \( D \) minus the index of Seifert graph \( \Gamma(D) \) associated with \( D \) and thus the MFW inequality is sharp (cf. Theorem 6).

The rest of this paper is organized as follows. In Section 2, we review the index of a graph and then calculate the index of a pretzel graph, which is a special sort of graphs and doing an essential role throughout this paper (Theorem 4). In Section 3, we briefly remind the Murasugi and Przytycki’s theory for the relationship between the index of a Seifert graph for a link \( L \) and the \( \nu \)-degree of the Homfly-pt polynomial \( P_{1}(v, z) \) and applications from [19] for our convenience and then give four lemmas (Lemmas 8–11). In Section 4, we first recall a representation of knots and links by integral matrices from [15] and then introduce a new class \( K \) of knots and links for which the MFW inequality is sharp and give an explicit formula for the braid index of a knot or link \( L \in K \) (Theorem 12). We also discuss some speculations for concerning the Jones conjecture (Theorem 13) and the braid index of certain periodic links with rational quotients (Corollary 14). In addition, we give examples that distinguish our class \( K \) from previously known classes as mentioned above. In the final Section 5, we give an algorithm to find a minimal braid representative for a given knot or link that belongs to the class \( K \).

2. Index of a pretzel graph

Let \( G \) be a graph. Let \( V(G) \) and \( E(G) \) be the sets of the vertices and edges of \( G \), respectively. \( G \) is called a signed graph if it is a graph equipped with a sign function \( f_G : E(G) \to \{-1, +1\} \). G is said to be separable if there are two subgraphs \( H \) and \( K \) such that \( G = H \cup K \) and \( H \cap K = \{v_0\} \), where both \( H \) and \( K \) have at least one edge and \( v_0 \) is a vertex. Otherwise, \( G \) is said to be non-separable. A block of \( G \) is a maximal non-separable connected subgraph of \( G \). If \( G_1, G_2, \ldots, G_k \) are all blocks of \( G \), we write \( G = G_1 \ast G_2 \ast \cdots \ast G_k \) and \( G \) is called the block sum of \( G_1, G_2, \ldots, G_k \). For a connected subset \( X \) of \( G \), \( G/X \) is defined to be the graph obtained from \( G \) by identifying all points in \( X \) to one point. For \( v \in V(G) \), star \( v \) is the smallest subgraph containing \( v \) and all edges of \( G \) which are incident to \( v \) and the valence of \( v \) is the number of edges incident to \( v \). For \( e \in E(G) \) with different ends \( v_1 \) and \( v_2 \), \( e \) is called a singular edge of \( G \) if there is no other edges between \( v_1 \) and \( v_2 \). A subgraph \( C \) of \( G \) is called a cycle if each vertex of \( C \) has an even valence. A cycle \( C \) of \( G \) is said to be simple if the valence of each vertex of \( C \) is 2. \( G \) is said to be bipartite if any cycle of \( G \) has an even length. For more details, we refer to [19].

Definition 1. ([19]) Let \( G \) be a graph.

(1) A family \( F = \{e_1, e_2, \ldots, e_k\} \) of edges of \( G \) is said to be independent if
   (i) all \( e_j \) (\( j = 1, 2, \ldots, k \)) are singular and
   (ii) there is an edge \( e_i \) in \( F \) and a vertex \( v \), one of the ends of \( e_i \), such that \( \{\phi(e_1), \ldots, \phi(e_{i-1}), \phi(e_{i+1}), \ldots, \phi(e_k)\} \) is an independent set of \( k - 1 \) edges in the graph \( G/\star v \), where \( \phi : G \to G/\star v \) is the collapsing map.

We define that the empty set of edges is independent.

(2) \( \text{ind}(G) \) is defined to be the maximal number of independent edges in \( G \) and called the index of \( G \).
Hence there is only one simple path from

in Fig. 1, in which there are exactly

Each path

\( p_i \)

in independent,

\( C \)

is determined by a

\( |e_1|, \ldots, |e_k| \)

contain

\( n_j \)

\( \text{cyclically independent} \)

\( \text{cyclically dependent} \)

\( G \)

\( \alpha(G) \)

\( (1) \) Let \( S = \{e_1, e_2, \ldots, e_n\} \) be a set of \( n \) distinct edges in a graph \( G \).

\( (1) \) \( S \) is said to be cyclically independent if no \( k \) edges in \( S \) \((1 \leq k \leq n)\) occur on a simple cycle of length at most \( 2k \). Otherwise \( S \) is called cyclically dependent.

\( (2) \) The cycle index of \( G \), denoted by \( \alpha(G) \), is defined to be the maximal number of cyclically independent edges of \( G \).

\( (3) \) If \( G \) is a graph, then \( \text{ind}(G) \leq \alpha(G) \).

\( (4) \) If two graphs \( G_1 \) and \( G_2 \) are disjoint, then \( \text{ind}(G_1 \cup G_2) = \text{ind}(G_1) + \text{ind}(G_2) \).

\( (3) \) If \( G \) is a connected bipartite graph and \( G \) consists of blocks \( G_1, G_2, \ldots, G_k \), then \( \text{ind}(G) = \text{ind}(G_1) + \text{ind}(G_2) + \cdots + \text{ind}(G_k) \).

\( (4) \) If \( G \) is bipartite, then \( \text{ind}(G) = \alpha(G) \).

For given nonzero integers \( a_1, a_2, \ldots, a_n \) \((n \geq 2)\), let \( G(a_1, a_2, \ldots, a_n) \) (briefly, \( G(a; n) \)) be the signed graph as described in Fig. 1, in which there are exactly \( n \) simple paths \( p_1, p_2, \ldots, p_n \) in \( G(a; n) \) from the vertex \( v_1 \) to another vertex \( v_2 \) and each path \( p_i \) consists of \( |a_i| \) edges. If \( a_i \) is positive (resp. negative), then the sign of each edge in \( p_i \) is positive (resp. negative). We call \( G(a; n) \) the pretzel graph determined by \( a_1, a_2, \ldots, a_n \). \( G(a; n) \) is called a pretzel graph with positive (resp. negative) pattern if all \( a_1, a_2, \ldots, a_n \) are positive (resp. negative). It is known that every Seifert graph \( \Gamma(D) \) of a link diagram \( D \) is bipartite and each cycle in \( \Gamma(D) \) has even length. This shows that if \( G(a; n) \) is a Seifert graph of a link diagram, then \( a_1, a_2, \ldots, a_n \) have the same parity.

**Theorem 4.** Let \( a_1, a_2, \ldots, a_n \) \((n \geq 2)\) be nonzero integers with the same parity and \( \eta = \frac{1 + (-1)^k}{2} \). If \( G(a; n) \) is the pretzel graph determined by \( a_1, a_2, \ldots, a_n \), then

\[
\text{ind}(G(a; n)) = \sum_{i=1}^{n} \left\lfloor \frac{|a_i| - 1}{2} \right\rfloor + \eta.
\]

Moreover, if \( G(a; n) \) is a pretzel graph with positive pattern, then

\[
\text{ind}_+(G(a; n)) = \sum_{i=1}^{n} \left\lfloor \frac{|a_i| - 1}{2} \right\rfloor + \eta \quad \text{and} \quad \text{ind}_-(G(a; n)) = 0,
\]

(2.2)

and if \( G(a; n) \) is a pretzel graph with negative pattern, then

\[
\text{ind}_-(G(a; n)) = \sum_{i=1}^{n} \left\lfloor \frac{-|a_i| - 1}{2} \right\rfloor + \eta \quad \text{and} \quad \text{ind}_+(G(a; n)) = 0.
\]

(2.3)

**Proof.** First we suppose that \( a_i = 2n_i \) for all \( i = 1, 2, \ldots, n \). Let \( S \) be a maximal set of cyclically independent edges of \( G(a; n) \). If \( C \) is a simple cycle in \( G(a; n) \), then \( C \) must be of length \( 2(|n_i| + |n_j|) \) for some \( i \) and \( j \). Since \( S \) is cyclically independent, \( C \) must contain at most \( |n_i| + |n_j| - 1 \) edges in \( S \). Since each simple cycle in \( G(a; n) \) consists of two simple paths from \( v_1 \) to \( v_2 \), the number of edges in \( S \) which belong to each simple path \( p_k \) from \( v_1 \) to \( v_2 \) is either \( |n_k| \) or \( |n_k| - 1 \). Hence there is only one simple path from \( v_1 \) to \( v_2 \) containing \( |n_i| \) edges in \( S \) and another simple paths from \( v_1 \) to \( v_2 \) contain \( |n_i| - 1 \) edges in \( S \). Thus
Next we suppose that \( a_i = 2n_i + 1 \) for all \( i = 1, 2, \ldots, n \). Let \( S \) be a maximal set of cyclically independent edges of \( G(a_i; n) \). If \( C \) is a simple cycle in \( G(a_i; n) \), then \( C \) must be of length \( 2n_i + 1 + 2n_j + 1 \) for some \( i \) and \( j \). Since \( S \) is cyclically independent, \( C \) must contain at most \((2n_i + 1 + 2n_j + 1)/2 - 1\) edges in \( S \). Since each simple cycle in \( G(a_i; n) \) consists of two simple paths from \( v_1 \) to \( v_2 \), the number of edges in \( S \) which belong to each simple path \( p_k \) from \( v_1 \) to \( v_2 \) is exactly \( \frac{2n_k + 1 - 1}{2} \). Hence

\[
\alpha(G(a_i; n)) = \sum_{i=1}^{n} \left( \frac{|n_i| - 1}{2} \right) + 1.
\]

Since \( a_1, a_2, \ldots, a_n \) have the same parity, \( G(a_i; n) \) is bipartite. By Theorem 3, \( \text{ind}(G(a_i; n)) = \alpha(G(a_i; n)) \). Hence

\[
\text{ind}(G(a_i; n)) = \sum_{i=1}^{n} \left\lfloor \frac{|a_i| - 1}{2} \right\rfloor + \eta.
\]

If each \( a_i \) is positive, then all edges in \( G(a_i; n) \) are positive. If each \( a_i \) is negative, then all edges in \( G(a_i; n) \) are negative. Hence we have (2.2) and (2.3). This completes the proof. \( \square \)

3. Index of a Seifert graph and Homfly-pt polynomial

The Homfly-pt polynomial \( P_L(v, z) \) (or \( P(L) \) for short) of an oriented link \( L \) in \( S^3 \) is defined by the following three axioms:

1. \( P_L(v, z) \) is invariant under ambient isotopy of \( L \).
2. If \( O \) is the trivial knot, then \( P_O(v, z) = 1 \).
3. If \( L_+ \), \( L_- \) and \( L_0 \) have diagrams \( D_+ \), \( D_- \) and \( D_0 \) which differ as shown in Fig. 2, then \( v^{-1}P_{L_+}(v, z) - vP_{L_-}(v, z) = zP_{L_0}(v, z) \).

It can be computed recursively by using a resolving tree, switching and smoothing crossings until the terminal nodes are labelled with trivial links. Note that

\[
P_{L_+}(v, z) = v^2P_{L_-}(v, z) + vzP_{L_0}(v, z),
\]
\[
P_{L_-}(v, z) = v^{-2}P_{L_+}(v, z) - v^{-1}zP_{L_0}(v, z).
\]

(3.4)

Set \( \delta = (v^{-1} - v)z^{-1} \). If \( L_1 \sqcup L_2 \) denotes the disjoint union of oriented links \( L_1 \) and \( L_2 \), then \( P_{L_1 \sqcup L_2}(v, z) = \delta P_{L_1}(v, z)P_{L_2}(v, z) \) [4].

Let \( D \) be an oriented link diagram. The writhe (or algebraic crossing number) \( w(D) \) of \( D \) is defined to be the sum of the signs of all crossings of \( D \). The Seifert circles of \( D \) are simple closed curves obtained from \( D \) by smoothing each crossing as described in Fig. 3. We denote by \( s(D) \) the number of the Seifert circles of \( D \). Let \( \Gamma(D) \) be the graph associated with \( D \) in which the vertices of \( \Gamma(D) \) correspond to the Seifert circles of \( D \) and the edges of \( G \) correspond to the crossings of \( D \). The ends of an edge \( e \) of \( \Gamma(D) \) correspond to Seifert circles connected by the crossing corresponding to \( e \). The sign of an edge of \( \Gamma(D) \) is the same as that of the corresponding crossing of \( D \). This signed graph \( \Gamma(D) \) is called a Seifert graph associated with \( D \).

For the Homfly-pt polynomial \( P_L(v, z) \) of a link \( L \), we denote the maximum degree in \( v \) of \( P_L(v, z) \) by \( \max \deg_v P_L(v, z) \) and the minimum degree in \( v \) of \( P_L(v, z) \) by \( \min \deg_v P_L(v, z) \).
Theorem 5. ([6,16]) Let $D$ be an oriented diagram of a link $L$. Then

$$w(D) - s(D) + 1 \leq \min_{v} \deg_{v} P_{L}(v, z) \leq \max_{v} \deg_{v} P_{L}(v, z) \leq w(D) + s(D) - 1.$$ 

Moreover, $\text{span}_{v} P_{L}(v, z) \leq 2b(L) - 2$. 

Theorem 6. ([19]) Let $D$ be an oriented diagram of a link $L$ and $\Gamma(D)$ the associated Seifert graph. Then

$$\max_{v} \deg_{v} P_{L}(v, z) \leq w(D) + s(D) - 1 - 2 \text{ind}_{+}(\Gamma(D)), \quad (3.5)$$

$$\min_{v} \deg_{v} P_{L}(v, z) \geq w(D) - s(D) + 1 + 2 \text{ind}_{-}(\Gamma(D)). \quad (3.6)$$

Cromwell [3] introduced the class of homogeneous links which contains all alternating links and positive links. Let $D$ be an oriented diagram of a link $L$. Suppose that the Seifert graph $\Gamma(D)$ associated with $D$ can be expressed as the block sum $\Gamma(D) = I_{1} \ast I_{2} \ast \cdots \ast I_{k}$. If each $I_{i}$ is either a positive or a negative graph, then $D$ is called a homogeneous diagram. If $L$ has a homogeneous diagram, then $L$ is called a homogeneous link.

Theorem 7. ([19]) If $D$ is a homogeneous diagram of a homogeneous link $L$ and the equalities of (3.5) and (3.6) hold, then $b(L) = s(D) - \text{ind}(\Gamma(D))$.

From now on, we are ready to state four lemmas.

Lemma 8. Let $D_{p,q}$ be the canonical diagram of a torus link $T_{p,q}$ as described in Fig. 4 (for $p = 4$ and $q = 5$). If $0 < p \leq q$, then

$$\max_{v} \deg_{v} P_{T_{p,q}} = w(D_{p,q}) + s(D_{p,q}) - 1 = (p - 1)(q + 1),$$

$$\min_{v} \deg_{v} P_{T_{p,q}} = w(D_{p,q}) - s(D_{p,q}) + 1 = (p - 1)(q - 1).$$

Proof. Since $w(D_{p,q}) = (p - 1)q$ and $s(D_{p,q}) = p$, we have $w(D_{p,q}) + s(D_{p,q}) - 1 = (p - 1)(q + 1)$ and $w(D_{p,q}) - s(D_{p,q}) + 1 = (p - 1)(q - 1)$. From [6, Corollary 2.4], the equalities follow. This completes the proof. 

For a given oriented link diagram $D$, in what follows, we fix the notations $\Phi_{+}(D) = w(D) + s(D) - 1 - 2 \text{ind}_{+}(\Gamma(D))$ and $\Phi_{-}(D) = w(D) - s(D) + 1 + 2 \text{ind}_{-}(\Gamma(D))$.

Lemma 9. Let $T(3)$, $T(1)$ and $T(\infty)$ be oriented links with the diagrams $D(3)$, $D(1)$ and $D(\infty)$, respectively, that are exactly the same except at a disk as described in Fig. 5. Suppose that the Seifert graph associated with $D(3)$ is a block sum of pretzel graphs with positive or negative pattern.

1. If $\max_{v} \deg_{v} P(T(1)) = \Phi_{+}(D(1))$, then $\max_{v} \deg_{v} P(T(3)) = \Phi_{+}(D(3))$.
2. If $\min_{v} \deg_{v} P(T(\infty)) = \Phi_{-}(D(\infty))$, then $\min_{v} \deg_{v} P(T(3)) = \Phi_{-}(D(3))$ and $\min_{v} \deg_{v} P(T(1)) = \Phi_{-}(D(1))$.
**Proof.** Let $\Gamma'(D(3))$ be the Seifert graph associated with $D(3)$. From hypothesis, we may assume that $\Gamma'(D(3)) = G_1 \ast G_2 \ast \cdots \ast G_k$, where each $G_i$ is a pretzel graph with positive or negative pattern. Let $e$ be an edge in $\Gamma'(D(3))$ corresponding to a crossing in the dotted circle of $D(3)$ in Fig. 5. Without loss of generality, we may assume that $G_1$ is the pretzel graph $G(a_1, a_2, \ldots, a_n)$ determined by $a_1, a_2, \ldots, a_n$ and $e$ is an edge in the path $p_1$ of $G_1$ corresponding to $a_1$. Then $a_1 \geq 3$. Since $e$ is a positive edge, $G(a_1, a_2, \ldots, a_n)$ is of positive pattern. It is easy to see that $G(a_1 - 2, a_2, \ldots, a_n)$ is also a pretzel graph with positive pattern and $\Gamma'(D(1)) = G(a_1 - 2, a_2, \ldots, a_n) \ast G_2 \ast \cdots \ast G_k$. By Theorem 3(3) and Theorem 4, we have

$$\text{ind}_+ \; \Gamma'(D(1)) = \text{ind}_+ \; \Gamma'(D(3)) - 1,$$

$$\text{ind}_- \; \Gamma'(D(1)) = \text{ind}_- \; \Gamma'(D(3)). \quad (3.7)$$

If $n \geq 3$, then $\Gamma'(D(\infty)) = G(a_2, \ldots, a_n) \ast H_1 \ast \cdots \ast H_{a_1 - 3} \ast G_2 \ast \cdots \ast G_k$. If $n = 2$, then $\Gamma'(D(\infty)) = H_1 \ast \cdots \ast I_1 \ast \cdots \ast I_{a_2} \ast G_2 \ast \cdots \ast G_k$. Here each $H_i$ or $I_j$ is a single edge graph. For example, see Fig. 6. Hence, by Theorem 3(3) and Theorem 4, we have

$$\text{ind}_+ \; \Gamma'(D(\infty)) \geq \text{ind}_+ \; \Gamma'(D(3)) - 1,$$

$$\text{ind}_- \; \Gamma'(D(\infty)) = \text{ind}_- \; \Gamma'(D(3)). \quad (3.8)$$

From (3.4), it follows that

$$P(T(3)) = v^2 P(T(1)) + vz P(T(\infty)). \quad (3.11)$$

Note that $w(D(\infty)) = w(D(3)) - 3$, $w(D(1)) = w(D(3)) - 2$, $s(D(\infty)) = s(D(3)) - 2$ and $s(D(1)) = s(D(3)) - 2$.

1. We suppose that $\text{max deg}_v \; P(T(1)) = \Phi_+(D(1))$. By (3.9), we get

$$\Phi_+(D(\infty)) = w(D(\infty)) + s(D(\infty)) - 1 - 2 \text{ind}_+ \; \Gamma'(D(\infty))$$

$$\leq w(D(3)) + s(D(3)) - 1 - 2 \text{ind}_+ \; \Gamma'(D(3)) - 3$$

$$= \Phi_+(D(3)) - 3. \quad (3.12)$$

From (3.5) and (3.12), we have

$$\text{max deg}_v \; P(T(\infty)) + 1 \leq \Phi_+(D(\infty)) + 1 < \Phi_+(D(3)). \quad (3.13)$$

By (3.7), we obtain

$$\Phi_+(D(1)) = w(D(1)) + s(D(1)) - 1 - 2 \text{ind}_+ \; \Gamma'(D(1))$$

$$= w(D(3)) + s(D(3)) - 1 - 2 \text{ind}_+ \; \Gamma'(D(3)) - 2$$

$$= \Phi_+(D(3)) - 2. \quad (3.14)$$

Since $\text{max deg}_v \; P(T(1)) = \Phi_+(D(1))$, it follows from (3.11), (3.13) and (3.14) that

$$\text{max deg}_v \; P(T(3)) = \text{max deg}_v \; P(T(1)) + 2 = \Phi_+(D(1)) + 2 = \Phi_+(D(3)). \quad (3.15)$$

2. We assume that $\text{min deg}_v \; P(T(\infty)) = \Phi_-(D(\infty))$. By (3.8), we have

$$\Phi_-(D(1)) = w(D(1)) - s(D(1)) + 1 + 2 \text{ind}_- \; \Gamma'(D(1))$$

$$= w(D(3)) - s(D(3)) + 1 + 2 \text{ind}_- \; \Gamma'(D(3))$$

$$= \Phi_-(D(3)). \quad (3.16)$$

From (3.6) and (3.15), we have

$$\text{min deg}_v \; P(T(1)) + 2 \geq \Phi_-(D(1)) + 2 > \Phi_-(D(3)). \quad (3.17)$$
By (3.10), it follows that
\[
\Phi_-(D(\infty)) = w(D(\infty)) - s(D(\infty)) + 1 + 2\text{ind}_- \Gamma'(D(\infty))
\]
\[
= w(D(3)) - s(D(3)) + 1 + 2\text{ind}_- \Gamma'(D(3)) - 1
\]
\[
= \Phi_-(D(3)) - 1.
\]
(3.17)

Since \(\min\text{deg}_v P(T(\infty)) = \Phi_-(D(\infty))\), we have from (3.11), (3.16) and (3.17) that
\[
\min\text{deg}_v P(T(3)) = \min\text{deg}_v P(T(\infty)) + 1 = \Phi_-(D(\infty)) + 1 = \Phi_-(D(3)).
\]

We also observe that
\[
\text{ind}_- \Gamma'(D(-1)) = \text{ind}_- \Gamma'(D(1)) + 1, \quad \text{(3.18)}
\]
\[
\text{ind}_- \Gamma'(D(\infty)) = \text{ind}_- \Gamma'(D(1)). \quad \text{(3.19)}
\]

From (3.4), we have
\[
P(T(1)) = v^2 P(T(-1)) + vz P(T(\infty)). \quad \text{(3.20)}
\]

Note that \(w(D(\infty)) = w(D(1)) - 1\), \(w(D(-1)) = w(D(1)) - 2\) and \(s(D(\infty)) = s(D(1)) = s(D(-1))\). By (3.18), it follows that
\[
\Phi_-(D(-1)) = w(D(-1)) - s(D(-1)) + 1 + 2\text{ind}_- \Gamma'(D(-1))
\]
\[
= w(D(1)) - s(D(1)) + 1 + 2\text{ind}_- \Gamma'(D(1))
\]
\[
= \Phi_-(D(1)). \quad \text{(3.21)}
\]

From (3.6) and (3.21), we have
\[
\min\text{deg}_v P(T(-1)) + 2 \geq \Phi_-(D(-1)) + 2 > \Phi_-(D(1)). \quad \text{(3.22)}
\]

By (3.19), we obtain
\[
\Phi_-(D(\infty)) = w(D(\infty)) - s(D(\infty)) + 1 + 2\text{ind}_- \Gamma'(D(\infty))
\]
\[
= w(D(1)) - s(D(1)) + 1 + 2\text{ind}_- \Gamma'(D(1)) - 1
\]
\[
= \Phi_-(D(1)) - 1. \quad \text{(3.23)}
\]

Since \(\min\text{deg}_v P(T(\infty)) = \Phi_-(D(\infty))\), by (3.20), (3.22) and (3.23), we get
\[
\min\text{deg}_v P(T(1)) = \min\text{deg}_v P(T(\infty)) + 1 = \Phi_-(D(\infty)) + 1 = \Phi_-(D(1)).
\]

This completes the proof. \(\square\)

For an oriented diagram \(D\) of a link \(L\), let \(D^*\) be the diagram obtained from \(D\) by reversing all of its crossings. Then \(D^*\) is a diagram of the mirror image \(L^*\) of \(L\). Since \(w(D^*) = -w(D)\), \(s(D^*) = s(D)\), \(\text{ind}_+ \Gamma'(D^*) = \text{ind}_- \Gamma'(D)\) and \(\text{ind}_- \Gamma'(D^*) = \text{ind}_+ \Gamma'(D)\), we have
\[
\Phi_+(D^*) = -\Phi_-(D), \quad \Phi_-(D^*) = -\Phi_+(D). \quad \text{(3.24)}
\]

Since \(P_L(v, z) = P_L(-v^{-1}, z)\), we also have
\[
\max\text{deg}_v P_L(v, z) = -\min\text{deg}_v P_L(v, z),
\]
\[
\min\text{deg}_v P_L(v, z) = -\max\text{deg}_v P_L(v, z). \quad \text{(3.25)}
\]

**Lemma 10.** Let \(T(-3)\), \(T(-1)\) and \(T(\infty)\) be oriented links with the diagrams \(D(-3)\), \(D(-1)\) and \(D(\infty)\), respectively, that are exactly the same except at a disk as described in Fig. 5. Suppose that the Seifert graph associated with \(D(-3)\) is a block sum of pretzel graphs with positive or negative pattern.

1. If \(\max\text{deg}_v P(T(\infty)) = \Phi_+(D(\infty))\), then \(\max\text{deg}_v P(T(-3)) = \Phi_+(D(-3))\) and \(\max\text{deg}_v P(T(-1)) = \Phi_+(D(-1))\).
2. If \(\min\text{deg}_v P(T(-1)) = \Phi_-(D(-1))\), then \(\min\text{deg}_v P(T(-3)) = \Phi_-(D(-3))\).
and there exist three diagrams \( D \) and \( D' \) with all of its crossings. Then \( D(-3)^*, \) \( D(-1)^* \) and \( D(\infty)^* \) are diagrams of \( T(-3)^* \), \( T(-1)^* \) and \( T(\infty)^* \), respectively, and there exist three diagrams \( D'(3), D'(1) \) and \( D'(\infty) \) that are the same except at the disk, in which \( D(-3)^* = D'(3), \) \( D(-1)^* = D'(1) \) and \( D(\infty)^* = D'(\infty) \). Since the Seifert graph associated with \( D(-3) \) is a block sum of pretzel graphs with positive or negative pattern, the Seifert graph associated with \( D(-3)^* \) is also a block sum of pretzel graphs with negative or positive pattern. If \( \max \deg_v P(T(\infty)) = \Phi_+(D(\infty)) \), then it follows from (3.24) and (3.25) that \( \min \deg_v P(T'(\infty)) = -\max \deg_v P(T(\infty)^*) = -\Phi_+(D(\infty)) = \Phi_-(D(\infty)^*) = \Phi_-(D'(\infty)) \). By (3.24), (3.25) and Lemma 9, we have

\[
\max \deg_v P(T(-3)) = -\min \deg_v P(T(-3)^*) = -\min \deg_v P(T'(3)) = \Phi_-(D'(3)) = \Phi_+(D(-3)),
\]

\[
\max \deg_v P(T(-1)) = -\min \deg_v P(T(-1)^*) = -\min \deg_v P(T'(1)) = \Phi_-(D'(1)) = \Phi_+(D(-1)).
\]

By a similar argument, we also obtain that if \( \min \deg_v P(T(-1)) = \Phi_-(D(-1)), \) then \( \min \deg_v P(T(-3)) = \Phi_-(D(-3)) \). This completes the proof.

**Lemma 11.** For two oriented link diagrams \( D \) and \( D' \), let \( D \# D' \) and \( D_n \) be the oriented link diagrams as described in Fig. 7. Then

\[
\Phi_+(D \# D') = \Phi_+(D_n) \quad \text{and} \quad \Phi_-(D \# D') = \Phi_-(D_n).
\]

Moreover, \( \Phi_+(D \# D') = \Phi_+(D) + \Phi_+(D') \) and \( \Phi_-(D \# D') = \Phi_-(D) + \Phi_-(D') \).

**Proof.** Since \( \Gamma(D \# D') = \Gamma(D) * \Gamma(D') \), it follows from Theorem 3 that \( \ind_+(D \# D') = \ind_+(D) + \ind_+(D') \) and \( \ind_-(D \# D') = \ind_-(D) + \ind_-(D') \). Since \( w(D \# D') = w(D) + w(D') \) and \( s(D \# D') = s(D) + s(D') - 1 \), we get

\[
\Phi_+(D \# D') = \Phi_+(D) + \Phi_+(D') \quad \text{and} \quad \Phi_-(D \# D') = \Phi_-(D) + \Phi_-(D').
\]

If \( n > 0 \), then \( \ind_+(D_n) = \ind_+(D) + \ind_+(D') + n \) and \( \ind_-(D_n) = \ind_-(D) + \ind_-(D') \) because \( \Gamma(D_n) = \Gamma(D) * I_1 * \cdots * I_n \). For each edge \( I_i \), \( 1 < i < n \), where each \( I_i \) is a single edge graph. Since \( w(D_n) = w(D) + w(D') + n \) and \( s(D_n) = s(D) + s(D') + n - 1 \), we have

\[
\Phi_+(D_n) = w(D_n) + s(D_n) - 1 - 2\ind_+(D_n)
\]

\[
= (w(D) + w(D') + n) + (s(D) + s(D') + n - 1) - 1 - 2(\ind_+(D) + \ind_+(D') + n)
\]

\[
= \Phi_+(D) + \Phi_+(D') = \Phi_+(D \# D')
\]

and

\[
\Phi_-(D_n) = w(D_n) - s(D_n) + 1 + 2\ind_-(D_n)
\]

\[
= (w(D) + w(D') + n) - (s(D) + s(D') + n - 1) + 1 + 2(\ind_-(D) + \ind_-(D'))
\]

\[
= \Phi_-(D) + \Phi_-(D') = \Phi_-(D \# D').
\]

If \( n < 0 \), then \( \ind_+(D_n) = \ind_+(D) + \ind_+(D') \) and \( \ind_-(D_n) = \ind_-(D) + \ind_-(D') - n \) because \( \Gamma(D_n) = \Gamma(D) * I_1 * \cdots * I_n \). For each edge \( I_i \), \( 1 < i < n \), where each \( I_i \) is a single edge graph. Since \( w(D_n) = w(D) + w(D') + n \) and \( s(D_n) = s(D) + s(D') - n - 1 \), we also have that

\[
\Phi_+(D_n) = \Phi_+(D \# D') \quad \text{and} \quad \Phi_-(D_n) = \Phi_-(D \# D').
\]

This completes the proof. \( \Box \)
4. A formula for the braid index

We first recall a description of knots and links in $S^3$ in terms of integral matrices [15]. Let $m \geq 1$ and $n \geq 1$ be integers and let $A = (a_{ij})_{1 \leq i \leq m, 1 \leq j \leq n}$ be an $m \times n$ integral matrix with nonzero entries $a_{ij}$. We define $L_A$ to be a link in $S^3$ represented by a diagram, $D(A)$, as shown in Fig. 8(a) when $m$ is even and in Fig. 8(b) when $m$ is odd. In Fig. 8, each tangle labeled an integer $a_{ij}$ (1 \leq i \leq m, 1 \leq j \leq n) denotes a 2-tangle as shown in Fig. 9(a). If all $a_{ij}$ are odd integers, we can choose an orientation of $L_A$ by assigning an orientation of each tangle labeled $a_{ij}$ as shown in Fig. 9(b).

Let $n_1, n_2, \ldots, n_m$ be a finite sequence of nonzero integers and let $N_1 = \{n_1, n_2, \ldots, n_{i_1}\}$, $N_2 = \{n_{i_1+1}, n_{i_1+2}, \ldots, n_{i_1+i_2}\}$, \ldots, $N_r = \{n_{i_1+\cdots+i_{r-1}+1}, n_{i_1+\cdots+i_{r-1}+2}, \ldots, n_{i_1+\cdots+i_{r-1}+i_r}\}$ be the partition of $n_1, n_2, \ldots, n_m$ such that $i_1 + i_2 + \cdots + i_r = m$ and the signs of the integers in $N_j$ (j = 1, 2, \ldots, r) are the same and the signs of $n_{i_k}$ and $n_{i_{k+1}}$ (k = 1, 2, \ldots, r − 1) are distinct. Let $\epsilon(n_1, n_2, \ldots, n_m)$ be an integer defined by $\epsilon(n_1, n_2, \ldots, n_m) = \max(\{i_1, i_2, \ldots, i_r\})$, that is, the maximum number of consecutive integers in the sequence $n_1, n_2, \ldots, n_m$ with the same sign.

**Theorem 12.** Let $A = (a_{ij})$ be an $m \times n$ integral matrix (m \geq 1, n \geq 2) such that all $a_{ij}$ are odd integers, $\epsilon(a_1, a_2, \ldots, a_n) + 1 \leq n$ and $a_1, a_2, \ldots, a_n$ have the same sign for each i = 1, 2, \ldots, m. Let $L_A$ be the oriented link in $S^3$ represented by $D(A)$ as shown in Fig. 8 in which each tangle has an orientation as shown in Fig. 9(b). Then

$$\max \deg_P P_L(v, z) = w(D(A)) + s(D(A)) - 1 - 2 \ind_\Gamma'(D(A)).$$

$$\min \deg_P P_L(v, z) = w(D(A)) - s(D(A)) + 1 + 2 \ind_\Gamma'(D(A)).$$

Moreover, the braid index $b(L_A)$ of $L_A$ is given by

$$b(L_A) = \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}| - \frac{1}{2} + m + 1.$$

**Proof.** Let $\Gamma'(D(A))$ be the Seifert graph associated with $D(A)$. It is easy to see that $\Gamma'(D(A)) = G(a_1, a_2, \ldots, a_n) * \cdots * G(a_1^n, a_2^n, \ldots, a_n^n)$ and each $G(a_1, a_2, \ldots, a_n)$ (i = 1, 2, \ldots, m) is a pretzel graph with positive or negative pattern according as $a_i$ is positive or negative. Let $N_1 = \{a_1, a_2, \ldots, a_{i_1}\}$, $N_2 = \{a_{i_1+1}, a_{i_1+2}, \ldots, a_{i_1+i_2}\}$, \ldots, $N_r = \{a_{i_1+\cdots+i_{r-1}+1}, a_{i_1+\cdots+i_{r-1}+2}, \ldots, a_{i_1+\cdots+i_{r-1}+i_r}\}$ be the partition of $a_1, a_2, \ldots, a_n$ such that all integers in $N_j$ (j = 1, \ldots, r) have the same sign, and $a_{i_1}$ and $a_{i_1+k}$ have different signs (k = 1, \ldots, r − 1). Without loss of generality, we may assume that $a_1$ is positive. Let $L_A^+$ be the link with diagram $D^+(A) = D_1 \# D_{i_1+1} \# \cdots \# D_{i_1+i_r}$, where $D_1 \# D_2$ denotes the connected sum of $D_1$ and $D_2$, and $D_{i_1 \# b}$ denotes the canonical diagram of a torus link $T_{a,b}$ as described in Fig. 4. Here $s = r$ if $r$ is odd and $s = r − 1$ otherwise.
By Lemma 9(1), Lemma 10(1) and Lemma 11, we have max deg\_v \, P_{L_A}(v, z) = \Phi_+(D(A)) if max deg\_v \, P_{L_A}(v, z) = \Phi_+(D^+(A)). For example, see Fig. 10.

From now on, we claim that max deg\_v \, P_{L_A}(v, z) = \Phi_+(D(A)) if max deg\_v \, P_{L_A}(v, z) = \Phi_+(D^+(A)). Since \Gamma(D^+(A)) = \Gamma(D_{i_1+1,n}) \ast \Gamma(D_{i_2+1,n}) \ast \cdots \ast \Gamma(D_{i_r+1,n})$, it follows from Lemma 11 that \Phi_+(D^+(A)) = \Phi_+(D_{i_1+1,n}) + \Phi_+(D_{i_2+1,n}) + \cdots + \Phi_+(D_{i_r+1,n}). Since \sum_{j=1}^{n} a_i^j + 1 \leq n, it follows from Lemma 8 that

max deg\_v \, P(D(i_j + 1, n)) = \Phi_+(D(i_j + 1, n))

for each \( j = 1, 2, \ldots, r \). Hence we obtain

max deg\_v \, P_{L_A}(v, z) = \Phi_+(D(i_1 + 1, n)) + \cdots + \Phi_+(D(i_r + 1, n))

Let \( L_A^- \) be the link with diagram \( D^-(A) = D_{i_2+1,n}^* \# D_{i_4+1,n}^* \# \cdots \# D_{i_r+1,n}^* \), where each \( D_{i_j+1,n}^* \) means the mirror image of \( D_{i_j+1,n} \) and \( t = r - 1 \) if \( r \) is odd and \( t = r \) if \( r \) is even. By Lemma 9(2), Lemma 10(2) and Lemma 11, we have min deg\_v \, P_{L_A}(v, z) = \Phi_-(D(A)) if min deg\_v \, P_{L_A}(v, z) = \Phi_-(D^-(A)). Similarly we can prove that min deg\_v \, P_{L_A}(v, z) = \Phi_-(D^-(A)). On the other hand, it is easy to see that

\[ s(D(A)) = \sum_{i=1}^{m} \sum_{j=1}^{n} (|a_i^j| - 1) + m + 1. \]

By Theorem 4, we have

\[ \text{ind} \, \Gamma(D(A)) = \sum_{i=1}^{m} \sum_{j=1}^{n} \frac{|a_i^j| - 1}{2}. \]

Since \( \Gamma(D(A)) \) is a block sum of pretzel graphs with positive or negative pattern, \( D(A) \) is a homogeneous diagram. Since the equalities (3.5) and (3.6) in Theorem 6 hold, it follows from Theorem 7 that

\[ b(L_A) = s(D(A)) - \text{ind} \, \Gamma(D(A)) = \sum_{i=1}^{m} \sum_{j=1}^{n} \frac{|a_i^j| - 1}{2} + m + 1. \]

This completes the proof.  

In [9], Jones conjectured that the algebraic crossing number of a minimal braid representative for a knot or link is a link type invariant. It is known that the following link has a unique algebraic crossing number in a minimal braid representative: torus links, closed positive braids with a full twist, the Lorenz links (Franks and Williams [6]), 2-bridge links and alternating fibered links (Murasugi [18]), and links with braid index \( \leq 3 \) (Birman and Menasco [2]). It is well known that if the MFW inequality for a knot or link is sharp, then Jones’ conjecture is true.

Theorem 13. Under the same assumption as Theorem 12, let \( \beta(L_A) \) be a minimal braid representative for a knot or link \( L_A \) and let \( e(\beta(L_A)) \) denote the algebraic crossing number of \( \beta(L_A) \). Then \( e(\beta(L_A)) \) is an invariant of \( L_A \). Moreover, \( e(\beta(L_A)) \) is given by

\[ e(\beta(L_A)) = \sum_{i=1}^{m} \sum_{j=1}^{n} \frac{a_i^j + \epsilon_j^i}{2}, \quad \text{where} \, \epsilon_j^i = \frac{a_i^j}{|a_i^j|}. \]
Proof. By Theorem 12, we know that the MFW inequality for \( L_A \) is sharp. Hence the algebraic crossing number \( e(\beta(L_A)) \) in a minimal braid representation \( \beta(L_A) \) is an invariant of \( L_A \). Since \( b(L_A) = s(D(A)) - \text{ind} \Gamma'(D(A)) \), we get

\[
s(\beta(L_A)) = s(D(A)) - \text{ind} \Gamma'(D(A)).
\]

Since the MFW inequality for \( L_A \) is sharp, we have

\[
\max \deg_v P_{L_A}(v, z) = w(\beta(L_A)) + s(\beta(L_A)) - 1.
\]

Since \( \max \deg_v P_{L_A}(v, z) = w(D(A)) + s(D(A)) - 1 - 2 \text{ind}_+ \Gamma'(D(A)) \) and \( \text{ind}_+ \Gamma'(D(A)) = \text{ind}_+ \Gamma'(D(A)) + \text{ind}_- \Gamma'(D(A)) \), we obtain

\[
w(\beta(L_A)) = w(D(A)) - \text{ind}_+ \Gamma'(D(A)) + \text{ind}_- \Gamma'(D(A)).
\]

Let \( I_{m}^1 = \{1, 2, \ldots, m\} \), \( I_{m}^+ = \{ i \in I_{m} | a_j^1 > 0 \} \) and \( I_{m}^- = \{ i \in I | a_j^1 < 0 \} \). Note that \( w(D(A)) = \sum_{i \in I_m} \sum_{j=1}^{n} a_j^1 \). From Theorem 4, we have

\[
\text{ind}_+ \Gamma'(D(A)) = \sum_{i \in I_{m}^+} \sum_{j=1}^{n} \frac{a_j^1 - 1}{2} \quad \text{and} \quad \text{ind}_- \Gamma'(D(A)) = \sum_{i \in I_{m}^-} \sum_{j=1}^{n} \frac{-a_j^1 - 1}{2}.
\]

If \( e_j^1 \) is the sign of \( a_j^1 \), i.e., \( e_j^1 = \frac{a_j^1}{|a_j^1|} \), then

\[
e(\beta(L_A)) = w(\beta(L_A)) = \sum_{i \in I_{m}^+} \sum_{j=1}^{n} a_j^1 - \sum_{i \in I_{m}^-} \sum_{j=1}^{n} a_j^1 - \sum_{i \in I_{m}^+} \sum_{j=1}^{n} \frac{a_j^1 - 1}{2} + \sum_{i \in I_{m}^-} \sum_{j=1}^{n} \frac{-a_j^1 - 1}{2}
\]

\[
= \sum_{i \in I_{m}^+} \sum_{j=1}^{n} \frac{a_j^1 + 1}{2} + \sum_{i \in I_{m}^-} \sum_{j=1}^{n} \frac{a_j^1 - 1}{2} = \sum_{i \in I_{m}^+} \sum_{j=1}^{n} \frac{a_j^1 + e_j^1}{2}.
\]

This completes the proof. \( \square \)

Let \( A = (a_j^1) \) be an \( m \times n \) integral matrix with \( a_1^1 = a_2^1 = \cdots = a_m^1 \) for each \( i = 1, 2, \ldots, m \). Then \( L_A \) is an \( n \)-periodic link \( L^{(n)} \) in \( S^3 \) with rational quotient \( L = \frac{C}{[a_1^1, a_2^1, \ldots, a_m^1]} \), that is, the 2-bridge link with Conway normal form \( C(2, a_1^1, -2, a_2^1, 2, \ldots, a_m^1, (-1)^m) \). We shall refer to \([8,12–14]\) for more details.

Corollary 14. For given odd integers \( n_1, n_2, \ldots, n_r \), let \( L^{(p)} \) be a \( p \)-periodic link with rational quotient \( L = \frac{C}{[n_1, n_2, \ldots, n_r]} \). Suppose that \( \epsilon(n_1, n_2, \ldots, n_r) + 1 \leq p \). Then the braid index of \( L^{(p)} \) is given by

\[
b(L^{(p)}) = p \sum_{i=1}^{r} \frac{|n_i| - 1}{2} + r + 1.
\]

Proof. Let \( A = (a_j^1)_{1 \leq i \leq r, 1 \leq j \leq p} \) be an \( r \times p \) integral matrix with \( a_1^1 = a_2^1 = \cdots = a_p^1 = n_i \) \((i = 1, 2, \ldots, r)\). Then \( L_A \) is equivalent to the link \( L^{(p)} \). It is straightforward from Theorem 12 that

\[
b(L^{(p)}) = p \sum_{i=1}^{r} \frac{|n_i| - 1}{2} + r + 1.
\]

This completes the proof. \( \square \)

Example 15. Let \( A = \left( \begin{array}{ll} 1 & 1 \\ 1 & 2 \\ 2 & 2 \end{array} \right) \). Then \( L_A \) is an oriented knot with a diagram \( D(A) \) as shown in Fig. 11. (Note that \( L_A \) is the prime knot \( 9_{49} \) in the Rolfsen’s table, which is the 3-periodic knot with rational quotient \( C[1, 2] \).) Since \( w(D(A)) = 9 \), \( s(D(A)) = 6 \) and \( \text{ind}_+ \Gamma'(D(A)) = 1 \), it follows that \( \Phi_+ (D(A)) = w(D(A)) + s(D(A)) - 1 - 2 \text{ind}_+ \Gamma'(D(A)) = 9 + 6 - 1 - 2 = 12 \). Observe that \( P_{L_A}(v, z) = (2z^2 + z) v^4 + (4 + 6z^2 + 2z^4) v^6 - (3 + 3z^2) v^8 \). Since \( b(L_A) = 3 \) and \( \text{span}_P P_{L_A} (v, z) = 4 \), the knot \( L_A \) is an example for which the MFW inequality is sharp. But \( \max \deg_v P_{L_A} (v, z) = 8 < 12 = \Phi_+ (D(A)) \). Therefore, the condition that all \( a_j^1 \) are odd integers is essential in Theorem 12.

Let \( K \) denote the set of all knots and links in \( S^3 \) represented by integral matrices \( A \) satisfying the conditions of Theorem 12. The following Example 16 shows that the class \( K \) is distinguished from previously known classes mentioned in the introduction.
Example 16. Let \( A = (-1 \ 1 -1 -1) \). Then \( L_A \) is an oriented knot with a diagram \( D(A) \) as shown in Fig. 12, which is the 4-periodic knot with rational quotient \( \mathcal{C}[-1, 3]\). It is immediate from Corollary 14 that \( b(L_A) = 7 \). Observe that \( P_L(v, z) = (z^2 + z^2) v^{-2} + (2z^2 - 2z^2 + (2 + 3z^2 - 3z^2 - 3z^2) v^2 + (-4 - 10z^2 - 14z^2 - 6z^2) v^4 + (5 + 5z^2 - 5z^2 - 4z^2) v^6 + (-1 + 3z^2 + z^2 - 3z^2) v^8 + (-1 + z^2 + z^2) v^{10} \) and so \( \text{span}_v P_L(v, z) = 12 \). Hence the MFW inequality for \( L_A \) is sharp. Moreover, it follows from Theorem 13 that the algebraic crossing number of a minimal braid representative for \( L_A \) is equal to 4. Since \( D(A) \) is a reduced alternating diagram, \( L_A \) is not a torus knot. It turns out that any 2-bridge knot which is not a torus knot has period 2 and no other [7, Theorem 6.1]. This implies that \( L_A \) is not a 2-bridge knot. On the other hand, Nakamura [20] and Stoimenow [23] independently showed that any reduced alternating diagram of a positive alternating link is a positive diagram. Since \( D(A) \) is reduced alternating but not positive, \( L_A \) is not a positive knot and hence \( L_A \) is not a positive closed braid. The authors [13, Theorem 10] showed that any \( p \)-periodic link with rational quotient \( \mathcal{C}[n_1, n_2, \ldots, n_r] \) is fibered if and only if \( n_k = \pm 1 \) for all \( k = 1, 2, \ldots, r \). Since \( L_A \) is the 4-periodic knot with rational quotient \( \mathcal{C}[-1, 3] \), \( L_A \) is not fibered.

5. An algorithm for a minimal braid representative

In this section, we give an algorithm to find a minimal braid representative for a given knot or link \( L \) in the class \( \mathcal{K} \). Let \( A = (a'_{ij}) \) be an \( m \times n \) integral matrix satisfying the conditions of Theorem 12. Then we can obtain a minimal braid representative \( \beta(L_A) \) for the knot or link \( L_A \in \mathcal{K} \) represented by \( A \) as follows. Suppose that \( b(L_A) = s \) for some integer \( s \geq 2 \). Let \( \sigma(1), \sigma(2), \ldots, \sigma(s - 1) \) be the generators of the \( s \)-string braid group \( B_s \) as shown in Fig. 13.

For any integers \( i, j \) with \( 1 \leq i \leq m \) and \( 1 \leq j \leq n \), we define the integers \( \epsilon_i, b'_j \) and \( c'_j \) by

\[
\epsilon_i = \frac{a'_{1i}}{|a'_{1i}|}, \quad b'_j = \frac{|a'_{ij}| - 1}{2}, \quad c'_j = \sum_{l=1}^{i-1} \left( \sum_{k=1}^{n} b'_k + 1 \right) + \sum_{k=1}^{j-1} b'_k + 1, \quad c''_{i+1} = c'_{i+1} = 1.
\]

Now we define the braid words \( A_{i,j}, B_{i,j}, C_{i,j} \) and \( W_{i,j} \) in \( B_s \) by

\[
A_{i,j} = \prod_{k=\epsilon_i+1}^{\epsilon_i+1} \sigma(k)^{-\epsilon_i}, \quad B_{i,j} = \prod_{k=\epsilon_i}^{\epsilon_i-1} \sigma(k)^{-\epsilon_i},
\]

where \( \epsilon_i = \frac{a'_{1i}}{|a'_{1i}|} \).
Example 17. Let $A$ be the $2 \times 4$ integral matrix of Example 16 and let $L_A$ be the knot represented by $A$. Recall that $b(L_A) = 7$. It follows from (5.26) and (5.27) that

$$
\begin{align*}
    &\epsilon_1 = -1, \quad \epsilon_2 = 1, \\
    &b_1^1 = b_2^1 = b_3^1 = b_4^1 = 0, \quad b_2^2 = b_2^3 = b_3^2 = b_3^3 = 1, \\
    &c_1 = c_1^1 = c_1^2 = c_1^3 = 1, \quad c_2^2 = 2, \quad c_2^3 = 3, \quad c_3^3 = 4, \quad c_4^3 = 5, \quad c_5^3 = 7, \\
    &A_{1,1} = A_{1,2} = A_{1,3} = A_{1,4} = e, \quad B_{1,1} = B_{1,2} = B_{1,3} = B_{1,4} = e, \\
    &A_{2,1} = \sigma(4)^{-1}\sigma(5)^{-1}\sigma(6)^{-1}, \quad A_{2,2} = \sigma(5)^{-1}\sigma(6)^{-1}, \quad A_{2,3} = \sigma(6)^{-1}, \quad A_{2,4} = e, \\
    &B_{2,1} = e, \quad B_{2,2} = \sigma(2)^{-1}, \quad B_{2,3} = \sigma(2)^{-1}\sigma(3)^{-1}, \quad B_{2,4} = \sigma(2)^{-1}\sigma(3)^{-1}\sigma(4)^{-1}, \\
    &C_{1,1} = C_{1,2} = C_{1,3} = C_{1,4} = \sigma(1)^{-1}, \\
    &C_{2,1} = \sigma(2)\sigma(3), \quad C_{2,2} = \sigma(3)\sigma(4), \quad C_{2,3} = \sigma(4)\sigma(5), \quad C_{2,4} = \sigma(5)\sigma(6), \\
    &W_{1,1} = W_{1,2} = W_{1,3} = W_{1,4} = \sigma(1)^{-1}, \\
    &W_{2,1} = \sigma(6)\sigma(5)\sigma(4)\sigma(2)\sigma(3)^{-1}\sigma(4)^{-1}\sigma(5)^{-1}\sigma(6)^{-1}, \\
    &W_{2,2} = \sigma(6)\sigma(5)\sigma(2)^{-1}\sigma(3)\sigma(4)\sigma(2)\sigma(5)^{-1}\sigma(6)^{-1}, \\
    &W_{2,3} = \sigma(6)\sigma(2)^{-1}\sigma(3)^{-1}\sigma(4)^{-1}\sigma(5)\sigma(3)\sigma(2)\sigma(6)^{-1}.
\end{align*}
$$

Then, using isotopy, we can transform the diagram $D(A)$ for $L_A$ as shown in Fig. 8 into a minimal braid representative $\beta(L_A)$ with a braid word in $B$, given by

$$
\beta(L_A) = \begin{cases} 
    \prod_{j=1}^{p}(\prod_{k=1}^{p} W_{2k-1,1}^j \prod_{k=1}^{p} W_{2k,j}) & \text{if } m = 2p, \\
    \prod_{j=1}^{p}(\prod_{k=1}^{p} W_{2k-1,1}^j \prod_{k=1}^{p} W_{2k,j}^{-1}) & \text{if } m = 2p - 1.
\end{cases}
$$

(5.28)

Here Fig. 14 illustrates a way for deformation of each tangle $a^j_i$ ($= 5$) into a braid presentation and the number of dotted arcs means $b^j_i$ ($= 2$). Applying these deformations to the diagram $D(A)$, we obtain a braid representative $\beta(L_A)$ in (5.28). For example, see Fig. 15 in Example 17.
\[ W_{2,4} = \sigma(2)^{-1} \sigma(3)^{-1} \sigma(4)^{-1} \sigma(5) \sigma(6) \sigma(4) \sigma(3) \sigma(2), \]

where \( e \) denotes the 7-string trivial braid. By (5.28), we then obtain a minimal braid representative \( \beta(L_A) \) with a braid word in \( B_7 \) given by

\[
\beta(L_A) = (1)^{-1} \sigma(6) \sigma(5) \sigma(4) \sigma(2) \sigma(3) \sigma(4)^{-1} \sigma(5)^{-1} \sigma(6)^{-1} \times (1)^{-1} \sigma(6) \sigma(5) \sigma(2)^{-1} \sigma(3) \sigma(4) \sigma(2) \sigma(5)^{-1} \sigma(6)^{-1} \times (1)^{-1} \sigma(6) \sigma(2)^{-1} \sigma(3)^{-1} \sigma(4) \sigma(5) \sigma(3) \sigma(2) \sigma(6)^{-1} \times (1)^{-1} \sigma(2)^{-1} \sigma(3)^{-1} \sigma(4)^{-1} \sigma(5) \sigma(6) \sigma(4) \sigma(3) \sigma(2),
\]

whose braid diagram is as shown in Fig. 15.

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**References**