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# The complexity of the characterization of networks supporting shortest-path interval routing

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**Abstract**

Interval Routing is a routing method that was proposed in order to reduce the size of the routing tables by using intervals and was extensively studied and implemented. Some variants of the original method were also defined and studied. The question of characterizing networks which support *optimal* (i.e., shortest path) Interval Routing has been thoroughly investigated for each of the variants and under different models, with only partial answers, both positive and negative, given so far. In this paper, we study the characterization problem under the most basic model (the one unit cost), and with the most restrictive memory requirements (one interval per edge). We prove that this problem is NP-hard (even for the restricted class of graphs of diameter at most 3). Our result holds for all variants of Interval Routing. It significantly extends some related NP-hardness result, and implies that, unless P = NP, partial characterization results of some classes of networks which support shortest path Interval Routing, cannot be pushed further to lead to efficient characterizations for these classes. © 2002 Elsevier Science B.V. All rights reserved.

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## 1. Introduction

In a distributed network, where processors communicate by sending and receiving messages, a *routing scheme* is employed in order to determine the path that a message

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will traverse from its source to its destination. A classical routing method is to keep in each node a table with $n$ entries (where $n$ is the number of nodes in the network); the $j$th entry in the table determines the link on which to send a message destined to node $j$. This routing method is *optimal* in the sense that it is always possible to construct routing tables that will guarantee a delivery of every message on a shortest length path. However, it has a prominent drawback; the space required in each node is proportional to the size of the network, which makes it very costly for large networks.

During the last decade, strategies to reduce the amount of space required for routing have been thoroughly investigated. It was shown that there exists a trade-off between the length of the routing paths and the memory which is required for the routing (see, e.g., [17, 1]). Many researchers focused on the problem of designing space-efficient optimal routing schemes. A common approach was to label the nodes and links of the network in a way that encodes some of the information on the network topology, and then to route messages according to the labeling.

One of the most popular such methods is Interval Routing, which was introduced in [19, 20] and was implemented in the C104 Router Chip of the INMOS T9000 Transputer design [14]. (See [11] for a surveys on Interval Routing.) The basic idea of Interval Routing is to reduce the amount of memory needed in a node by encoding sets of destinations using intervals. An interval is a consecutive sequence of nodes which might wrap-around over the end of the name segment to its beginning. Under Interval Routing, every node is assigned a unique number from $\{0,\dots,n-1\}$, and each link is labeled by a set of intervals. The routing of a message is performed according to the labeling; a message is sent on the link labeled by an interval which contains its destination. Some variants of the original method were proposed by posing more constraints on the interval labels. In linear interval routing wrap-around intervals are not allowed. In strict interval routing (and strict linear interval routing) an interval on a link outgoing a node must not include the label of that node.

Though Interval Routing can be implemented on any network, for some networks we might have to use $\Theta(n)$ intervals on some of the links in order to achieve optimality [10]. The *compactness* of an interval routing scheme is the maximum size of a set of intervals which labels a link. Interval routing schemes with compactness $k$ are termed $k$-interval routing schemes (in short, $k$-IRS). The abbreviation $k$-LIRS, $k$-SIRS, and $k$-SLIRS, are used for the other variants (linear, strict and strict linear, respectively). Fundamental questions are to characterize for a constant integer $k$ the class of graphs for which there exists an interval routing scheme with compactness $k$ (for each of the variants), or in general, to determine the minimal compactness for which there exists an interval routing scheme for a given graph. These questions were investigated quite extensively, under various models. So far, only partial answers (both positive and negative) have been given.

The class of graphs which admit optimal interval routing schemes with compactness $k$ was termed $k$-$\mathcal{IRS}$ (respectively, $k$-$\mathcal{SIRS}$, $k$-$\mathcal{LIRS}$, $k$-$\mathcal{SLIRS}$ for the other variants). It was first shown that many familiar classes of graphs such as complete graphs, meshes, hypercubes [3], complete bipartite graphs [15], and proper interval graphs [6] are in the class 1-$\mathscr{LIRS}$. Other graphs such as trees, tori, and unit-circular arc graphs are in the class 1-$\mathscr{IRS}$ [19, 15, 7, 16].

These positive results correspond to the basic and familiar model termed *one unit cost* which assumes a uniform cost on all the links. Another common model (which is a generalization of the first one) is the *fixed link cost* under which every link of the network is assigned a positive number, termed *its cost*. Costs of links reflect attributes such as propagation delay and congestion. The length of a path in this model is the sum of the costs of its links (optimality of a labeling for a network depends on these costs).

Yet another model is the *dynamic link cost* under which a labeling of the nodes of a network is optimal if it enables optimal routing for every assignment of costs to the links of the network. Under the last model few characterization results were established. Bakker et al. [3] fully characterized graphs with dynamic link cost in the class 1-$\mathscr{LIRS}$. In [8] a characterization of graphs with dynamic link cost in the class 1-$\mathscr{SIRS}$ was shown. This result was extended [2] to non-strict schemes and was generalized in [4], where it was shown that for each $k \geqslant 1$, the set of connected graphs in the class $k$-$\mathscr{IRS}$ under the dynamic link cost model are closed under taking minors, which implies, using the classical results in [18], that there are linear time algorithms recognizing these sets, though there might be no constructive way to design these algorithms. Furthermore, the model of dynamic link cost is considered too restrictive. A network admits an optimal labeling only if it does so regardless of the costs of its links; therefore most of the graphs that admit an optimal labeling under the one unit cost or fixed link cost models do not fall into that category when dynamic link costs are assumed.

Under the one unit cost model, the following partial characterization was given in [16, 12] for graphs in the class $k$-$\mathscr{IRS}$: a necessary and sufficient condition for a network to belong to $k$-$\mathscr{IRS}$ is that each of its biconnected components belongs to this class. This result reduces the characterization question for interval routing to the class of biconnected graphs. In the same spirit, a characterization was given in [16] for the graphs which in the class 1-$\mathscr{SLIRS}$ in terms of its biconnected components. These characterization results hold also under the more general fixed link cost model. However, none of these results imply a polynomial time algorithm which recognizes the corresponding graphs. Last, in an attempt to characterize one-unit cost graphs in the class 1-$\mathscr{LIRS}$ some conditions were given in [4]. It was conjectured in [22] that the class 1-$\mathscr{SLIRS}$ is constructible in the sense that it can be well described using simple graph operators and gave some properties of graphs in this class.

On the negative side, it was shown [7] that the optimization problem of determining the minimal $k$ such that a given network with fixed link costs belongs to $k$-$\mathscr{IRS}$ (resp., $k$-$\mathscr{LIRS}$, etc.), is NP-hard. They also showed that the related problem of determining the minimal integer $K$ such that there is an optimal interval labeling which uses a total of at most $K$ intervals on all the links is NP-hard. Recently, it was proved in [9] that the problem of characterizing the class of graphs 2-$\mathscr{IRS}$ (and respectively, 2-$\mathscr{LIRS}$, 2-$\mathscr{SIRS}$ and 2-$\mathscr{SLIRS}$) is NP-Hard under the one unit cost model (and

| Model / Problem | One unit cost | Fixed link cost | Dynamic link cost |
|---|---|---|---|
| $G \in 1\text{-}\mathcal{IRS}$? | NPC this paper | NPC this paper | P [2] |
| $G \in 1\text{-}\mathcal{LIRS}$? | NPC this paper | NPC this paper | P [3] |
| $G \in 2\text{-}\mathcal{IRS}$? | NPC [9] | NPC [9] | P [4] |
| $G \in 2\text{-}\mathcal{LIRS}$? | NPC [9] | NPC [9] | P [4] |
| $G \in k\text{-}\mathcal{IRS}$? for any fixed $k > 2$ | ? | ? | P [4] |
| $G \in k\text{-}\mathcal{LIRS}$? for any fixed $k > 2$ | ? | ? | P [4] |
| Minimal $k$ s.t. $G \in k\text{-}\mathcal{IRS}$ | NPH this paper, [9] | NPH [7] | ? |
| Minimal $k$ s.t. $G \in k\text{-}\ \mathcal{LIRS}$ | NPH this paper, [9] | NPH [7] | ? |

Fig. 1. Complexity results under the various models.

hence under the fixed link cost model). In that paper the problem of characterizing the class $1\text{-}\mathcal{IRS}$ (and its variants) is mentioned as the main open problem in this area.

In this paper we study the characterization question under the most basic model—the one unit cost, and with the most restrictive memory requirements—one interval per link—and settle the above problem. Specifically, we prove that the characterization of graphs which admit optimal (strict or non-strict) interval labeling and optimal (strict or non-strict) linear interval labeling with compactness 1 in this model is NP-hard (these results are easily generalized to the fixed-cost model. Our results significantly extend the related NP-hardness results of [7], and imply that, unless $P = NP$, the results of [3, 16, 22] cannot be further pushed to lead to efficient characterizations of these classes of graphs. It is worth noting that, at least for the (strict or non-strict) $1\text{-}\mathcal{LIRS}$ with unit-cost case, our result could come as a surprise, since it was shown in [3] that this class of graphs is very poor; for example, among other restrictions, a graph in this class may not contain a sub-graph of minimum paths which is a cycle of length greater than four.

The complexity results under the various models are summarized in Fig. 1, where every entry refers to both the strict and non-strict versions of the problem. Following is a brief explanation of it. Entry "Minimal $k$ s.t. $G \in k\text{-}\mathcal{IRS}$" (and "minimal $k$ s.t. $G \in k\text{-}\mathcal{LIRS}$") refers to the optimization problem of determining the minimal $k$ for which a given graph admits an optimal labeling with compactness $k$. The entry "$G \in k\text{-}\mathcal{IRS}$? for any fixed $k > 2$" (and "$G \in k\text{-}\mathcal{LIRS}$? for ...") actually refers to a family of problems—the $k$th problem is to recognize the graphs in the class $k\text{-}\mathcal{IRS}$. To the best of our knowledge, these problems are open for all $k>2$. In the table NPH denotes an NP-hard optimization problem and NPC denotes an NP-complete problem.

The rest of the paper is organized as follows. In Section 2, we give a precise definition of the model and the characterization problems. In Section 3, we prove that two problems which we define in the sequel, the Acyclic Graph Orientation problem and the Constrained Order problem are NP-complete. We then use these results in Section 4, where we prove the NP-completeness of both the 1-$\mathscr{LIRS}$ and the 1-$\mathscr{SLIRS}$ problems under the one unit cost model (and hence under the fixed link cost model) and in Section 5, where we prove the same result for the 1-$\mathscr{IRS}$ and 1-$\mathscr{SIRS}$ problems. Implications of our results and directions for further research are discussed in Section 6.

## 2. Preliminaries, definitions and notations

### 2.1. The model

We consider two different models of a network; the *one unit cost* and the *fixed link cost*. Under both, the network is modeled as an undirected finite graph $G=(V,E)$, $|V|=n$, where the set of vertices represent the nodes of the network and the set of edges represent the bidirectional links. Graphs are connected, loopless and do not contain parallel edges. Under the fixed link cost model, every edge of the graph is associated with a positive number, which is its *cost*; under the one-unit cost model we assume that the cost of every edge is one unit. The length of the path in the graph under both models is the sum of costs of its edges (for the one unit cost model this is equal to the number of edges in it). The basic idea of Interval Routing is to label each vertex of the graph by a unique number from the set $N=\{0,\dots,n-1\}$, and then at each vertex $v$ to label each adjacent edge by a set of intervals over the set $N$, such that intervals assigned to edges emanating $v$ are mutually disjoint and their union covers the set $N$. The routing is performed according to the labeling; in a vertex $v$, a message will be sent on the (unique) edge whose label contains an interval which contains the destination of the message. The *compactness* of an interval routing scheme is the maximum number of intervals that label any edge. In this paper we study only interval routing schemes with compactness 1 (i.e., every edge in each direction is labeled with one interval). Formally we define an *interval* over $N$ as follows.

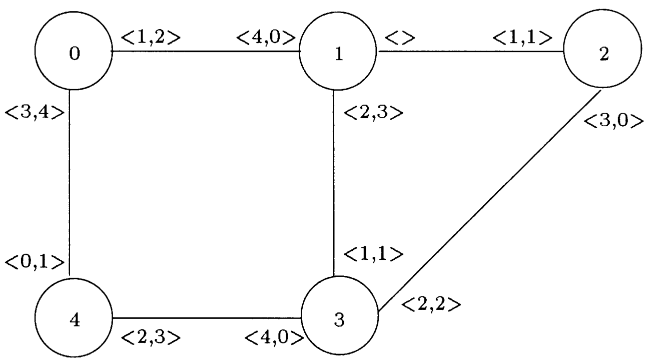
**Definition 1.** An *interval of N* is one of the following:
1. A *linear interval* $\langle p,q\rangle=\{p,p+1,\dots,q\}$, where $p,q\in N$ and $p\leqslant q$.
2. A *wrap-around interval* $\langle p,q\rangle=\{p,p+1,\dots,n-1,0,\dots,q\}$, where $p,q\in N$ and $p>q$.
3. The *null interval* $\langle\rangle$ is the empty set.

We say that a vertex $u\in V$ is *contained* in an interval $\langle p,q\rangle$ if $u\in\langle p,q\rangle$. An interval labeling scheme with compactness 1 is termed 1-interval labeling scheme and is formally defined as follows.

**Definition 2.** A 1-*interval labeling scheme* (in short, 1-ILS), $\mathscr{L}_G = (L, \{I_v\}_{v\in V})$, of a graph $G=(V,E)$, is defined by

1. A one-to-one function $L: V \to N$ that labels the vertices of $V$.
2. For every vertex $v$, an edge labeling function $I_v: E_v \to I$, where $E_v$ is the set of edges adjacent to $v$ and I is the set of intervals of $N$, that satisfies the following properties for every $v\in V$:
   *union property*: $N\backslash\{L(v)\} \subseteq \bigcup_{e\in E_v} I_v(e)$.
   *disjunction property*: For any two distinct edges $e_1, e_2$ in $E_v$, $I_v(e_1)\cap I_v(e_2)=\emptyset$.
   (In other words, the non-empty intervals associated with all the edges outgoing any vertex $v$ form a partition of $N$ or of $N - \{L(v)\}$.)

For simplicity, in the sequel we will not distinguish between a vertex $v$ and its label $L(v)$.



**Example 1.** The above Figure shows a graph together with a 1-ILS for it. A message with source 0 and destination 2 will traverse the path 0–1–3–2.

Few variants of interval routing are *linear interval routing* in which wrap-around intervals are not allowed, and *strict* interval routing (respectively, *strict* Linear Interval Routing) in which an interval on an edge outgoing a vertex may not contain the label of this vertex. Their definitions follows.

**Definition 3.** A 1-*linear-interval labeling scheme* (in short, 1-LILS), of a graph $G=(V,E)$, is a 1-interval labeling scheme in which for every $v\in V$ and for every $e\in E_v$, the interval $I_v(e)$ is either linear or null.

**Definition 4.** A 1 *strict interval labeling scheme* (in short, 1-SIRS) (resp. 1-SLIRS), of a graph $G=(V,E)$, is a 1-interval labeling scheme (respectively, 1-linear interval labeling scheme) in which for every $v\in V$ and for every $e\in E_v$, $L(v)\notin I_v(e)$.

By *a labeling scheme* we mean 1-ILS or any of its variants. Note that the path a message with source $u$ and destination $v$ traverses is completely determined by the labeling scheme of the graph. If the labeling is arbitrary then this path could contain a cycle; in other words, the message can cycle forever without reaching its destination. A *valid* labeling scheme is a labeling that guarantees that every message will eventually arrive at its destination. A valid labeling scheme induces, for every ordered pair of vertices $(u,v)$ in the graph, a simple path from $u$ to $v$. A valid 1-ILS is termed 1-IRS (interval routing scheme) and respectively, valid 1-LILS, valid 1-SILS and valid 1-SLILS are termed 1-LIRS, 1-SIRS and 1-SLIRS. An *optimal* labeling scheme is a labeling scheme that guarantees that for every ordered pair of vertices, the induced path is a shortest length path in the graph (that is, the labeling guarantees that every message will traverse a shortest length path).

We define four classes of graphs; 1-$\mathscr{IRS}$ contains all graphs for which there exists an optimal 1-IRS, and 1-$\mathscr{LIRS}$ contains all the graphs for which there exists an optimal 1-LIRS. (Clearly, 1-$\mathscr{LIRS} \subseteq$ 1-$\mathscr{IRS}$.) Correspondingly, 1-$\mathscr{SIRS}$ and 1-$\mathscr{SLIRS}$ are the classes of graphs which admit optimal 1-SIRS and optimal 1-SLIRS. (Clearly, 1-$\mathscr{SIRS} \subseteq$ 1-$\mathscr{IRS}$ and 1-$\mathscr{SLIRS} \subseteq$ 1-$\mathscr{LIRS}$.)

The problems of determining, for a given graph, whether it belongs to either of these classes, are formally defined as follows.

*The* 1-$\mathscr{LIRS}$ *problem*

*Input*: A graph $G$.

*Question*: does $G \in$ 1-$\mathscr{LIRS}$?

and

*The* 1-$\mathscr{IRS}$ *problem*

*Input*: A graph $G$.

*Question*: does $G \in$ 1-$\mathscr{IRS}$?

The problems 1-$\mathscr{SIRS}$ *and* 1-$\mathscr{SLIRS}$ are defined similarly. Also, if the model assumed is the fixed link cost then the input to each of the above four problems is a graph $G=(V,E)$ and a cost function $cost: E \to R^+$, which assigns a positive number to every edge in the graph. Clearly, all of these problems are in the class NP; given a graph (with or without a cost function) and any labeling for it, it can be verified in polynomial time whether the labeling is optimal for the graph.

In this paper we prove the following two theorems.

**Theorem 1.** *The* 1-$\mathscr{LIRS}$ *and the* 1-$\mathscr{SLIRS}$ *problems are NP-complete under the one unit cost model* (*and hence under the fixed link cost model*).

**Theorem 2.** *The* 1-$\mathscr{IRS}$ *and the* 1-$\mathscr{SIRS}$ *problems are NP-complete under the one unit cost model* (*and hence under the fixed link cost model*).

We will prove Theorems 1 and 2 by a sequence of polynomial transformations from the 3-SAT problem. In the next subsection we will define two problems that will be used in the proofs.

## 2.2. The acyclic graph orientation and the constrained order problems

Informally, the input to the acyclic graph orientation problem is a directed graph and a partition of its edges to equivalence classes. The problem is to determine whether there exists a subset of the equivalence classes such that switching the directions of the edges in them will result in a directed acyclic graph (DAG). Formally,

*The acyclic graph orientation* ($AGO$) *problem*

*Input*: A directed graph $G=(V,E)$ and a partition of its edges to equivalence classes $\{E_i\}_{0\leqslant i\leqslant t}$, (the sets $E_i$ are pairwise disjoint and $E=\bigcup_{0\leqslant i\leqslant t} E_i$).

*Question*: Is there a subset of indices $J\subseteq\{0,\ldots,t\}$ such that the directed graph $G_J=(V,E_J)$, which is obtained by reversing the directions of all the edges in $\bigcup_{j\in J} E_j$, is a DAG?

In the Constrained Order problem, the question is to determine whether there exists a partial order $<_o$ over a given set of items $U$ that satisfies a given set of *order constraints* (in short *constraints*).[1] A constraint is an ordered pair of sets of items of the form $(Set_1 \mid Set_2)$. Such a constraint is satisfied by a partial order $<_o$ iff either $Set_1 <_o Set_2$ or $Set_2 <_o Set_1$ ($A <_o B$ means that $u <_o v$ for every $u\in A$ and every $v\in B$). Formally,

*The constrained order* ($CO$) *problem*

*Input*: A set of constraints $Cons=\{Cons_i\}_{1\leqslant i\leqslant t}$, over a set of items $U$, where the constraint $cons_i$ is of the form $Cons_i=(Set_i^1 \mid Set_i^2)$, $Set_i^1, Set_i^2\subseteq U$, $Set_i^1, Set_i^2\neq\emptyset$, and $Set_i^1\cap Set_i^2=\emptyset$, for every $i=1,\ldots,t$.

*Question*: Is there a partial order $<_o$ over the set $U$ such that for every constraint $Cons_i$, $i=1,\ldots,t$, either $Set_i^1 <_o Set_i^2$ or $Set_i^2 <_o Set_i^1$.

Note that both the AGO problem and the CO problem are in NP. In the course of proving Theorems 1 and 2 we prove that the AGO problem and the CO problem are NP-complete. Specifically, the proof of Theorem 1 is by a sequence of polynomial transformations. The first transformation is from the well-known NP-complete 3-SAT problem (see, e.g., [13]) to the AGO problem. The second transformation is from the AGO problem to the CO problem. The last transformation is from the CO problem to the 1-$\mathscr{LIRS}$ problem. In fact, the last transformation will imply that both 1-$\mathscr{LIRS}$ and 1-$\mathscr{SLIRS}$ are NP-complete under both the fixed cost model and the one unit cost model.

In order to prove Theorem 2, we show a polynomial transformation from the CO problem. Again, the same transformation will imply the NP-completeness of both variations and under both models. The structure of the proofs of Theorems 1 and 2 is illustrated in Fig. 2.

[1] In this paper, a partial order is a transitive, asymmetric relation.

$$\begin{tikzcd} & & & \text{1-}\mathcal{LIRS} \\ \text{3-SAT} \arrow[r, "1"] & \text{AGO} \arrow[r, "2"] & \text{CO} \arrow[ur, "3"] \arrow[dr, "4"'] & \\ & & & \text{1-}\mathcal{IRS} \end{tikzcd}$$

$\longrightarrow$ **denotes a polynomial transformation**

Fig. 2. The structure of the proofs.

## 3. The NP-completeness of the AGO and the CO problems

We prove that the AGO problem is NP-complete by a polynomial transformation from the 3-SAT problem. We then prove the NP-completeness of the CO problem by a polynomial transformation from the AGO problem.

### 3.1. A transformation from 3-SAT to AGO

An instance of the 3-SAT problem is a CNF formula $\varphi$ over a set of variables $Var = \{v_1, \dots, v_t\}$, that is, $\varphi = \bigwedge_{i=1,\dots,m} C_i$ where $C_i = (l_i^1 \vee l_i^2 \vee l_i^3)$ is a clause of three literals, the literal $l_i^j$ is either a variable $v_k$ or a negation of a variable $\neg v_k$ for some $v_k \in Var$. The question is whether there is a truth assignment to the variables in $Var$ which satisfies $\varphi$. This instance is transformed to an instance of the AGO problem, which consists of a directed graph $G = (V, E)$ and a partition of $E$ to equivalence classes $\{E_j\}_{j \in \{0,\dots,t\}}$ as follows.

The graph $G = (V, E)$ consists of $m$ components $H_1, \dots, H_m$, where $H_i$ corresponds to clause $C_i = (l_i^1 \vee l_i^2 \vee l_i^3)$. $H_i$ contains four directed edges $e_i, e_i^1, e_i^2, e_i^3$, whose underlying graph is a cycle; The orientations of the edges are determined as follows: the edge $e_i$, $i = 1, \dots, m$, is a *compass* edge which is always oriented counterclockwise. An edge $e_i^j$ $(1 \leqslant j \leqslant 3)$ is oriented clockwise if the literal $l_i^j$ is a variable $v_k$ and it is oriented counterclockwise if $l_i^j$ is a negation of a variable $\neg v_k$.
The partition of the edges to equivalence classes is as follows. $E_0 = \{e_i\}_{i=1,\dots,m}$ consists of all the compass edges. In addition, there is a class $E_k$ for each variable $v_k \in Var$, which consists of all edges $e_i^j$, where $l_i^j$ is either the variable $v_k$ or its negation $\neg v_k$.

Following is an example of the transformation.

**Example 2.** Fig. 3(a) depicts an example of the transformation from a CNF formula $\varphi$ to a directed graph and a set of equivalence classes on its set of edges. The vertices of $H_i$, $i = 1, 2$, are $a_i, b_i, c_i$ and $d_i$; The compass edge $e_i$ is the edge $(a_i, d_i)$. The edge $e_i^1$ (which correspond to the literal $l_i^1$) connects the vertices $a_i$ and $b_i$, similarly the edges $e_i^2$ and $e_i^3$ connect the vertices $b_i, c_i$ and $c_i, d_i$ respectively. The edge $e_1^1$, for example, is oriented clockwise (i.e., from $a_1$ to $b_1$) since the corresponding literal $l_1^1$ is a variable ($v_1$). Similarly, $e_1^2$ is oriented counterclockwise since the corresponding literal $l_1^2$

$\varphi = (v_1 \vee \neg v_2 \vee v_3) \wedge (\neg v_2 \vee \neg v_1 \vee \neg v_3)$

$V = \{a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2\}$
$E_0 = \{(a_1, d_1), (a_2, d_2)\}$
$E_1 = \{(a_1, b_1), (c_2, b_2)\}$
$E_2 = \{(c_1, b_1), (b_2, a_2)\}$
$E_3 = \{(c_1, d_1), (d_2, c_2)\}$

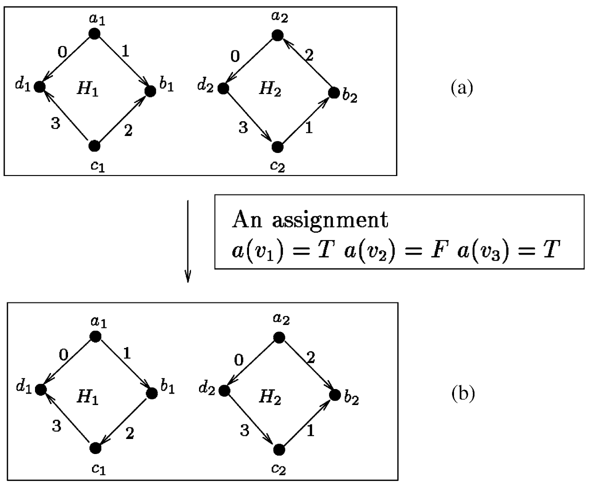


Fig. 3. An example of the transformation. (a) The directed graph $G$ and the equivalence classes of its edges, constructed by the transformation from the formula $\varphi$. (b) The legal orientation of $G$'s edges which corresponds to the truth-assignment $a$.

is a negation of a variable ($\neg v_2$). All other edges are oriented by similar rules. The numbers on the edges $(0,1,2,3)$ denote the equivalence class $E_i$, $(i=0,1,2,3)$ to which the edge belongs. For example, the literal $l_1^1$ is $v_1$ and the literal $l_2^2$ is $\neg v_1$ and therefore $E_1 = \{(a_1,b_1),(c_2,b_2)\}$.

In this transformation, we interpret a clockwise orientation of an edge $e_i^j$ as a truth assignment $a$ under which the truth value $\bar{a}(l_i^j)$ of the literal $l_i^j$ is $T$ and a counterclockwise orientation of this edge as an assignment $a$ such that $\bar{a}(l_i^j)=F$. Thus, the initial orientation of the edges in the graph $G$ corresponds to an assignment of the value $T$ to all the variables (therefore, a literal which is a variable is oriented clockwise and a literal which is a negation of a variable is oriented counterclockwise). Denote an orientation which is obtained by reversing all the edges in a set of equivalence classes $\{E_j\}_{j\in J}$ as *legal* orientation. Then the definition of the equivalence classes $E_j$ guarantees that in each legal orientation, all the edges which correspond to a variable $v_k$ have the same orientation, which is the opposite orientation of the edges which correspond to $\neg v_k$. The counterclockwise orientation of the compass edges $e_i$ guarantees that in the initial graph $G$ a cycle $H_i$ is a directed cycle iff clause $C_i$ contains only negated variables (thus, this clause is not satisfied by the initial assignment). For a set $J \subseteq \{1,\ldots,t\}$, the graph $G_J$ defined by reversing the edges in the classes $\{E_j\}_{j\in J}$ corresponds to a truth assignment which assigns the value $F$ to each variable in the set $\{v_j\}_{j\in J}$, and the value $T$ to all other variables.

**Example 3.** Fig. 3(b) is the graph $G_J$ where $J=\{2\}$. It corresponds to the assignment of the value $T$ to $v_1$ and $v_3$ and the value $F$ to $v_2$. Note that $\varphi$ is satisfied under this assignment and indeed the corresponding graph $G_J$ is a DAG.

We now prove the correctness of the transformation.

**Proposition 1.** *A CNF formula $\varphi=\bigwedge_{i=1,\dots,m} C_i$ over $Var=\{v_1,\dots,v_t\}$ has a satisfying truth assignment iff there exists a subset $J\subseteq\{0,\dots,t\}$ such that the directed graph $G_J=(V,E_J)$ is a DAG.*

**Proof.** Assume first that there is a truth-assignment $a: Var\to\{F,T\}$ which satisfies $\varphi$. We take $J$ to be the set of indices $J=\{k \mid a(v_k)=F\}$. By the discussion above, an edge $e_i^j$ in $G_J$ is oriented clockwise iff $\bar{a}(l_i^j)=T$. By the assumption that $a$ satisfies $\varphi$, each component $H_i$ in $G_J$ has at least one edge which is oriented clockwise. Since the compass edge of $H_i$ is oriented counterclockwise in $G_J$, we have that $G_J$ contains no directed cycle, hence it is acyclic, as required.

Assume now that there is a set of indices $J$ such that the graph $G_J=(V,E_J)$, is acyclic. We can assume without loss of generality that $0\notin J$ (otherwise, replace $J$ by $\bar{J}=\{0,1,\dots,t\}\backslash J$, noting that $G_{\bar{J}}$ is obtained by reversing all the edges in $G_J$, and hence it is also acyclic). Thus, for each component $H_i$ in $G_J$, the compass edge $e_i$ is directed counterclockwise. Now, consider the truth assignment $a$ that assigns $F$ to all variables $\{v_j\}_{j\in J}$ (and $T$ to the rest of the variables). By the definition of $G_J$, $\bar{a}(l_i^j)=T$ only if in the graph $G_J$, the edge $e_i^j$ is directed clockwise. Since the graph $G_J$ is acyclic, each component $H_i$ contains an edge which is oriented clockwise. This edge corresponds to a literal in clause $C_i$ with value $T$ under the assignment $a$, hence all clauses are satisfied by $a$, as claimed. □

## 3.2. A transformation from AGO to CO

Let a directed graph $G=(V,E)$ and a partition of its edges to equivalence classes $\{E_j\}_{j=0,\dots,t}$ be an instance to the AGO problem, where $V=\{v_1,\dots,v_n\}$. We transform it to a set of constraints $S$ over a set of items $U$ as follows.

The set $U$ is given by $U=\{u_1,\dots,u_n\}\cup\{l_j,r_j \mid 0\leqslant j\leqslant t\}$. The set of constraints is defined as follows. For every directed edge $(x,y)\in E$ there exists a unique $j$, $0\leqslant j\leqslant t$ such that the edge $(x,y)$ belongs to the equivalence class $E_j$. We add the constraint $(\{l_j,x\}\mid\{r_j,y\})$ to the set of constraints (i.e., the total number of constraints is $\sum_{j=0}^{t}|E_j|=|E|$). The transformation is demonstrated by the following example.

**Example 4.** Let the instance of the AGO be as constructed in Example 2 (Fig. 3(a)). The transformation will construct from it the following set of constraints over the set of items $U = \{a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2, l_0, r_0, l_1, r_1, l_2, r_2, l_3, r_3\}$.

$$
\begin{array}{l}
(\ \{l_0, a_1\} \mid \{r_0, d_1\}\ ),\\
(\ \{l_0, a_2\} \mid \{r_0, d_2\}\ ),\\
(\ \{l_1, a_1\} \mid \{r_1, b_1\}\ ),\\
(\ \{l_1, c_2\} \mid \{r_1, b_2\}\ ),\\
(\ \{l_2, c_1\} \mid \{r_2, b_1\}\ ),\\
(\ \{l_2, b_2\} \mid \{r_2, a_2\}\ ),\\
(\ \{l_3, c_1\} \mid \{r_3, d_1\}\ ),\\
(\ \{l_3, d_2\} \mid \{r_3, c_2\}\ ).
\end{array}
$$

**Proposition 2.** *There exists a subset $J \subseteq \{0, \ldots, t\}$ such that the directed graph $G_J = (V, E_J)$ is a DAG iff there exists a partial order of the items in U which satisfies all the constraints in S.*

**Proof.** Assume that there is a partial order $<_o$ that satisfies all the constraints in $S$. We claim that if we orient the edges in $E$ such that an edge is oriented from $v_i$ to $v_j$ if $u_i <_o u_j$, we obtain the desired acyclic orientation. Consider the set of constraints which corresponds to the class $E_j$. If $l_j <_o r_j$ then $x <_o y$ for each edge $(x, y) \in E_j$, hence these edges preserve their original orientation; otherwise $(r_j <_o l_j)$, all the edges in $E_j$ are reversed. This means that the resulted graph is $G_J$, where $J = \{j \mid r_j <_o l_j\}$. Since $<_o$ is a partial order, $G_J$ cannot contain a cycle, as required.

Conversely, given a set of indices $J \subseteq \{0, \ldots, t\}$ such that $G_J$ is acyclic, we define the partial order $<_o$ on the items of $U$ to be the transitive closure of the following relation $R$: For each $j \notin J$, and for each $(x, y) \in E_j$, $x\,R\,y, l_j\,R\,y, x\,R\,r_j$, and in addition $l_j\,R\,r_j$. For each $j \in J$, and for each $(x, y) \in E_j$, $y\,R\,x, y\,R\,l_j, r_j\,R\,x$, and in addition $r_j\,R\,l_j$. Clearly, the relation $R$ satisfies all the constraints in $S$. Moreover, since the graph $G_J$ is acyclic, it is easily seen that the transitive closure of the relation $R$ is a partial order on the items of $U$. □

## 4. The NP-completeness of the 1-$\mathscr{LIRS}$ problem

In this section we complete the proof that the 1-$\mathscr{LIRS}$ and the 1-$\mathscr{SLIRS}$ problems are NP-complete (Theorem 1), by a polynomial transformation form the CO problem.

### 4.1. *Polynomial transformation from the constrained order problem*

For a set of constraints $S = \{Cons_1, Cons_2, \ldots, Cons_t\}$ over a set of items $U = \{u_1, u_2, \ldots, u_n\}$, where $Cons_i = (Set_i^1 \mid Set_i^2)$, $i = 1, \ldots, t$, we construct the graph $G_S = (V_S, E_S)$ as follows. For each constraint $Cons_i \in S$ we will have three vertices in $V$, $a_i, b_i$, and $m_i$ which are termed the *ith triple*. Additionally, to every item $u_i \in U$ we will have in

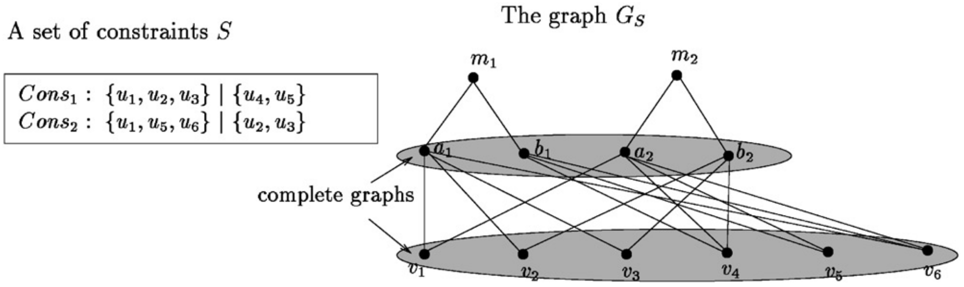


Fig. 4. A set of constraints $S$ over $U = \{u_1, \ldots, u_6\}$ and the corresponding graph $G_S$.

$G$ a vertex $v_i$. Formally,

$$V_S = \{m_i, a_i, b_i \mid 1 \leqslant i \leqslant t\} \cup \{v_j \mid 1 \leqslant j \leqslant n\}.$$

The vertices $\{m_i\}_{i=1,\ldots,t}$ are termed *high-level* vertices, the vertices $\{a_i, b_i\}_{i=1,\ldots,t}$ are termed *intermediate* vertices, and the vertices $\{v_j\}_{j=1,\ldots,n}$ are termed *low-level* vertices.

The set of edges $E_S$ is defined so that the following holds. The subgraph induced by the intermediate vertices $\{a_i, b_i\}_{i=1,\ldots,t}$ and the subgraph induced by the low-level vertices $\{v_j\}_{j=1,\ldots,n}$ are both complete graphs. In addition, each high level vertex $m_i$ is connected only to the two other vertices in the $i$th triple, $a_i$ and $b_i$; $a_i$ is also connected to all low-level vertices except those which correspond to the set $Set_i^2$, and $b_i$ is connected to all low-level vertices except those which correspond to the set $Set_i^1$. Formally, $E = \bigcup_{i=1}^{4} E_i$, where

$$\begin{aligned}
E_1 &= \{\{m_i, a_i\}, \{m_i, b_i\} \mid 1 \leqslant i \leqslant t\},\\
E_2 &= \{\{a_i, b_j\} \mid 1 \leqslant i, j \leqslant t\} \cup \{\{a_i, a_j\}, \{b_i, b_j\} \mid 1 \leqslant i, j \leqslant t, i \neq j\},\\
E_3 &= \{\{a_i, v_j\} \mid u_j \notin Set_i^2\} \cup \{\{b_i, v_k\} \mid u_k \notin Set_i^1\}, \text{and}\\
E_4 &= \{\{v_i, v_j\} \mid 1 \leqslant i, j \leqslant n, i \neq j\}.
\end{aligned}$$

The transformation is demonstrated by the following example.

**Example 5.** Fig. 4 is an example of a set of constraints $S$ over a set of items $U = \{u_1, u_2, \ldots, u_6\}$ and the corresponding graph $G_S$. For simplicity, the complete graph induced by the set of vertices $\{a_i\} \cup \{b_i\}$ and the complete graph induced by the set $\{v_j\}$ are depicted only implicitly.

Informally, the transformation is based on the following idea. For each constraint $Cons_i = (Set_i^1 \mid Set_i^2)$, all the shortest paths from $m_i$ to vertices that correspond to the items in $Set_i^1$ start with the edge $\{m_i, a_i\}$, and all the shortest paths from $m_i$ to vertices that corresponds to the items in $Set_i^2$ starts with the edge $\{m_i, b_i\}$. It follows that for each $i = 1, \ldots, t$, the order on the low-level vertices $\{v_j\}$ induced by an optimal 1-LIRS for the graph must satisfy the constraint $Cons_i$. Therefore, if there is an optimal 1-LIRS for the graph, there is an order of the items in $S$ that satisfies all the constraints in the set $S$. Conversely, we will show that, given a partial order $<_o$ which satisfies all the constraints in $S$, we can construct an optimal 1-LIRS for the graph. It is worth to note that it is not hard to prove the first direction using a simpler construction; connect every vertex $a_i$ to all vertices which correspond to items in $Set_i^1$ and every vertex $b_i$ to all vertices that correspond to items in $Set_i^2$, however, we do not know how to construct an optimal 1-LIRS for this graph.

## *4.2. Correctness of the transformation*

First note that the transformation is polynomial. Given a set of constraints $S$ over a set of items $U$, the number of vertices in the graph $G_S$ is exactly $|V_S| = n + 3t$, where $n$ is the number of items in the set $U$ and $t$ is the number of constraints in the set $S$. We prove the following proposition.

**Proposition 3.** *There exists a partial order which satisfies all the constraints in S iff* $G_S \in 1\text{-}\mathscr{LIRS}$ *(that is, there exists an optimal* 1*-LIRS for* $G_S$*).*

The rest of this section is the proof of Proposition 3.

We first prove that if $G_S \in 1\text{-}\mathscr{LIRS}$ then there exists a partial order that satisfies all the constraints in $S$. Let $\mathscr{L}_{G_S} = (L, \{I_v\})$ be an optimal 1-LIRS for the graph $G_S$. Then the vertex labeling $L$ induces a natural order on the items in $U$, namely $u_i <_o u_j$ iff $L(v_i) < L(v_j)$. We argue that $<_o$ is a partial (in fact—total) order on $U$ which satisfies all the constraints in $S$.

Consider the constraint $Cons_i = (Set_i^1 \mid Set_i^2)$. By the construction of the graph $G_S$, every shortest path from $m_i$ to any vertex $v_j$ that corresponds to an item $u_j$ in $Set_i^1$ starts with the edge $\{m_i, a_i\}$ (in fact, $\langle m_i, a_i, v_j\rangle$ is the only shortest path from $m_i$ to $v_j$). It follows that $\{v_j \mid u_j \in Set_i^1\} \subseteq I_{m_i}(m_i, a_i)$. By the same arguments, $\{v_j \mid u_j \in Set_i^2\} \subseteq I_{m_i}(m_i, b_i)$. Thus, there are two disjoint intervals, one of which contains all vertices that correspond to items in $Set_i^1$ and the other contains all vertices that correspond to the items in the set $Set_i^2$. Since the order of the low-level vertices is equivalent to the order of the corresponding items in $U$, either $Set_i^1 <_o Set_i^2$ or $Set_i^2 <_o Set_i^1$; in either case, the constraints $Cons_i$ is satisfied by the order $<_o$.

The proof of the other direction is more intricate. Given a partial order $<_o$ which satisfies the constraints in $S$ we will construct an optimal 1-LIRS $\mathscr{L}_{G_S}$ for $G_S$.

*The labeling of the vertices*
First we extend the partial order $<_o$ on the items in $U$ which satisfies all the constraints in $S$ to a total order $<_{to}$ in an arbitrary manner. We then construct a vertex-labeling function $L : V_S \to \{0, \dots, |V_S| - 1\}$ that satisfies the following conditions.

1. For every two low-level vertices $v_i$ and $v_j$: $L(v_i) < L(v_j)$ iff $u_i <_{to} u_j$. That is, the order of the low-level vertices is equivalent to the order $<_{to}$ on the items in the set $U$. Having this in mind, we will identify the set of items $Set_i^j$ with the set of low-level vertices $\{v_j \mid u_j \in Set_i^j\}$.
2. For every $i$th triple $\{m_i, a_i, b_i\}$, $1 \leqslant i \leqslant t$, there are two cases to consider.

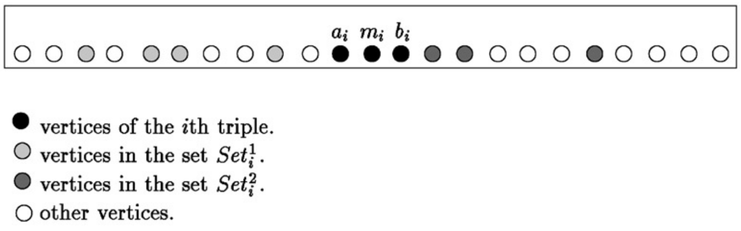

$a_i$ $m_i$ $b_i$
vertices of the $i$th triple.
vertices in the set $Set_i^1$.
vertices in the set $Set_i^2$.
other vertices.

Fig. 5. The order of the vertices in the case: $Set_i^1 < Set_i^2$.

(a) $Set_i^1 <_{to} Set_i^2$. Then $L(Set_i^1) < L(a_i) < L(m_i) < L(b_i) < L(Set_i^2)$ (where $L(A) = \{L(v) \mid v \in A\}$ for a set of vertices $A$). (See Fig. 5.)

(b) $Set_i^2 <_{to} Set_i^1$. Then $L(Set_i^2) < L(b_i) < L(m_i) < L(a_i) < L(Set_i^1)$.

Note that by the assumption that the order $<_{to}$ satisfies the constraints in $S$, exactly one of the two cases (a) and (b) holds.

3. For $i = 1, \ldots, t$, the vertices of the $i$th triple $(m_i, a_i, b_i)$ are ordered consecutively. Formally, $\{L(m_i), L(a_i), L(b_i)\} = \{k_i, k_i + 1, k_i + 2\}$ for some integer $k_i$.

Note that clearly a labeling that satisfies all three conditions exists: first arrange all low-level vertices by the order $<_{to}$, and then insert every $i$th triple anywhere between $Set_i^1$ and $Set_i^2$.

An observation which we use later in the proof is that for the $i$th triple $\{m_i, a_i, b_i\}$, if case 2(a) holds then $a_i$ is adjacent (in $G_S$) to all low-level vertices that are smaller than it (and possibly to some low-level vertices that are larger than it) and $b_i$ is adjacent to all low-level vertices which are larger than it. Case 2(b) implies the opposite observation.

*The labeling of the edges*

Now we show how to label the edges so as to obtain an optimal 1-LIRS (actually, 1-SLIRS). In our proof we use the observation that in order to show optimality of a labeling scheme, it suffices to show that for each pair of vertices $u, v$, the (unique) edge $e$ ($e$ adjacent to $u$) for which $L(v) \in I_u(e)$, lies on a shortest path from $u$ to $v$.

For brevity, we will not distinguish between a vertex $v$ and the number assigned to it $L(v)$. We will denote by *min* the vertex with minimal label and by *max* the vertex with maximal label.

1. *Labeling of the edges of a high-level vertex*: Consider any vertex $m_i$ ($1 \leqslant i \leqslant t$). $m_i$ is at distance 3 from any other high-level vertex $m_j$, at distance 1 from $a_i$ and $b_i$, and at distance 2 from all other vertices. Moreover, the vertex labeling implies that the edge $(m_i, a_i)$ [resp. $(m_i, b_i)$] lies on a shortest path from $m_i$ to each of the other vertices, except $b_i$ [resp. $a_i$] and except the low-level vertices in $Set_i^2$ [resp. $Set_i^1$]. Thus, if $a_i < m_i < b_i$, (case 2(a)), then the labeling $I_{m_i}(m_i, a_i) = \langle min, a_i \rangle$ and $I_{m_i}(m_i, b_i) = \langle b_i, max \rangle$ guarantees optimal routing from $m_i$. In the other case

$(b_i < m_i < a_i)$, the labeling $I_{m_i}(m_i, a_i) = \langle a_i, max\rangle$ and $I_{m_i}(m_i, b_i) = \langle min, b_i\rangle$ guarantees optimal routing from $m_i$.

2. *Labeling of the edges of a low-level vertex*: Consider any vertex $v_j$, $1 \leqslant j \leqslant n$. Then $v_j$ is adjacent to all other low-level vertices; also, for any $i$th triple $\{m_i, a_i, b_i\}$ $(1 \leqslant i \leqslant t)$, $v_j$ is adjacent to one or two vertices in $\{a_i, b_i\}$ (since it cannot be both in $Set_i^1$ and $Set_i^2$). Moreover, an edge from $v_j$ to any adjacent vertex in the $i$th triple is on a shortest path from $v_j$ to any non-adjacent vertex in this triple.
   Therefore, optimal routing for any other low-level vertex $v_k$, is guaranteed by the labeling $I_{v_j}(v_j, v_k) = \langle v_k, v_k\rangle$. Optimal routing to vertices in any $i$th triple $\{m_i, a_i, b_i\}$ $(1 \leqslant i \leqslant t)$ is guaranteed by the following labeling: Assume w.l.o.g. that $a_i < m_i < b_i$ (the other case is symmetric). If $v_j$ is adjacent to both $a_i$ and $b_i$ then $I_{v_j}(v_j, a_i) = \langle a_i, m_i\rangle$ and $I_{v_j}(v_j, b_i) = \langle b_i, b_i\rangle$, otherwise, if $v_j$ is connected only to $a_i$ then $I_{v_j}(v_j, a_i) = \langle a_i, b_i\rangle$, otherwise, $v_j$ is connected only to $b_i$ and $I_{v_j}(v_j, b_i) = \langle a_i, b_i\rangle$.
3. *Labeling of the edges of an intermediate vertex*: Consider any intermediate vertex $a_i$ (a vertex $b_i$ is treated similarly). $a_i$ is adjacent to $m_i$, to all other intermediate vertices, and to all low-level vertices except those in $Set_i^2$, and it is at distance 2 from all other vertices. A shortest path from $a_i$ to a non-adjacent high level vertex $m_j$ passes through either $a_j$ or $b_j$, and the shortest paths to a non-adjacent low-level vertex $v_j$ passes through a low-level vertex or an intermediate-level vertex which is adjacent to both $a_i$ and $v_j$. Using these facts, the labeling is constructed as follows: First, $I_{a_i}(a_i, m_i) = \langle m_i, m_i\rangle$. For a vertex $a_j$ $(j \neq i)$ the interval $I_{a_i}(a_i, a_j)$ will include $a_j$ and $m_j$ and for a vertex $b_j$ (possibly $j = i$) the interval $I_{a_i}(a_i, b_j)$ will include the vertex $b_j$. This guarantees optimal labeling for all high-level and intermediate vertices. Now we extend this labeling so as to guarantee optimal routing from $a_i$ to all low-level vertices. Assume w.l.o.g that $a_i < m_i < b_i$ (the other case is treated similarly). We have to construct a labeling which will send messages destined to a neighbor $v_j$ of $a_i$ directly to $v_j$, and messages destined to a vertex $v_k \in Set_i^2$ to a neighbor of $v_k$. Thus, we first insert vertex $v_j$ to $I_{a_i}(a_i, v_j)$ for every neighbor $v_j$ of $a_i$. Next we extend the labeling obtained so far to include also the vertices of $Set_i^2$ as follows. Denote as a *block* a maximal set of vertices in $Set_i^2$ which are ordered consecutively by $L$. Consider such a block $B$, and assume that its vertices are mapped by $L$ on the integers $l, l+1, \ldots, l+|B|-1$. Since $B$ does not contain all the vertices of $G$, there is a vertex $x \notin Set_i^2$ which is mapped on either $l-1$ or on $l+|B|$ (i.e., $x$ appears immediately before or immediately after the vertices in $B$). If $x$ appears immediately before $B$, then $x = v_k$ for some low-level vertex $v_k$, or $x$ is an intermediate-level vertex (i.e., $a_j$ or $b_j$) which is adjacent to all the low-level vertices which are larger than it. Similarly, if $x$ appears immediately after $B$, then $x = v_k$ for some low-level vertex $v_k$, or $x$ is an intermediate-level vertex (i.e., $a_j$ or $b_j$) which is adjacent to all the low-level vertices which are smaller than it. In both cases, $x$ is adjacent to both $a_i$ and to all the vertices in $B$. In both cases, we set $I_{a_i}(a_i, x)$ to include the segment containing $B \cup \{x\}$. This procedure is repeated for every block $B$.

This completes the proof of Proposition 3. □

Note that the above constructed labeling is strict. Thus, if the set of constraints $S$ has the desired partial order, then the graph $G_S$ has an optimal strict linear interval routing. Also, since the graph constructed admits no non-uniform costs on the links, we proved the NP-completeness of the 1-$\mathscr{LIRS}$ problem and the 1-$\mathscr{SLIRS}$ problem under the one unit cost model and hence under the fixed link cost model (Theorem 1).

## 5. The NP-completeness of the 1-$\mathscr{IRS}$ problem

In this section we prove that the 1-$\mathscr{IRS}$ and the 1-$\mathscr{SIRS}$ problems are NP-complete under the one unit cost model (and hence under the fixed link cost model) by a polynomial transformation from the CO problem.

### 5.1. Polynomial transformation from the CO problem

The transformation is accomplished in two steps; given any set of constraints $S$ we first construct from it a new set of constraints $S'$, we then use exactly the same transformation we used in Section 4 to construct from $S'$ the graph $G_{S'}$. We prove that the original set of constraints $S$ is satisfiable if the graph $G_{S'}$ has an optimal interval routing scheme and that $G_{S'}$ has an optimal strict interval routing scheme if $S$ is satisfiable.

*Step* 1: Let $U$ be a set of items and $S=\{Cons_i\}_{1\leqslant i\leqslant t}$ a set of constraints over $U$ such that $Cons_i=(Set_i^1 \mid Set_i^2)$ $(i=1,\dots,t)$. We construct a set of constraints $S'$ over the set $U'=U\cup\{a\}$ $(a\notin U)$ as follows. For every constraint $Cons_i\in S$ there are two constraints in $S'$;
$Cons_i^l=(\,Set_i^1\cup\{a\}\mid Set_i^2\,)$,
$Cons_i^r=(\,Set_i^1\mid Set_i^2\cup\{a\}\,)$.
The set of constraints $S'$ is
$S'=\{Cons_i^l, Cons_i^r \mid 1\leqslant i\leqslant t\}$.

**Example 6.** The following set of constraints is constructed from the set of constraints presented in Example 5.

$$\begin{array}{llll}
( & \{u_1,u_2,u_3,a\} & \mid \{u_4,u_5\} & ),\\
( & \{u_1,u_2,u_3\} & \mid \{u_4,u_5,a\} & ),\\
( & \{u_1,u_5,u_6,a\} & \mid \{u_2,u_3\} & ),\\
( & \{u_1,u_5,u_6\} & \mid \{u_2,u_3,a\} & ).
\end{array}$$

*Step* 2: Now we construct from $U'$ and $S'$ the graph $G_{S'}=(V_{S'},E_{S'})$ by the transformation of Section 4.

### 5.2. Correctness of the transformation

It is easily seen that the transformation is polynomial. We prove the following proposition.

**Proposition 4.** *There is a partial order on the items in $U$ that satisfies the constraints in the set $S$ iff $G_{S'} \in 1\text{-}\mathcal{IRS}$.*

**Proof.** We use the following well-known (see, e.g., [21]) observation. Given any 1-IRS $\mathcal{L}_G$ for a graph $G$ with $n$ vertices, there are $n-1$ equivalent 1-IRSs that are obtained from $\mathcal{L}_G$ by cyclically shifting the labels of the vertices and intervals. That is, the $i$th 1-interval labeling is obtained by assigning every vertex $v$ the label $L(v)+i \pmod n$ and replacing the intervals on the edges correspondingly. The paths that a message will traverse under any of the $n$ equivalent 1-IRSs are identical. Note that this observation enables us to assume, given a 1-IRS for the graph $G$ that a specific vertex, say $a$, is the vertex with maximal label ($L(a)=n-1$).

Assume that there is an optimal 1-IRS $\mathcal{L}_G$ for $G_{S'}$. Then (by the observation above) we can assume w.l.o.g. that the vertex $v_a$ which corresponds to the item $a$ in $U'$ is the vertex with maximal label. We denote by $\{m_i^l, a_i^l, b_i^l\}$ the triple of vertices which correspond to the constraint $Cons_i^l$ in $S'$ and by $\{m_i^r, a_i^r, b_i^r\}$ the triple which corresponds to the constraint $Cons_i^r$. By the same considerations of the proof in Section 4, we get $Set_i^1 \cup \{v_a\} \subseteq I_{m_i^l}(m_i^l, a_i^l)$, $Set_i^2 \subseteq I_{m_i^l}(m_i^l, b_i^l)$, $Set_i^1 \subseteq I_{m_i^r}(m_i^r, a_i^r)$ and $Set_i^2 \cup \{v_a\} \subseteq I_{m_i^r}(m_i^r, b_i^r)$. We argue that the order of the vertices determined by the 1-IRS $\mathcal{L}_G$ satisfies either $Set_i^1 < Set_i^2$ or $Set_i^2 < Set_i^1$ ($i=1,\ldots,t$). Assume that this is not the case. Then either there are three low-level vertices $v_x, v_y \in Set_i^1$ and $v_z \in Set_i^2$ such that $L(v_x) < l(v_z) < l(v_y)$, or a similar inequality holds where $v_x, v_y \in Set_i^2$ and $v_z \in Set_i^1$. Assume w.l.o.g that the first case holds. Then by the assumption that $v_a$ is the vertex with maximal label we get $L(v_x) < L(v_z) < L(v_y) < L(v_a)$ and clearly this implies that the intervals $I_{m_i^r}(m_i^r, a_i^r)$ and $I_{m_i^r}(m_i^r, b_i^r)$ on the edges outgoing the vertex $m_i^r$ are not disjoint. The other case will imply, by the same arguments, that the intervals on the edges outgoing $m_i^l$ are not disjoint. It follows that the order of the low-level vertices by the labeling $\mathcal{L}_G$ naturally induces a total order on the items of the set $U$ that satisfies all the constraints in $S$.

In the other direction we have to construct, given a partial order $<_o$ that satisfies all the constraints in the set $U$, an optimal 1-IRS for the graph $G_{S'}$. The construction proceeds as follows: First, the vertex $a$ and all its adjacent edges are ignored, and the construction is done as in Section 4 (note that when $a$ is ignored, the graph $G_{S'}$ is similar to the graph $G_S$ constructed in Section 4, except that now each constraint is replaced by two identical constraints, which are represented by two triples, $\{m_i^l, a_i^l, b_i^l\}$ and $\{m_i^r, a_i^r, b_i^r\}$). We set $L(a) = max + 1$ where $max$ is the maximal label of all other vertices. Next we assign the labels $I_a(a,x)$ to the edges leaving $a$ by the same rules as the labeling $I_v$ for any other low-level vertex $v$. Then, for each vertex $x$ which is adjacent to $a$, $I_x(x,a)$ is set to $\langle a,a\rangle$. Last, we modify the edge labeling of high-level and intermediate-level vertices which are not adjacent to $a$. For each edge leaving a high-level vertex $m_i$ which was labeled by $\langle *, max\rangle$ we replace $max$ by $max+1 = L(a)$. Each intermediate-level vertex $x$ ($x \in \{a_i^l, a_i^r, b_i^l, b_i^r\}$) which is not adjacent to $a$ (there is exactly one such vertex in each triple) must be adjacent to a vertex $y$ such that $L(y) = min$ or $L(y) = max$; we add $L(a) = max+1$ to the interval $I_x(x,y)$ (if $L(a) = min$

then this transforms $I_x(x, y)$ to a cyclic interval). It is easy to see that the resulting labeling is optimal and uses one interval per edge. □

As in Section 4, here also we actually construct an optimal 1-SIRS for the graph $G$, thus proving that both the 1-$\mathscr{IRS}$ and the 1-$\mathscr{SIRS}$ are NP-complete under the one unit cost model and hence under the fixed link cost model. This completes the proof of Theorem 2.

## 6. Summary and open problems

We proved that recognizing networks that admit either optimal 1-IRS or optimal 1-LIRS (strict or non-strict), is NP-hard under the one-unit cost model (and hence under the fixed link cost model). These results clearly imply that the problems of determining the minimal $k$ such that a network $G$ belongs to $k$-$\mathscr{IRS}$ or to $k$-$\mathscr{LIRS}$ (entries "Minimal $k$ s.t. $G \in k$-$\mathscr{IRS}$" and "Minimal $k$ s.t. $G \in k$-$\mathscr{LIRS}$", respectively, in Fig. 1) are NP hard. They also imply that determining the minimal integer $K$, for which a network $G$ has an optimal labeling (IRS or LIRS) which uses a total of at most $K$ intervals is NP hard: Under the one unit cost model, a simple graph $G = (V, E)$ admits an optimal 1-LIRS (respectively, 1-IRS) iff it admits an optimal LIRS (respectively, IRS) which uses a total of at most $2|E|$ intervals.

There are still many open questions. Our transformations from the AGO problem to the 1-$\mathscr{LIRS}$ problem and the 1-$\mathscr{IRS}$ problem imply that the same problems remain NP-complete even for the restricted class of graphs with diameter at most 3. Do these problems remain NP-complete for other restricted classes of graphs, e.g., bounded degree graphs? In a different direction, recently it was proved [9] that the same characterization problems are NP-hard also for $k = 2$; it remains an open question to determine the complexity of the problems $k$-$\mathscr{IRS}$ and $k$-$\mathscr{LIRS}$ for fixed $k$, $k > 2$. It seems that our technique does not directly imply that any of these problems is NP-hard. Another question is to find an approximation algorithm to the optimization problem of finding the minimal $k$ such that a graph belongs to $k$-$\mathscr{IRS}$ (resp. $k$-$\mathscr{LIRS}$). Our results imply that there is no such algorithm with approximation ratio less than 2 (since the result of such algorithm will be a number strictly less than 2 iff there is an optimal 1-IRS). The existence of such algorithm with constant ratio is another interesting open question.

A different direction of research, which is motivated by our results, is to study networks that admit near optimal interval labeling schemes, that is, networks which admit a labeling scheme (of a given type) that guarantees a small stretch factor (where *a stretch factor* of a routing scheme is the maximum ratio between the length of a shortest path and the length of a routing path between two endpoints). Recently, it was shown in [5] that every graph admits an interval routing scheme with stretch factor at most 5 (and an average stretch factor at most 3) and with compactness $2\sqrt{n(1 + \ln n)}$. It was also shown in [5] that there are some graphs for which the compactness of any interval routing scheme that guarantees any constant stretch factor is at least $\sqrt{n}$. An interesting direction for further research is recognizing classes of networks for which a better trade-off can be achieved.

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