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GROUP EXPLICIT METHODS FOR THE NUMERICAL SOLUTION OF THE WAVE EQUATION

M. S. SAHIMI† and D. J. EVANS

Department of Computer Studies, Loughborough University of Technology, Loughborough, Leicestershire, England

Abstract—The applicability of the group explicit (GE) methods to the numerical solution of hyperbolic partial differential equations of second-order is discussed in this paper following closely many of the ideas presented in an earlier related paper.

1. INTRODUCTION

The group explicit (GE) methods have been used successfully to solve numerical problems involving parabolic and hyperbolic partial differential equations [1, 2].

In this paper we extend the techniques discussed in Ref. 1 so that they are applicable to a system of first order hyperbolic equations of the form,

$$\frac{\partial U}{\partial t} + A\frac{\partial U}{\partial x} = 0,$$

where A is a real $(n \times n)$ matrix and U is an n component column vector. In particular the method is suitable for the solution of the wave equation,

$$\frac{\partial^2 U}{\partial t^2} = \frac{\partial^2 U}{\partial x^2}, \quad 0 \leqslant x \leqslant 1,\ t \geqslant 0.$$

The new techniques are shown to be clearly more superior to an earlier strategy presented in Ref. 3.

2. GE METHODS FOR THE SECOND-ORDER WAVE EQUATION

Let us now consider solving the following second-order wave equation:

$$\frac{\partial^2 U}{\partial x^2} = \frac{\partial^2 U}{\partial t^2}, \tag{1}$$

subject to the initial conditions,

$$U(x, 0) = f_1(x),$$
$$\frac{\partial U}{\partial t}(x, 0) = f_2(x) \tag{2}$$

and the boundary conditions

$$U(0, t) = g_1(t)$$

and

$$U(1, t) = g_2(t). \tag{3}$$

The wave equation (1) can be reduced to a system of simultaneous differential equations of *first order* by the following substitutions:

$$U^{(1)} = \frac{\partial U}{\partial t}$$

†Present address: National University of Malaysia, Bangi, Selangor, Malaysia.

and

$$U^{(2)} = \frac{\partial U}{\partial x}. \tag{4}$$

In the more general case, first-order systems of equations can be written in matrix form as

$$\frac{\partial \hat{\mathbf{U}}}{\partial t} + A\frac{\partial \hat{\mathbf{U}}}{\partial x} = \mathbf{0}, \tag{5}$$

where A is an $n \times n$ real matrix (not necessarily symmetric) and $\hat{\mathbf{U}}$ is an n-component column vector $\hat{\mathbf{U}} = [U^{(1)}, U^{(2)}, \ldots, U^{(n)}]^{\mathrm{T}}$.

A non-singular matrix P exists through *the similarity transformation*,

$$PAP^{-1} = D \tag{6}$$

where D is a diagonal matrix having the real eigenvalues of A as its elements (i.e. $D = \mathrm{diag}(\mu_i)$, the μ_i being the eigenvalues of A). On premultiplying equation (5) by P, we get

$$\frac{\partial}{\partial t}(P\hat{\mathbf{U}}) + PAP^{-1}\frac{\partial}{\partial x}(P\hat{\mathbf{U}}) = \mathbf{0},$$

i.e.

$$\frac{\partial \mathbf{V}}{\partial t} + D\frac{\partial \mathbf{V}}{\partial x} = 0, \tag{7}$$

where $\mathbf{V} = P\hat{\mathbf{U}}$. Hence, *the decoupled scalar form* of equation (7) is given by

$$\frac{\partial V^{(i)}}{\partial t} + \mu_i\frac{\partial V^{(i)}}{\partial x} = 0, \quad i = 1, 2, \ldots, n. \tag{8}$$

For our particular problem, if

$$P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix},$$

then equation (5) takes the form of equation (7), where

$$\mathbf{V} = [V^{(1)}, V^{(2)}]^{\mathrm{T}} = [U^{(1)} - U^{(2)}, U^{(1)} + U^{(2)}]^{\mathrm{T}},$$

i.e.

$$V^{(1)} = U^{(1)} - U^{(2)}, \quad U^{(1)} = \tfrac{1}{2}(V^{(1)} + V^{(2)})$$

and

$$V^{(2)} = U^{(1)} + U^{(2)}, \quad U^{(2)} = C\tfrac{1}{2}(V^{(2)} - V^{(1)}), \tag{9}$$

and

$$D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Hence, the decoupled scalar equations for $V^{(1)}$ and $V^{(2)}$ are

$$\frac{\partial V^{(1)}}{\partial t} + \frac{\partial V^{(1)}}{\partial x} = 0 \tag{10a}$$

and

$$\frac{\partial V^{(2)}}{\partial t} - \frac{\partial V^{(2)}}{\partial x} = 0, \tag{10b}$$

respectively. System (10) can be rewritten as

$$\frac{\partial V^{(p)}}{\partial t} = a\frac{\partial V^{(p)}}{\partial x}, \tag{11a}$$

where

$$a = \begin{cases} -1, & \text{when } p = 1 \\ 1, & \text{when } p = 2 \end{cases}. \tag{11b}$$

These *first order* differential equations in $\mathbf{V} = [V^{(1)}, V^{(2)}]^T$ will be solved by using the *weighted difference analogues*,

$$\lambda\{\theta[(1-w)v_{i+1,j+1}^{(p)} + (2w-1)v_{i,j+1}^{(p)} - wv_{i-1,j+1}^{(p)}] + (1-\theta)[(1-w)v_{i+1,j}^{(p)} + (2w-1)v_{ij}^{(p)} - wv_{i-1,j}^{(p)}]\}$$
$$= a[v_{i,j+1}^{(p)} - v_{ij}^{(p)}]; \quad 0 \leqslant \theta,\ w \leqslant 1. \tag{12}$$

These equations reduce to

$$\lambda\theta v_{i-1,j+1}^{(p)} + (a - \lambda\theta)v_{i,j+1}^{(p)} = -\lambda(1-\theta)v_{i-1,j}^{(p)} + [a + \lambda(1-\theta)]v_{ij}^{(p)} \tag{13}$$

and

$$(a + \lambda\theta)v_{i,j+1}^{(p)} - \lambda\theta v_{i+1,j+1}^{(p)} = [a - \lambda(1-\theta)]v_{ij}^{(p)} + \lambda(1-\theta)v_{i+1,j}^{(p)}, \tag{14}$$

when w takes the values 1 and 0, respectively. The local truncation errors of equations (13) and (14) at the point $(x_i, t_{j+1/2})$ are given respectively by,

$$\begin{aligned} T_{13} = {} & \Delta x\left[-\frac{1}{2}\frac{\partial^2 V^{(p)}}{\partial x^2} - \frac{(\Delta t)^2}{16}\frac{\partial^4 V^{(p)}}{\partial x^2 \partial t^2}\right]_{i,j+1/2} + \Delta t\left[\tfrac{1}{2}(1-2\theta)\frac{\partial^2 V^{(p)}}{\partial x \partial t} + \frac{(\Delta x)^2}{12}(1-2\theta)\frac{\partial^4 V^{(p)}}{\partial x^3 \partial t}\right]_{i,j+1/2} \\ & + (\Delta x)(\Delta t)\left[\tfrac{1}{4}(1-2\theta)\frac{\partial^3 V^{(p)}}{\partial x^2 \partial t}\right]_{i,j+1/2} + (\Delta x)^2\left[\frac{1}{6}\frac{\partial^3 V^{(p)}}{\partial x^3} + \frac{\Delta x}{24}\frac{\partial^4 V^{(p)}}{\partial x^4}\right]_{i,j+1/2} \\ & + (\Delta t)^2\left[\tfrac{1}{8}(1-2\theta)\frac{\partial^3 V^{(p)}}{\partial x \partial t^2} + \frac{a}{24}\frac{\partial^3 V^{(p)}}{\partial t^3} + \frac{\Delta t}{48}(1-2\theta)\frac{\partial^4 V^{(p)}}{\partial x \partial t^3}\right]_{i,j+1/2} + 0[(\Delta x)^{\alpha_1}(\Delta t)^{\alpha_2}] \end{aligned} \tag{15}$$

and

$$\begin{aligned} T_{14} = {} & \Delta x\left[\frac{1}{2}\frac{\partial^2 V^{(p)}}{\partial x^2} + \frac{(\Delta t)^2}{16}\frac{\partial^4 V^{(p)}}{\partial x^2 \partial t^2}\right]_{i,j+1/2} + \Delta t\left[\tfrac{1}{2}(1-2\theta)\frac{\partial^2 V^{(p)}}{\partial x \partial t} + \frac{(\Delta x)^2}{12}(1-2\theta)\frac{\partial^4 V^{(p)}}{\partial x^3 \partial t}\right]_{i,j+1/2} \\ & + (\Delta x)(\Delta t)\left[\tfrac{1}{4}(1-2\theta)\frac{\partial^3 V^{(p)}}{\partial x^2 \partial t}\right]_{i,j+1/2} + (\Delta x)^2\left[\frac{1}{6}\frac{\partial^3 V^{(p)}}{\partial x^3} - \frac{\Delta x}{24}\frac{\partial^4 V^{(p)}}{\partial x^4}\right]_{i,j+1/2} \\ & + (\Delta t)^2\left[-\tfrac{1}{8}(1-2\theta)\frac{\partial^3 V^{(p)}}{\partial x \partial t^2} + \frac{a}{24}\frac{\partial^3 V^{(p)}}{\partial t^3} + \frac{\Delta t}{48}(1-2\theta)\frac{\partial^4 V^{(p)}}{\partial x \partial t^3}\right]_{i,j+1/2} + 0[(\Delta x)^{\alpha_1}(\Delta t)^{\alpha_2}] \end{aligned} \tag{16}$$

with $\alpha_1 + \alpha_2 = 4$ and $0 \leqslant \theta \leqslant 1$. If we apply the formula (14), at the point $(x_{i-1}, t_{j+\theta})$, we obtain,

$$(a + \lambda\theta)v_{i-1,j+1}^{(p)} - \lambda\theta v_{i,j+1}^{(p)} = [a - \lambda(1-\theta)]v_{i-1,j}^{(p)} + \lambda(1-\theta)v_{ij}^{(p)}. \tag{17}$$

By coupling equations (13) and (17), we arrive at the following set of explicit equations (we have omitted the details to avoid repetition):

$$v_{i-1,j+1}^{(p)} = \frac{(a-\lambda)}{a}v_{i-1,j}^{(p)} + \frac{\lambda}{a}v_{ij}^{(p)} \tag{18a}$$

and

$$v_{i,j+1}^{(p)} = -\frac{\lambda}{a}v_{i-1,j}^{(p)} + \frac{(a+\lambda)}{a}v_{ij}^{(p)}. \tag{18b}$$

These equations must be solved simultaneously to give the values of $v^{(1)}$ and $v^{(2)}$ at the grid points along each j-line. From equation (14), the equation determining the values of $v^{(p)}$ at the ungrouped point adjacent to the right boundary is given by

$$v^{(p)}_{m-1,j+1} = \{[a - \lambda(1-\theta)]v^{(p)}_{m-1,j} + \lambda(1-\theta)v^{(p)}_{mj} + \lambda\theta v^{(p)}_{m,j+1}\}/(a+\lambda\theta), \tag{19}$$

whilst from equation (13) we obtain the following formula for the ungrouped point at the left end:

$$v^{(p)}_{1,j+1} = \{-\lambda(1-\theta)v^{(p)}_{0j} - \lambda\theta v^{(p)}_{0,j+1} + [a + \lambda(1-\theta)]v^{(p)}_{1j}\}/(a-\lambda\theta), \quad a \neq \lambda\theta. \tag{20}$$

The GE schemes are then constructed along a similar line as before—and without loss of generality we assume that we will be using an even number of intervals of the line segment $0 \leqslant x \leqslant 1$.

(i) The GER scheme

By means of equations (13), (17) and (19), the group explicit with right ungrouped point (GER) scheme is represented by the formula

$$(aI + \lambda\theta G_1)\mathbf{v}^{(p)}_{j+1} = [aI - \lambda(1-\theta)G_1]\mathbf{v}^{(p)}_j + \mathbf{b}_1, \tag{21}$$

where

$$G_1 = \left[\begin{array}{cc|cc|c|cc|c}
1 & -1 & & & & & & \\
1 & -1 & & & & & & \\
\hline
 & & 1 & -1 & & & & \\
 & & 1 & -1 & & & & \\
\hline
 & & & & \ddots\;0 & & & \\
\hline
 & & & & 0 & 1 & -1 & \\
 & & & & & 1 & -1 & \\
\hline
 & & & & & & & 1
\end{array}\right]_{[(m-1)\times(m-1)]} \tag{22}$$

and

$$\mathbf{b}_1 = [0, 0, \ldots, \lambda(1-\theta)v^{(p)}_{mj} + \lambda\theta v^{(p)}_{m,j+1}]^{\mathrm{T}}.$$

(ii) The GEL scheme

The group explicit with left ungrouped point (GEL) scheme is determined by the equations (20), (13) and (17) which can be expressed in a more compact form as

$$(aI + \lambda\theta G_2)\mathbf{v}^{(p)}_{j+1} = [aI - \lambda(1-\theta)G_2]\mathbf{v}^{(p)}_j + \mathbf{b}_2, \tag{23}$$

where

$$G_2 = \left[\begin{array}{c|cc|c|cc}
-1 & & & & & \\
\hline
 & 1 & -1 & & & \\
 & 1 & -1 & & 0 & \\
\hline
 & & & \ddots & & \\
\hline
 & & & & 1 & -1 \\
 & & & 0 & 1 & -1
\end{array}\right]_{[(m-1)\times(m-1)]} \tag{24}$$

and

$$\mathbf{b}_2 = [-\lambda(1-\theta)v^{(p)}_{0j} - \lambda\theta v^{(p)}_{0,j+1}, 0, 0, \ldots, 0]^{\mathrm{T}}.$$

(iii) The (S)AGE and D(AGE) schemes

The alternative use of the GER and the GEL methods leads to the following (S)AGE formulae:

$$\left.\begin{aligned}(aI+\lambda\theta G_1)\mathbf{v}_{j+1}^{(p)}&=[aI-\lambda(1-\theta)G_1]\mathbf{v}_j^{(p)}+\mathbf{b}_1\\(aI+\lambda\theta G_2)\mathbf{v}_{j+2}^{(p)}&=[aI-\lambda(1-\theta)G_2]\mathbf{v}_{j+1}^{(p)}+\mathbf{b}_2\end{aligned}\right\},\quad j=0,2,4,\ldots,\tag{25}$$

and the (D)AGE four-step process,

$$\left.\begin{aligned}(aI+\lambda\theta G_1)\mathbf{v}_{j+1}^{(p)}&=[aI-\lambda(1-\theta)G_1]\mathbf{v}_j^{(p)}+\mathbf{b}_1\\(aI+\lambda\theta G_2)\mathbf{v}_{j+2}^{(p)}&=[aI-\lambda(1-\theta)G_2]\mathbf{v}_{j+1}^{(p)}+\mathbf{b}_2\\(aI+\lambda\theta G_2)\mathbf{v}_{j+3}^{(p)}&=[aI-\lambda(1-\theta)G_2]\mathbf{v}_{j+2}^{(p)}+\mathbf{b}_2\\(aI+\lambda\theta G_1)\mathbf{v}_{j+4}^{(p)}&=[aI-\lambda(1-\theta)G_1]\mathbf{v}_{j+3}^{(p)}+\mathbf{b}_1\end{aligned}\right\},\quad j=0,4,\ldots.\tag{26}$$

All of the GE schemes employed above provide us with the values of $V^{(1)}$ and $V^{(2)}$ at the mesh points. The solution u of the wave equation (1) can then be computed using the relations in equations (9), i.e.

$$U^{(1)}\approx\tfrac{1}{2}[v^{(1)}+v^{(2)}]\tag{27}$$

and

$$U^{(2)}\approx\tfrac{1}{2}[v^{(2)}-v^{(1)}].\tag{28}$$

From expression (27), for example, we have, at the point (x_i, t_j),

$$\left(\frac{\partial U}{\partial t}\right)_{i,j}\approx\tfrac{1}{2}[v_{ij}^{(1)}+v_{ij}^{(2)}]\tag{29}$$

and a first-order explicit approximation is obtained from the equation,

$$\frac{(u_{i,j+1}-u_{ij})}{\Delta t}=\tfrac{1}{2}[v_{ij}^{(1)}+v_{ij}^{(2)}]$$

or

$$u_{i,j+1}=u_{ij}+\tfrac{1}{2}\Delta t[v_{ij}^{(1)}+v_{ij}^{(2)}].\tag{30}$$

On the other hand, if we add equations (27) and (28), we find that,

$$U^{(1)}+U^{(2)}\approx v^{(2)}\tag{31}$$

and at the point (x_i, t_j) we have

$$\left(\frac{\partial U}{\partial t}\right)_{i,j}+\left(\frac{\partial U}{\partial x}\right)_{ij}\approx v_{ij}^{(2)}.\tag{32}$$

This can be solved by the second-order Lax–Wendroff explicit analogue given by,

$$u_{i,j+1}=\tfrac{1}{2}\lambda(1+\lambda)u_{i-1,j}+(1-\lambda^2)u_{ij}-\tfrac{1}{2}\lambda(1-\lambda)u_{i+1,j}+\Delta t v_{ij}^{(2)}.\tag{33}$$

At the point $(x_i, t_{j+1/2})$ however, expression (31) becomes

$$\left(\frac{\partial U}{\partial t}\right)_{i,j+1/2}+\left(\frac{\partial U}{\partial x}\right)_{i,j+1/2}\approx v_{i,j+1/2}^{(2)}\tag{34}$$

and the following second-order accurate Crank–Nicolson type implicit approximation can be used,

$$-\tfrac{1}{4}\lambda u_{i-1,j+1}+u_{i,j+1}+\tfrac{1}{4}\lambda u_{i+1,j+1}=\tfrac{1}{4}\lambda u_{i-1,j}+u_{ij}-\tfrac{1}{4}\lambda u_{i+1,j}+\tfrac{1}{2}\Delta t(v_{ij}^{(2)}+v_{i,j+1}^{(2)}).\tag{35}$$

In employing the method of solution to expression (31), we must of course bear in mind its stability requirements as well as its order of accuracy.

3. TRUNCATION ERROR ANALYSIS OF THE GE SCHEMES

(*i*) *Truncation error for the GER scheme*

The set of explicit equations obtained by coupling equations (13) and (17) are

$$v_{i-1,j+1}^{(p)} - \frac{(a-\lambda)}{a} v_{i-1,j}^{(p)} - \frac{\lambda}{a} v_{ij}^{(p)} = 0 \tag{36}$$

and

$$v_{i,j+1}^{(p)} + \frac{\lambda}{a} v_{i-1,j}^{(p)} - \frac{(a+\lambda)}{a} v_{ij}^{(p)} = 0. \tag{37}$$

The truncation errors for any two grouped points are given by the truncation errors of equations (36) and (37) for $i = 2, 4, \ldots, m-2$. Thus we have,

$$\begin{aligned} T_{36} = {} & \Delta x\left[\frac{1}{2}\frac{\partial^2 V^{(p)}}{\partial x^2} + \frac{(\Delta t)^2}{16}\frac{\partial^4 V^{(p)}}{\partial x^2 \partial t^2}\right]_{i-1,j+1/2} + \Delta t\left[\frac{1}{2}\frac{\partial^2 V^{(p)}}{\partial x \partial t} + \frac{(\Delta x)^2}{12}\frac{\partial^4 V^{(p)}}{\partial x^3 \partial t}\right]_{i-1,j+1/2} \\ & + (\Delta x)(\Delta t)\left[\frac{1}{4}\frac{\partial^3 V^{(p)}}{\partial x^2 \partial t}\right]_{i-1,j+1/2} + (\Delta x)^2\left[-\frac{1}{6}\frac{\partial^3 V^{(p)}}{\partial x^3} - \frac{\Delta x}{24}\frac{\partial^4 V^{(p)}}{\partial x^4}\right]_{i-1,j+1/2} \\ & + (\Delta t)^2\left[-\frac{1}{8}\frac{\partial^3 V^{(p)}}{\partial x \partial t^2} + \frac{a}{24}\frac{\partial^3 V^{(p)}}{\partial t^3} + \frac{\Delta t}{48}\frac{\partial^4 V^{(p)}}{\partial x \partial t^3}\right]_{i-1,j+1/2} + 0[(\Delta x)^{\alpha_1}(\Delta t)^{\alpha_2}], \quad \alpha_1 + \alpha_2 = 4 \end{aligned} \tag{38}$$

$$\begin{aligned} T_{37} = {} & \Delta x\left[-\frac{1}{2}\frac{\partial^2 V^{(p)}}{\partial x^2} - \frac{(\Delta t)^2}{16}\frac{\partial^4 V^{(p)}}{\partial x^2 \partial t^2}\right]_{i,j+1/2} + \Delta t\left[\frac{1}{2}\frac{\partial^2 V^{(p)}}{\partial x \partial t} + \frac{(\Delta x)^2}{12}\frac{\partial^4 V^{(p)}}{\partial x^3 \partial t}\right]_{i,j+1/2} \\ & + (\Delta x)(\Delta t)\left[-\frac{1}{4}\frac{\partial^3 V^{(p)}}{\partial x^2 \partial t}\right]_{i,j+1/2} + (\Delta x)^2\left[-\frac{1}{6}\frac{\partial^3 V^{(p)}}{\partial x^3} + \frac{\Delta x}{24}\frac{\partial^4 V^{(p)}}{\partial x^4}\right]_{i,j+1/2} \\ & + (\Delta t)^2\left[-\frac{1}{8}\frac{\partial^3 V^{(p)}}{\partial x \partial t^2} + \frac{a}{24}\frac{\partial^3 V^{(p)}}{\partial t^3} + \frac{\Delta t}{48}\frac{\partial^4 V^{(p)}}{\partial x \partial t^3}\right]_{i,j+1/2} + 0[(\Delta x)^{\alpha_1}(\Delta t)^{\alpha_2}], \quad \alpha_1 + \alpha_2 = 4. \end{aligned} \tag{39}$$

The truncation error for the single ungrouped point near the right end is obtained from equation (16) by putting $i = m - 1$. This gives

$$\begin{aligned} T_R = {} & \Delta x\left[\frac{1}{2}\frac{\partial^2 V^{(p)}}{\partial x^2} + \frac{(\Delta t)^2}{16}\frac{\partial^4 V^{(p)}}{\partial x^2 \partial t^2}\right]_{m-1,j+1/2} + \Delta t\left[\tfrac{1}{2}(1-2\theta)\frac{\partial^2 V^{(p)}}{\partial x \partial t} + \frac{(\Delta x)^2}{12}(1-2\theta)\frac{\partial^4 V^{(p)}}{\partial x^3 \partial t}\right]_{m-1,j+1/2} \\ & + (\Delta x)(\Delta t)\left[\tfrac{1}{4}(1-2\theta)\frac{\partial^3 V^{(p)}}{\partial x^2 \partial t}\right]_{m-1,j+1/2} + (\Delta x)^2\left[-\frac{1}{6}\frac{\partial^3 V^{(p)}}{\partial x^3} - \frac{\Delta x}{24}\frac{\partial^4 V^{(p)}}{\partial x^4}\right]_{m-1,j+1/2} \\ & + (\Delta t)^2\left[-\tfrac{1}{8}(1-2\theta)\frac{\partial^3 V^{(p)}}{\partial x \partial t^2} + \frac{a}{24}\frac{\partial^3 V^{(p)}}{\partial t^3} + \frac{\Delta t}{48}(1-2\theta)\frac{\partial^4 V^{(p)}}{\partial x \partial t^3}\right]_{m-1,j+1/2} \\ & + 0[(\Delta x)^{\alpha_1}(\Delta t)^{\alpha_2}], \quad \alpha_1 + \alpha_2 = 4. \end{aligned} \tag{40}$$

(*ii*) *Truncation error for the GEL scheme*

The truncation error for the single ungrouped point near the left boundary is obtained from equation (15) with $i = 1$ and this gives the expression

$$\begin{aligned} T_L = {} & \Delta x\left[-\frac{1}{2}\frac{\partial^2 V^{(p)}}{\partial x^2} - \frac{(\Delta t)^2}{16}\frac{\partial^4 V^{(p)}}{\partial x^2 \partial t^2}\right]_{1,j+1/2} + \Delta t\left[\tfrac{1}{2}(1-2\theta)\frac{\partial^2 V^{(p)}}{\partial x \partial t} + \frac{(\Delta x)^2}{12}(1-2\theta)\frac{\partial^4 V^{(p)}}{\partial x^3 \partial t}\right]_{1,j+1/2} \\ & + (\Delta x)(\Delta t)\left[-\tfrac{1}{4}(1-2\theta)\frac{\partial^3 V^{(p)}}{\partial x^2 \partial t}\right]_{1,j+1/2} + (\Delta x)^2\left[-\frac{1}{6}\frac{\partial^3 V^{(p)}}{\partial x^3} + \frac{\Delta x}{24}\frac{\partial^4 V^{(p)}}{\partial x^4}\right]_{1,j+1/2} \\ & + (\Delta t)^2\left[-\tfrac{1}{8}(1-2\theta)\frac{\partial^3 V^{(p)}}{\partial x \partial t^2} + \frac{a}{24}\frac{\partial^3 V^{(p)}}{\partial t^3} + \frac{\Delta t}{48}(1-2\theta)\frac{\partial^4 V^{(p)}}{\partial x \partial t^3}\right]_{1,j+1/2} \\ & + 0[(\Delta x)^{\alpha_1}(\Delta t)^{\alpha_2}], \quad \alpha_1 + \alpha_2 = 4. \end{aligned} \tag{41}$$

The truncation errors for any two grouped points are given by T_{36} and T_{37}, respectively.

(iii) Truncation error for the (S)AGE and (D)AGE schemes

As we have already seen, the truncation errors of the GER and GEL schemes (in their appropriate order of alternation) constitute the overall truncation errors of the two- and four-step processes. Thus, there will be cancellations of errors at most points leading to some improvement in the solutions of the methods when compared with the constituent GER and GEL schemes.

4. STABILITY ANALYSIS OF THE GE SCHEMES

It is clear from equations (1) and (9) that to reach an *overall stability*, the GE schemes applied to the decoupled equations in equations (10a) and (10b) must be stable simultaneously.

(i) Stability of the GER scheme

From equation (21), we have

$$\mathbf{v}_{j+1}^{(p)} = \Gamma_{\mathrm{GER}}\mathbf{v}_j^{(p)} + \hat{\mathbf{b}}_1, \tag{42}$$

where Γ_{GER} is the amplification matrix given by

$$\Gamma_{\mathrm{GER}} = (aI + \lambda\theta G_1)^{-1}[aI - \lambda(1-\theta)G_1] \tag{43}$$

and

$$\hat{\mathbf{b}}_1 = (aI + \lambda\theta G_1)^{-1}\mathbf{b}_1. \tag{44}$$

For the case $p = 1$ (and $a = -1$) we have already established in Evans and Sahimi [1] that the GER scheme for equation (10a) is absolutely unstable in the range $0 \leqslant \theta \leqslant \frac{1}{2}$ and is conditionally stable for $\lambda \geqslant 2/(2\theta - 1)$ when $\frac{1}{2} < \theta \leqslant 1$. For the case $p = 2$ (and $a = 1$) we have

$$\Gamma_{\mathrm{GER}} = \left[\begin{array}{cc|cc|c|cc|c}
(1-\lambda) & \lambda & & & & & & \\
-\lambda & (1+\lambda) & & & & & & \\
\hline
 & & (1-\lambda) & \lambda & & & & \\
 & & -\lambda & (1+\lambda) & & & 0 & \\
\hline
 & & & & \ddots & & & \\
\hline
 & & & & & (1-\lambda) & \lambda & \\
 & & 0 & & & -\lambda & (1+\lambda) & \\
\hline
 & & & & & & & 1 - \dfrac{\lambda}{(1+\lambda\theta)}
\end{array}\right]_{[(m-1)\times(m-1)]}, \tag{45}$$

whose eigenvalues are 1 [of multiplicity $(m-2)$] and $1 - \lambda/(1+\lambda\theta)$. For stability, we require that

$$\left|1 - \frac{\lambda}{(1+\lambda\theta)}\right| \leqslant 1 \quad \text{giving} \quad 0 \leqslant \frac{\lambda}{(1+\lambda\theta)} \leqslant 2.$$

Hence we deduce that the scheme is conditionally stable for $\lambda \leqslant 2/(1-2\theta)$ with $0 \leqslant \theta < \frac{1}{2}$ and it is absolutely stable for all values of λ when $\frac{1}{2} \leqslant \theta \leqslant 1$. From the two stability requirements above, we therefore conclude that for *overall stability*, the GER scheme is stable only for $\lambda \geqslant 2/(2\theta - 1)$ when $\frac{1}{2} < \theta \leqslant 1$.

(ii) Stability of the GEL scheme

From equation (23), the GEL scheme can be explicitly expressed as

$$\mathbf{v}_{j+1}^{(p)} = \Gamma_{\mathrm{GEL}}\mathbf{v}_j^{(p)} + \hat{\mathbf{b}}_2, \tag{46}$$

where Γ_{GEL} is the amplification matrix given by $\Gamma_{\mathrm{GEL}} = (aI + \lambda\theta G_2)^{-1}[aI - \lambda(1-\theta)G_2]$ and $\hat{\mathbf{b}}_2 = (aI + \lambda\theta G_2)^{-1}\mathbf{b}_2$. We have already seen in Evans and Sahimi [1] that the GEL scheme when applied to the differential equation (10a) (when $p = 1$ and $a = -1$) is conditionally stable for $\lambda \leqslant 2/(1-2\theta)$ and is always stable when $\frac{1}{2} \leqslant \theta \leqslant 1$. The amplification matrix of the GEL scheme for equation (10b) (when $p = 2$ and $a = 1$) is

$$
\Gamma_{\mathrm{GEL}} = \left[\begin{array}{c|cc|c|cc}
1+\dfrac{\lambda}{(1-\lambda\theta)} & & & & & \\
\hline
 & (1-\lambda) & \lambda & & & \\
 & -\lambda & (1+\lambda) & & 0 & \\
\hline
 & & & \ddots & & \\
\hline
 & & & & (1-\lambda) & \lambda \\
 & & 0 & & -\lambda & (1+\lambda)
\end{array}\right]_{[(m-1)\times(m-1)]} \tag{47}
$$

whose eigenvalues are 1 [of multiplicity $(m-2)$] and $1+\lambda/(1-\lambda\theta)$. For stability, we require

$$\left|1+\frac{\lambda}{(1+\lambda\theta)}\right| \leqslant 1$$

or

$$-2 \leqslant \frac{\lambda}{(1-\lambda\theta)} \leqslant 0$$

which is the same inequality given by Evans and Sahimi [1]. From the argument that followed, we deduce that the GEL scheme for equation (10b) is always unstable when $0 \leqslant \theta \leqslant \frac{1}{2}$ and is stable only for $\lambda \geqslant 2(2\theta-1)$ when $\frac{1}{2} < \theta \leqslant 1$. Again for *overall stability*, we are led to the same stability conditions as that which was concluded for the GER scheme above.

(iii) Stability of the (S)AGE and (D)AGE schemes

From equations (25) and (26) we find that the amplification matrices of the S(AGE) and D(AGE) schemes are given respectively by

$$\Gamma_{\mathrm{SAGE}} = \Gamma_{\mathrm{GEL}}\Gamma_{\mathrm{GER}}$$

and

$$\Gamma_{\mathrm{DAGE}} = \Gamma_{\mathrm{GER}}\Gamma_{\mathrm{GEL}}\Gamma_{\mathrm{SAGE}}.$$

It has already been proved that the (S)AGE scheme for equation (10a) (when $p=1$ and $a=-1$) is conditionally stable for $\lambda \leqslant 1$. Similarly, the (D)AGE scheme is found to have conditional stability only for $\lambda \leqslant \frac{1}{2}$. For the case $p=2$ (and $a=1$), we have,

$$
\Gamma_{\mathrm{SAGE}} = \begin{bmatrix}
a' & b' & & & & & & & \\
c' & d' & -c' & e' & & & & & \\
e' & f' & d' & -f' & & & & & \\
 & & c' & d' & -c' & e' & & 0 & \\
 & & e' & f' & d' & -f' & & & \\
 & & & \ddots & \ddots & \ddots & \ddots & & \\
 & 0 & & & c' & d' & -c' & e' & \\
 & & & & e' & f' & d' & -f' & \\
 & & & & & & c' & d' & g' \\
 & & & & & & e' & f' & h'
\end{bmatrix}_{[(m-1)\times(m-1)]}, \tag{48}
$$

where

$$
a' = (1-\lambda)\left[1+\frac{\lambda}{(1-\lambda\theta)}\right],\quad b' = \lambda\left[1+\frac{\lambda}{(1-\lambda\theta)}\right],\quad c' = -\lambda(1-\lambda),\quad d' = 1-\lambda^2,
$$

$$
e' = \lambda^2,\quad f' = -\lambda(1+\lambda),\quad g' = \lambda\left(1-\frac{\lambda}{(1+\lambda\theta)}\right) \text{ and } h' = (1+\lambda)\left[1-\frac{\lambda}{(1+\lambda\theta)}\right] \tag{49}
$$

and

$$|a'| + (m-3)|d'| + |h'| \leqslant \psi(\lambda)$$

where

$$\psi(\lambda) = (1-\lambda)\left[1 + \frac{\lambda}{(1-\lambda\theta)}\right] + (m-3)(1-\lambda^2) + (1+\lambda)\left[1 - \frac{\lambda}{(1-\lambda\theta)}\right]$$

for $\lambda \leqslant 1$, $\theta \in [0, 1]$ and $\lambda\theta \neq 1$. It can be shown that the (S)AGE method is stable for $\lambda \leqslant 1$.

The (D)AGE amplification matrix (for $p = 2$ and $a = 1$), however, takes the form

$$\Gamma_{\mathrm{DAGE}} = \begin{bmatrix}
p' & q' & r' & s' & & & & & & & & \\
-q' & t' & u' & v' & & & & & & & & \\
r' & -u' & w' & x' & r' & s' & & & & & & \\
-s' & v' & -x' & z' & u' & v' & & & & & & \\
 & & r' & -u' & w' & x' & r' & s' & & 0 & & \\
 & & -s' & v' & -x' & z' & u' & v' & & & & \\
 & & & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & \\
 & & & 0 & & r' & -u' & w' & x' & r' & s' & \\
 & & & & & -s' & v' & -x' & z' & u' & v' & \\
 & & & & & & & r' & -u' & w' & x' & y' \\
 & & & & & & & -s' & v' & -x' & z' & q_1' \\
 & & & & & & & & & y' & -q_1' & p_1'
\end{bmatrix}_{[(m-1)\times(m-1)]} \tag{50}$$

where

$$p' = (1-\lambda)^2\left[1 + \frac{\lambda}{(1-\lambda\theta)}\right]^2 - \lambda^2(1-2\lambda),$$

$$q' = -\left\{-\lambda(1-\lambda)\left[1 + \frac{\lambda}{(1-\lambda\theta)}\right]^2 - \lambda(1+\lambda)(1-2\lambda)\right\}, \quad r' = 2(1-\lambda)\lambda^2, \quad s' = 2\lambda^3,$$

$$t' = -\lambda^2\left[1 + \frac{\lambda}{(1-\lambda\theta)}\right]^2 + (1+\lambda)^2(1-2\lambda), \quad u' = 2\lambda(1-\lambda^2), \quad v' = 2(1+\lambda)\lambda^2,$$

$$w' = (1-\lambda)^2(1+2\lambda) - \lambda^2(1-2\lambda), \quad x' = -2\lambda(2\lambda^2-1), \quad y' = 2\lambda^2\left[1 - \frac{\lambda}{(1+\lambda\theta)}\right],$$

$$z' = (1+\lambda)^2(1-2\lambda) - \lambda^2(1+2\lambda), \quad q_1' = 2\lambda(1+\lambda)\left[1 - \frac{\lambda}{(1+\theta\lambda)}\right],$$

$$p_1' = (1+2\lambda)\left[1 - \frac{\lambda}{(1+\theta\lambda)}\right]^2 \tag{51}$$

and

$$|p'| + |t'| + \tfrac{1}{2}(m-4)|w'| + \tfrac{1}{2}(m-4)|z'| + |p_1'| \leqslant \psi(\lambda),$$

where

$$\psi(\lambda) = (1-\lambda)^2\left[1+\frac{\lambda}{(1-\lambda\theta)}\right]^2 - \lambda^2(1-2\lambda) + \lambda^2\left[1+\frac{\lambda}{(1-\lambda\theta)}\right]^2 + (1+\lambda)^2(1-2\lambda)$$
$$+ \tfrac{1}{2}(m-4)[(1-\lambda)^2(1+2\lambda)+\lambda^2(1-2\lambda)] + \tfrac{1}{2}(m-4)[(1+\lambda)^2(1-2\lambda)+\lambda^2(1+2\lambda)]$$
$$+ (1+2\lambda)\left[1-\frac{\lambda}{(1+\lambda\theta)}\right]^2$$
$$= [\lambda^2+(1-\lambda)^2]\left[1+\frac{\lambda}{(1-\lambda\theta)}\right]^2 + [\lambda^2+(1+\lambda)^2](1-2\lambda) + (m-4)(1-2\lambda^2)$$
$$+ (1+2\lambda)\left[1-\frac{\lambda}{(1+\lambda\theta)}\right]^2$$

for $\lambda \leqslant \frac{1}{2}$, $\theta \in [0, 1]$ and $\lambda\theta \neq 1$. It can be shown in a similar manner as before that the (D)AGE method is stable for $\lambda \leqslant \frac{1}{2}$. For an overall stability, we conclude that the (S)AGE and (D)AGE processes are conditionally stable for $\lambda \leqslant 1$ and $\lambda \leqslant \frac{1}{2}$, respectively. Therefore, it is recommended that for practical purposes, only (S)AGE is used.

5. NUMERICAL RESULTS

In this experiment, we proceeded with the application of the GE techniques on the second-order wave equation,

$$\frac{\partial^2 U}{\partial x^2} = \frac{\partial^2 U}{\partial t^2}, \tag{52}$$

subjected to

$$U(x, 0) = \tfrac{1}{8}\sin(\pi x),$$
$$\frac{\partial U}{\partial t}(x, 0) = 0,$$
$$U(0, t) = 0$$

and

$$U(1, t) = 0.$$

The analytical solution is given by

$$U(x, t) = \tfrac{1}{8}\sin(\pi x)\cos(\pi t). \tag{53}$$

Again, we display the absolute errors of the numerical solutions along the mesh line $t = 1.0$ for $\lambda = 0.5$ and $\theta = 0.5$ in Table 1.

To arrive at the solution of the second-order wave equation, Experiment 1 necessitates us to solve two different sets of first-order differential equations. The first set involves $V^{(1)}$ and $V^{(2)}$ whose approximations at the mesh points are obtained by applying the GE techniques on equations (10a)

Table 1. Absolute errors of the numerical solutions to the wave equation

	x									
Method	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	Average of all absolute errors
(S)AGE–EXP	4.02×10^{-3}	2.6×10^{-3}	1.61×10^{-3}	3.74×10^{-3}	6.18×10^{-4}	3.38×10^{-3}	1.63×10^{-3}	1.86×10^{-3}	4.06×10^{-3}	2.61×10^{-3}
(D)AGE–EXP	7.66×10^{-4}	6.25×10^{-4}	1.27×10^{-3}	1.27×10^{-3}	1.22×10^{-3}	1.55×10^{-3}	7.35×10^{-4}	1.46×10^{-3}	1.14×10^{-4}	1.0×10^{-3}
(S)AGE–LW	8.35×10^{-4}	1.11×10^{-3}	4.34×10^{-4}	1.17×10^{-3}	3.61×10^{-4}	2.52×10^{-4}	7.79×10^{-4}	1.5×10^{-3}	2.09×10^{-3}	9.47×10^{-4}
(D)AGE–LW	1.13×10^{-3}	8.85×10^{-4}	1.83×10^{-3}	1.66×10^{-3}	1.24×10^{-3}	1.15×10^{-3}	3.88×10^{-4}	1.76×10^{-4}	1.98×10^{-3}	1.16×10^{-3}
(S)AGE–CN	1.46×10^{-3}	1.78×10^{-3}	4.69×10^{-3}	2.44×10^{-3}	1.09×10^{-2}	1.67×10^{-3}	1.2×10^{-2}	6.54×10^{-4}	1.02×10^{-2}	5.09×10^{-3}
(D)AGE–CN	1.95×10^{-3}	4.35×10^{-4}	5.96×10^{-3}	9.9×10^{-4}	9.17×10^{-3}	1.49×10^{-3}	9.3×10^{-3}	1.42×10^{-3}	7.86×10^{-3}	4.29×10^{-3}
Exact solution	−0.0386272	−0.0734732	−0.1011271	−0.1188821	−0.125000	−0.1188821	−0.1011271	−0.0734732	−0.0386272	—

and (10b). The solutions u are then computed by means of the explicit Lax–Wendroff and the Crank–Nicolson type formulae. These solutions are compared in Table 1. No attempt is made to compute the GER and the GEL solutions as these schemes have a rather rigid stability requirement. It becomes apparent from the table that the (S)AGE–LW methods provide the most accurate solution. The stability restrictions of the (S)AGE–CN and (D)AGE–CN methods are $\lambda \leqslant 1$ and $\lambda \leqslant \frac{1}{2}$, respectively, and besides incurring a comparatively heavier computational load, these methods also happen to produce a less accurate solution for our particular problem. Hence, for its simplicity and accuracy, the (S)AGE–LW (also stable for $\lambda \leqslant 1$) combination is favoured.

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