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Rooted trees and iterated wreath products of cyclic groups

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Abstract

Let $W_{n,r}$ denote the n-fold iterated wreath product of $\mathbb{Z}/r\mathbb{Z}$ with itself. In this paper, we are interested in the tower of groups $W_{1,r} \subset W_{2,r} \subset \cdots$. We show that the irreducible representations of $W_{n,r}$ are indexed by a set of labeled rooted trees. By adding a partial order on this set of rooted trees, we obtain the Bratteli diagram for this tower of groups. In particular, we give the branching rules. This approach yields combinatorial rules for the decomposition of restricted and induced representations.

Introduction

Wreath products arise naturally as automorphisms of regular hierarchical combinatorial structures (see [19] for a nice history and extensive bibliography). For example, iterated wreath products comprise the symmetries of rooted trees and of nested designs (see, e.g., [1,2]); they occur in chemistry as the symmetry groups of certain regularly branching molecules of non-rigid molecules [3,20]; and have even been suggested as descriptors for the way in which the human visual system processes information [12].

In this paper, we concern ourselves only with the (iterated) wreath products of cyclic groups. We let $W_{n,r}$ be the n-fold iterated wreath product of $\mathbb{Z}/r\mathbb{Z}$ with itself. For instance, $W_{1,r} = \mathbb{Z}/r\mathbb{Z}$, $W_{2,r} = \mathbb{Z}/r\mathbb{Z} \text{ wr } \mathbb{Z}/r\mathbb{Z}$, and more generally, $W_{n,r} = W_{n-1,r} \text{ wr } \mathbb{Z}/r\mathbb{Z}$. In

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terms of the symmetries of a rooted tree, the former consists of only the cyclic shifts of the leaves of a rooted tree of depth one, while the latter, those automorphisms generated by permitting only cyclic shifts of nodes at a given depth.

These groups and their close relatives appear in various contexts. The groups $W_{n,4}$ have been used for signal analysis and image processing [7,8]. When $r = p$ a prime, $W_{n,p}$ is distinguished as the Sylow p-subgroup of S_{p^n} (see, e.g., [17]). When the index r does not remain fixed, and we allow n to go to infinity, we arrive at profinite groups important to the theory of renormalizable dynamical systems [4].

In this paper, we show that the representation theory of these groups, examined through the use of the tower of groups

$$W_{1,r} \subset W_{2,r} \subset W_{3,r} \cdots$$

has a rich combinatorial structure.

Our main result shows that the irreducible representations of $W_{n,r}$ are indexed by a family of trees which we call *r-trees*. These are labeled complete r-ary trees (defined in Section 3) of height n.

By adding a partial order to the set of r-trees we find a new interpretation of the branching rules for the irreducible representations of these groups. This in turn gives an alternate derivation, as well as new interpretation, of the *Bratteli diagram* of this sequence of groups. Bratteli diagrams are a construction first used in operator algebras [5]. Their graphical representation of the branching rules corresponding to towers of groups have, over the past several years, come to be recognized as a tremendously useful tool in the construction of efficient algorithms for computing the Fourier transform for finite and even compact groups including wreath products, where they are crucial to the development of, what is to date, the most efficient "FFT" available for these groups [14–16].

This paper is organized as follows. In the first section, we define Bratteli diagrams and iterated wreath products. We also describe the inclusion $W_{n-1,r} \subset W_{n,r}$ and describe how these groups are interpreted as automorphism groups of trees. We also give a presentation of this group in terms of generators and relations. In Section 2, we describe the irreducible representations and give a formula for the number of conjugacy classes and irreducible representations of these groups.

In Section 3, we introduce r-trees and define an order on these trees. We enumerate these trees via a recursive formula. The proof of the formula is given by establishing a bijection between r-trees of height $n-1$ and the irreducible representations of $W_{n,r}$.

In Section 4, we use the bijection of the r-trees with the irreducible representations of $W_{n,r}$ to construct the Bratteli diagram for the sequence of groups $W_{1,r} \subset W_{2,r} \subset \cdots$. This Bratteli diagram yields the branching rules for this sequence of groups. Furthermore, counting Hasse walks on the Bratteli diagram yields the degrees of the irreducible representations. We end this section by giving a bijection between r-trees and conjugacy classes of $W_{n,r}$.

We hope that the Bratteli diagram together with the presentation of these iterated wreath products will serve as a foundation for defining the analogous theory that exits for the symmetry groups and for the wreath products $\mathbb{Z}/r\mathbb{Z}$ wr S_n. In particular, it would be interesting to define positive traces on the sequence of group algebras.

1. Preliminaries

A *Bratteli diagram* B is a weighted graph. It is described by a set of vertices from a disjoint collection of sets $B_m,\ m \geqslant 0$, and edges that connect vertices in B_m to vertices in B_{m+1}. We assume that the set B_0 contains a unique vertex. The edges are labeled by positive integer weights. The set B_m is the set of vertices at *level* m. If $T_1 \in B_m$ is connected to a vertex $T_2 \in B_{m+1}$, then we write $T_1 \leqslant T_2$.

Bratteli diagrams first arose in the theory of operator algebras [5] and more precisely, the theory of multimatrix algebras [9]. We are interested in the construction of such diagrams as relates to the representation theory of finite groups.

Given a tower of subgroups $\langle 1\rangle = G_0 < G_1 < \cdots < G_n$, the corresponding Bratteli diagram has vertices of set B_i labeling the irreducible representations of G_i. If ρ and η are irreducible representations of G_i, and G_{i-1} respectively, then the corresponding vertices are connected by an edge weighted by the multiplicity of η in ρ when restricted to G_{i-1}.

Example 1. The Young lattice is an example of a Bratteli diagram. In the Young lattice, the vertices are Young diagrams (or partitions) and the edge joining a partition of m to a partition of $m+1$ has weight 1 (if the multiplicity is 1, we omit the labels). Let S_n be the symmetric group. Figure 1 shows the Bratteli diagram for the sequence $S_1 < S_2 < S_3$, where S_j permutes only the symbols $1, \ldots, j$.

The distinct edges in these diagrams, viewed as directed from level $m-1$ to level m, have an interpretation of mutually orthogonal G_{m-1}-equivariant morphisms from $C[G_{m-1}]$ into $C[G_m]$. With this interpretation, the paths from root to leaf give a natural indexing of so-called "Gel'fand–Tsetlin" bases for the underlying towers of subgroups. These bases correspond to those whose matrix representations have the property that when restricted through the tower of subgroups they are block diagonal with irreducible blocks at each step, with equivalent irreducibles actually equal. It is this sort of recursive structure that is so critical to efficient harmonic analysis for these groups (see, e.g., [14]).

We consider the Bratteli diagram of a particular tower of subgroups of the iterated wreath product. In this paper, we will describe the Bratteli diagram for iterated wreath products of cyclic groups. In this case, we obtain a Bratteli diagram that is not multiplicity-free—i.e., a given irreducible of $W_{n,r}$ may contain multiple copies of some irreducibles when restricted to $W_{n-1,r}$. By encoding this not by multiple edges, but rather, by weighted edges, we arrive at the r-trees of Section 3.

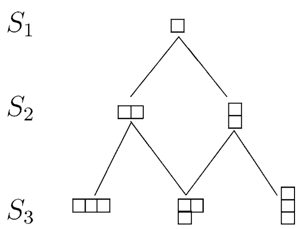


Fig. 1.

1.1. Wreath products

In this section, we define the wreath product of two groups. We give some basic results regarding these wreath products and a combinatorial interpretation of these groups as subgroups of the automorphism groups of complete r-ary trees. We also describe the inclusion $W_{n-1,r} \subset W_{n,r}$ and give a pictorial interpretation of this inclusion.

In general, let G be a finite group and H be a subgroup of S_n. We can define an action of H on $G^n = G \times G \times \cdots \times G$ (n times), where if $a = (a_1, a_2, \ldots, a_n) \in G^n$, then

$$a^\pi = (a_{\pi^{-1}(1)}, a_{\pi^{-1}(2)}, \ldots, a_{\pi^{-1}(n)})$$

where $\pi \in H$.

The *wreath product* of G with H, denoted by $G \text{ wr } H$, is the set $G^n \times H$ with the multiplication:

$$(a;\pi)(b;\sigma) := \bigl(ab^\pi;\pi\sigma\bigr).$$

If e denotes the identity in G^n and ι denotes the identity in H, then $H \cong \{(e;\pi) \mid \pi \in H\}$ and $G^n \cong \{(a;\iota) \mid a \in G^n\}$. With these identifications, we can think of G^n as a normal subgroup of $G \text{ wr } H$. Thus, with the action of H on G as given above, this wreath product is just the semi-direct product of G^n by H. The order of this group is $|G|^n|H|$.

The wreath product as defined above is associative for finite groups (for a proof, see [17, Theorem 7.26]). The *iterated wreath product* of $\mathbb{Z}/r\mathbb{Z}$ is the n-fold wreath product defined recursively as follows:

$$W_{1,r} \cong \mathbb{Z}/r\mathbb{Z}$$

and

$$W_{n,r} \cong W_{n-1,r} \text{ wr } \mathbb{Z}/r\mathbb{Z},$$

where r and n are any two positive integers. It will be convenient to think of $\mathbb{Z}/r\mathbb{Z}$ as the cyclic subgroup of the symmetry group S_r generated by the cycle $(1\ 2\ \ldots\ r)$ of length r. In this case, we think of $\mathbb{Z}/r\mathbb{Z}$ as acting on $(W_{n-1,r})^r$ via cyclic permutations.

Perhaps the best way to understand wreath products is by interpreting them as automorphism groups of rooted trees. An automorphism of a tree, Γ, with vertex set V, is a bijection $\phi : V \to V$ such that $u, v \in V$ are adjacent if and only if $\phi(u)$ and $\phi(v)$ are adjacent. For a more detailed description of how this is done, see [17]. Under composition, the set $\mathrm{Aut}(\Gamma)$ of automorphisms of Γ is a group.

In this context, $W_{n,r}$ can be identified with the subgroup of the automorphism group of the complete r-ary tree of height n that cyclically permutes the r children of each node. In the special case $r = 2$, we get that $W_{n,2}$ is isomorphic to the automorphism group of the complete binary tree of height n.

Example. $\mathrm{Aut}(\Gamma) \cong \mathbb{Z}/2\mathbb{Z} \text{ wr } \mathbb{Z}/2\mathbb{Z}$ where Γ is the rooted tree in Fig. 2. In this case,

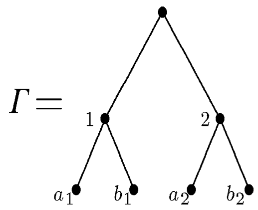


Fig. 2.

we think of $\mathbb{Z}/2\mathbb{Z}$ as permuting the vertices labeled 1 and 2, and $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ as independently permuting a_1 with b_1 and a_2 with b_2. For $\mathbb{Z}/2\mathbb{Z}$ wr $\mathbb{Z}/2\mathbb{Z}$ wr $\mathbb{Z}/2\mathbb{Z}$ we would add two "children" to each of the last 4 vertices in the tree in Fig. 2. Note that using this identification gives an easy way to see that the wreath product is associative.

1.2. Embedding $W_{n-1,r} \subset W_{n,r}$

We have defined $W_{1,r} = \mathbb{Z}/r\mathbb{Z}$, $W_{2,r} = \mathbb{Z}/r\mathbb{Z}$ wr $\mathbb{Z}/r\mathbb{Z}$ (a group of order $r \cdot r^r$), and more generally,

$$W_{n,r} = W_{n-1,r} \text{ wr } \mathbb{Z}/r\mathbb{Z}.$$

Thus, $|W_{n,r}| = r \cdot |W_{n-1,r}|^r$, so that

$$|W_{n,r}| = r^{(r^n-1)/r-1}$$

implying that the index of $W_{n-1,r}$ in $W_{n,r}$ is $r^{r^{n-1}}$.

We have defined the iterated wreath product by adding a factor of $\mathbb{Z}/r\mathbb{Z}$ on the right. This definition is essentially via the embedding of $W_{n-1,r}$ in $W_{n,r}$ as the subgroup of automorphisms of the rooted regular r-ary tree of height n which permute the leftmost (or any fixed) subtree of height $n-1$ from the root while keeping the other nodes of the tree fixed.

Another way to describe the iterated wreath product is via the embedding of $W_{n-1,r}$ in $W_{n,r}$, as the subgroup of automorphisms which leave fixed the relative ordering of any of the "bottom-most" subtrees, consisting of groups of r leaves with a common parent. Collapsing each of these "leaf trees" gives the rooted r-ary tree of height $n-1$, hence the embedding. This embedding gives a different recursive definition of the iterated wreath product $\widetilde{W}_{n,r}$ as

$$\widetilde{W}_{n,r} = \mathbb{Z}/r\mathbb{Z} \text{ wr } \widetilde{W}_{n-1,r}.$$

Algebraically, we identify $\widetilde{W}_{n,r}$ with a subgroup of S_{r^n} (see [17] for the permutation version of G wr H). With this identification, $\widetilde{W}_{n-1,r}$ acts by permutations on $(\mathbb{Z}/r\mathbb{Z})^{r^{n-1}}$. Notice that this implies that $\widetilde{W}_{n,r}$ is isomorphic to the semi-direct product of $(\mathbb{Z}/r\mathbb{Z})^{r^{n-1}}$ by $\widetilde{W}_{n-1,r}$.

Remark. The groups $\widetilde{W}_{n,r}$ and $W_{n,r}$ are isomorphic since the wreath product is associative.

1.3. Presentation of $W_{n,r}$

We now give a presentation of $W_{n,r}$ in terms of generators and relations. A group G generated by a set X satisfying a set of relations R will be written $G = \langle X \mid R\rangle$. The presentation of the wreath product of H by G is given in the following theorem. The proof can be found in [10].

Theorem 1.1 [10, p. 176]. *Let G and H be finite groups with presentations $G = \langle X \mid R\rangle$ and $H = \langle Y \mid S\rangle$ and let $L \subset H$ be a subset such that $L \cup L^{-1} = H/\{e\}$ where e is the identity in H. Then G wr H has a presentation*

$$G \text{ wr } H = \bigl\langle X, Y \mid R, S, \bigl[X, X^L\bigr]\bigr\rangle$$

where X^L denotes the action of L on X and $[\cdot,\cdot]$ *denotes the commutator.*

We now use this theorem to write a presentation of $W_{n,r}$.

Theorem 1.2. *The group $W_{n,r}$ is given by generators $t_1, \dots, t_n$ satisfying the following relations*:

(1) $t_i^r = 1$, $i = 1, \dots, n$;
(2) $t_i t_j^{-k} t_i t_j^k = t_j^{-k} t_i t_j^k t_i$, $1 \leqslant i < j \leqslant n$, $k = 1, \dots, r-1$;
(3) $t_i t_j^{-k} t_m t_j^k = t_j^{-k} t_m t_j^k t_i$, $1 \leqslant i < j \leqslant n$, $m < j$, *and* $k = 1, 2, \dots, r-1$.

Proof. The proof is by induction on n. We apply Theorem 1.1 at each step of the induction. It is clear that we will have n generators corresponding to each factor of $\mathbb{Z}/r\mathbb{Z}$ and that each generator will have order r. Relations (2) and (3) correspond to the commutators. $\square$

One way to visualize the t_i's as automorphisms of the complete r-ary tree is as the cyclic permutations on the leftmost vertices at a given level of the r-ary tree.

Example. A presentation of $W_{n,2}$ is given by generators $t_1, \dots, t_n$ satisfying the following relations: $t_i^2 = 1$, $1 \leqslant i \leqslant n$; $(t_i t_j)^4 = 1$, $i \neq j$, $1 \leqslant i, j \leqslant n$; $(t_i t_j t_i t_{j+k})^2 = 1$, $j > i$, and $k = 1, \dots, n-i$. It is well known that $W_2 = \mathbb{Z}_2$ wr $\mathbb{Z}_2$ is isomorphic to the dihedral group, D_8, which has presentation:

$$\bigl\langle t_1, t_2 \mid t_1^2 = t_2^2 = 1, (t_1 t_2)^4 = 1\bigr\rangle.$$

Notice that the iterated wreath products $W_{n,2}$ are not Coxeter groups for $n > 2$.

2. Representation theory of $W_{n,r}$

In this section we describe the representation theory of the group $W_{n,r}$. The irreducible representations can be constructed in a recursive fashion by applying Clifford theory [6]. Recall that we have defined $W_{n,r} = W_{n-1,r}$ wr $\mathbb{Z}/r\mathbb{Z}$ as the semi-direct product of $W^r_{n-1,r}$ by $\mathbb{Z}/r\mathbb{Z}$. Here $W^r_{n-1,r}$ denotes the r-fold direct product of $W_{n-1,r}$ with itself. Therefore, we have that $W^r_{n-1,r}$ is a normal subgroup of $W_{n,r}$. The general representation theory of wreath products can be found in [11].

If $\{\rho_i\}$ forms a complete set of irreducible representations of $W_{n-1,r}$, then the irreducible representations of $W^r_{n-1,r}$ are of the form $\rho_1 \otimes \cdots \otimes \rho_r$, where the ρ_i are not necessarily distinct. The action of $\mathbb{Z}/r\mathbb{Z}$ on $W^r_{n-1,r}$ by cyclic permutations translates into an action of $\mathbb{Z}/r\mathbb{Z}$ on the representation space of $W^r_{n-1,r}$. This group action provides a decomposition of the representation space into a disjoint union of orbits.

Let O_i be an orbit of the representation space. Fix a representative $\sigma_i \in O_i$ and let $G_i \leqslant \mathbb{Z}/r\mathbb{Z}$ denote the corresponding stabilizer subgroup of σ_i. Then in the terminology of Clifford theory, $W_{n-1,r}$ wr G_i is an inertia group of $W_{n,r}$. In particular, we know that every irreducible representation of $W^r_{n-1,r}$ can be extended (trivially) to an irreducible representation of $W_{n-1,r}$ wr G_i. This extension is then tensored with any irreducible representation of G_i to yield an irreducible representation of $W_{n-1,r}$ wr G_i. Finally, each of these (twisted) extensions is induced from $W_{n-1,r}$ wr G_i to $W_{n,r}$. These induced representations are irreducible. Furthermore, every irreducible representation of $W_{n,r}$ is obtained in this way.

We summarize the discussion above in the following theorem:

Theorem 2.1. *Let $\{\rho_i\}$ be a complete set of inequivalent irreducible representations of $W_{n-1,r}$, and let d_i be the dimension of ρ_i. Let $\sigma = \rho_1 \otimes \cdots \otimes \rho_r$ be an irreducible representa tion of $W^r_{n-1,r}$, let $\mathbb{Z}/d\mathbb{Z}$ be its stabilizer under cyclic permutations by $\mathbb{Z}/r\mathbb{Z}$, and let $\sigma^1, \ldots, \sigma^{r/d}$ be the representations of $W^r_{n-1,r}$ in the orbit of σ. Set $\overline{d_i} = (d_1 d_2 \cdots d_r)/d_i$. If τ is an irreducible representation of $\mathbb{Z}/d\mathbb{Z}$, then $\sigma \otimes \tau$ is an irreducible representation of $W_{n-1,r}$ wr $\mathbb{Z}/d\mathbb{Z}$ and $\mathrm{Ind}^{W_{n,r}} \sigma \otimes \tau$ is an irreducible representation of $W_{n,r}$. Moreover, all irreducible representations of $W_{n,r}$ may be constructed this way,*

$$\mathrm{Res}_{\,W^r_{n-1,r}} \left(\mathrm{Ind}^{\,W_{n,r}}_{\,W_{n-1,r}\ \mathrm{wr}\ \mathbb{Z}/d\mathbb{Z}}\, \sigma \otimes \tau \right) = \sigma^1 \oplus \cdots \oplus \sigma^{r/d},$$

and

$$\mathrm{Res}_{\,W_{n-1,r}} \left(\mathrm{Ind}^{\,W_{n,r}}_{\,W_{n-1,r}\ \mathrm{wr}\ \mathbb{Z}/d\mathbb{Z}}\, \sigma \otimes \tau \right) = \bigoplus_{j=1}^{r/d} \overline{d_j} \rho_j .$$

2.1. The number of irreducible representations of $W_{n,r}$

Given the construction of $W_{n,r}$, we may recursively count the number $k_n(r)$ of irreducible representations of $W_{n,r}$ as follows. The number of irreducible representations of G_i is $|G_i|$ since G_i is abelian. Summing over the orbit representatives, we see that the number of irreducible representations of $W_{n,r}$ is

$$k_n(r) = \sum_{\sigma_i} |G_i|,$$

since each orbit representative is tensored by an irreducible representation of G_i to obtain an irreducible representation of $W_{n,r}$. Let R be the complete set of irreducible representations of $W^r_{n-1,r}$. Let $\sigma \in R$ and let $G_\sigma \subset \mathbb{Z}/r\mathbb{Z}$ be the stabilizer of σ. The orbit containing σ has size $|\mathbb{Z}/r\mathbb{Z}|/|G_\sigma| = r/|G_\sigma|$. Thus we also have that

$$k_n(r) = \sum_{\sigma_i} |G_i| = \sum_{\sigma \in R} \left(\frac{r}{|G_\sigma|}\right)^{-1} |G_\sigma| = \frac{1}{r} \sum_{\sigma \in R} |G_\sigma|^2.$$

For every divisor d of r, there is a unique subgroup of $\mathbb{Z}/r\mathbb{Z}$ that is isomorphic to $\mathbb{Z}/d\mathbb{Z}$. Moreover, all subgroups of $\mathbb{Z}/r\mathbb{Z}$ are of this form. If we let $f(d)$ be the number of $\sigma \in R$ such that $\mathbb{Z}/d\mathbb{Z} = G_\sigma$, then we may write

$$k_n(r) = \frac{1}{r} \sum_{d|r} f(d) |\mathbb{Z}/d\mathbb{Z}|^2 = \frac{1}{r} \sum_{d|r} f(d) d^2.$$

To compute $f(d)$, we use Möbius inversion (see, e.g., [18]). First we define

$$g(d) = \sum_{d|c|r} f(c).$$

Note that the number $g(d)$ is the number of elements of R that are stabilized by $\mathbb{Z}/d\mathbb{Z}$, which is simply $k_{n-1}(r)^{r/d}$. By Möbius inversion, we have that

$$f(d) = \sum_{d|c|r} \mu(c/d) g(c) = \sum_{d|c|r} \mu(c/d) k_{n-1}(r)^{r/c}$$

where

$$\mu(c/d) = \begin{cases} (-1)^t & \text{if } c/d \text{ is the product of } t \text{ distinct primes,} \\ 0 & \text{otherwise.} \end{cases}$$

Thus, we have the following theorem:

Theorem 2.2. *The number $k_n(r)$ of irreducible representations of $W_{n,r}$ is given by the recursion*

$$k_n(r) = \frac{1}{r} \sum_{d|r} f(d) d^2 = \frac{1}{r} \sum_{d|c|r} \mu(c/d) k_{n-1}(r)^{r/c} d^2.$$

where $k_1(r) = r$.

In particular, if $r = p$ is prime, then

$$k_n(p) = \frac{1}{p} \sum_{d|c|p} \mu(c/d) k_{n-1}(p)^{p/c} d^2 = \frac{1}{p} \big[k_{n-1}(p) p^2 + \big(k_{n-1}(p)\big)^p - k_{n-1}(p) \big]$$

and

$$k_n(p^m) = \frac{1}{p^m} \left[k_{n-1}\big(p^m\big) p^{2m} + \sum_{i=0}^{m-1} \Big[k_{n-1}\big(p^m\big)^{p^{m-i}} - k_{n-1}\big(p^m\big)^{p^{m-i-1}} \Big] p^{2i} \right].$$

Example.

$$k_n(2) = \frac{1}{2} \big[2^2 k_{n-1}(2) + k_{n-1}(2)^2 - k_{n-1}(2) \big] = 2k_{n-1}(2) + \binom{k_{n-1}(2)}{2}$$

and

$$k_n(4) = +\frac{1}{4} \big[k_{n-1}(4)^4 + 3k_{n-1}(4)^2 + 12k_{n-1}(4) \big].$$

3. *r*-trees

In this section, we define a family of labeled trees which we call *r-trees*. We also give a recursive formula for the number of such trees. We then conclude this section with the definition of a partial order on this family of labeled trees.

Recall that a *rooted tree* is a connected simple graph without cycles and with a distinguished vertex called the *root*. A vertex is said to be at *level* j if the distance from it to the root is j. If x is a vertex at level j that is connected to vertex y at level $j+1$, then y is said to be a *child* of x and x is the *parent* of y. The *branching factor* of a vertex is its number of children. A *leaf* is a vertex with branching factor zero.

If x is a vertex of a rooted tree T, then the *subtree of* T *with root* x is the connected component containing x of the forest obtained by removing the edge between x and its parent. In general, a *subtree* of T is a subtree with root y for some vertex y of T, and the *maximal subtrees* of T are the subtrees obtained by removing the root of T.

A *labeled* rooted tree is a rooted tree whose vertices have been labeled using the elements of some set. If H is a subgroup of the group of automorphisms of a rooted tree T, then the action of H on T induces an action of H on the labellings of T. We say that two

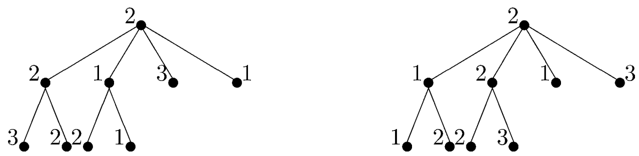


Fig. 3. Labeled rooted trees that are equivalent under the action of the full automorphism group.

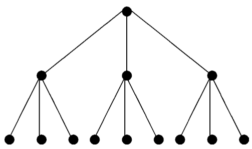


Fig. 4. Complete 3-ary tree of height 2.

labellings of T are equivalent with respect to H if they are in the same orbit under the action of H. See Fig. 3.

The *complete r-ary tree of height n* is the rooted tree whose leaves are all at level n, and whose vertices that are not leaves all have branching factor r. We are now ready to recursively define *r-trees*.

Definition. An *r-tree of height* 0 is simply a vertex labeled with an integer from $\{1, \dots, r\}$. An *r-tree of height* $n + 1$ is a labeled complete r-ary tree T of height $n + 1$ whose maximal subtrees are r-trees of height n, and whose root is labeled with an integer from $\{r/d, 2(r/d), \dots, d(r/d)\}$ where $\mathbb{Z}/d\mathbb{Z}$ is the stabilizer of the labeled maximal subtrees of T under cyclic permutation.

Finally, we say that two r-trees are equivalent if they are in the same orbit under the action of $W_{n,r}$ as described in Section 1.1. For example, notice that the trees in Fig. 5 are equivalent under the action of $W_{2,3}$.

Proposition 3.1. *There exists a* 1–1 *correspondence between r-trees of height n and the irreducible representations of $W_{n+1,r}$.*

Proof. Let z be a generator of $\mathbb{Z}/r\mathbb{Z}$ and let ω be a primitive rth root of unity. For $i = 1, \dots, r$, let ρ_i be the irreducible representation of $\mathbb{Z}/r\mathbb{Z}$ such that $\rho_i(z) = \omega^i$. Note that if $\mathbb{Z}/d\mathbb{Z}$ is a subgroup of $\mathbb{Z}/r\mathbb{Z}$, then $z^{r/d}$ is a generator for $\mathbb{Z}/d\mathbb{Z}$, and a complete set of irreducible representations of $\mathbb{Z}/d\mathbb{Z}$ are those ρ_i (restricted to $\mathbb{Z}/d\mathbb{Z}$) where $i = 1, \dots, d$. In this sense, we may view the labels of an r-tree as corresponding to irreducible representations, where the label $i(r/d)$ corresponds to the representation ρ_i (for the appropriate stabilizer).

By construction, we therefore have a bijection between r-trees of height n and the irreducible representations of $W_{n+1,r}$. In particular, the maximal subtrees of an r-tree of height n correspond to irreducible representations of $W_{n,r}$. Recall that we tensor these representations to give a representation for $W_{n,r}^r$. We then extend (trivially) to a

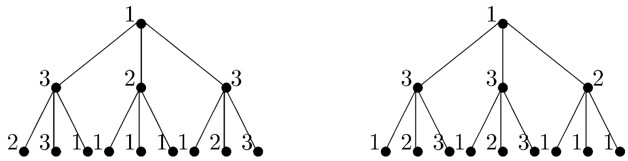


Fig. 5. Example of equivalent 3-ary trees of height 2.

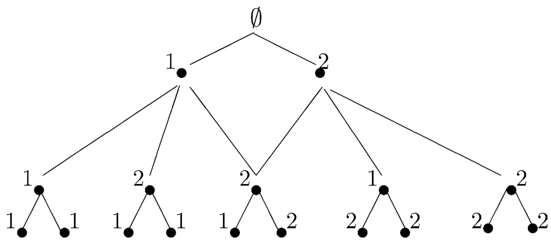


Fig. 6. The first three levels of the Hasse diagram of the poset $\mathcal{T}^{(2)}$.

representation of the subgroup $W_{n,r}$ wr $\mathbb{Z}/d\mathbb{Z}$ of $W_{n+1,r}$, where $\mathbb{Z}/d\mathbb{Z}$ is now the stabilizer of the maximal subtrees under cyclic permutation. This extension is tensored with an irreducible representation of $\mathbb{Z}/d\mathbb{Z}$, which we use to label the root of the r-tree, to give an irreducible representation of $W_{n,r}$ wr $\mathbb{Z}/d\mathbb{Z}$. Finally, this (twisted) extension is induced to give an irreducible representation of $W_{n+1,r}$. □

This bijection yields the following corollary:

Corollary 3.2. *The number $h_n(r)$ of r-trees of height n is given by the recursion*

$$h_n(r) = \frac{1}{r}\sum_{d|c|r} \mu(c/d)h_{n-1}(r)^{r/c}d^2$$

where $h_0(r) = r$.

Finally, let $\mathcal{T}^{(r)}$ denote the set of all r-trees. For each $n \geqslant 0$, let $\mathcal{T}_n^{(r)}$ be the subset of $\mathcal{T}^{(r)}$ of r-trees of height n. Note that the set $\mathcal{T}^{(r)}$ decomposes as the disjoint union of the sets $\mathcal{T}_n^{(r)}$:

$$\mathcal{T}^{(r)} := \bigcup_{n\geqslant 0} \mathcal{T}_n^{(r)} \quad \text{(disjoint union)}.$$

We define a partial order on $\mathcal{T}^{(r)}$ by saying that, if $S, T \in \mathcal{T}^{(r)}$, then $S \leqslant T$ if S is a subtree of T.

In Fig. 6, the next level would have twenty 2-trees. Note that we have added the empty set to the set $\mathcal{T}^{(r)}$. This is done for convenience. We will think of $\emptyset$ as the tree of height -1.

Proposition 3.3. *$(\mathcal{T}, \leqslant)$ is a locally finite, graded poset with rank function $h : \mathcal{T} \to \mathbb{N}$, where $h(T)$ is the height of T.*

Proof. We have already observed that $\mathcal{T}^{(r)}$ is the disjoint union of subsets according to the height function h. Moreover, by definition, if $S, T \in \mathcal{T}^{(r)}$ and $S \leqslant T$, then it must be the case that $h(S)$ is less than or equal to $h(T)$. Thus $(\mathcal{T}, \leqslant)$ is a graded poset with rank function h. Lastly, that $(\mathcal{T}, \leqslant)$ is locally finite follows directly from the fact that every r-tree has only a finite number of subtrees. □

The posets $\mathcal{T}^{(r)}$ are not lattices and they are not differentiable posets as in the case of the Young lattice.

4. r-trees and iterated wreath products

In Section 3, we observed that there is a bijection between r-trees of height n and the irreducible representations of $W_{n+1,r}$. In this section, we will use the posets $\mathcal{T}^{(r)}$ to describe the Bratteli diagrams for the sequence of groups $W_{1,r} \subset W_{2,r} \subset \cdots$. We also construct a correspondence between the r-trees of height n and the conjugacy classes of $W_{n+1,r}$.

4.1. Bratteli diagram for $W_{n-1,r} \subset W_{n,r}$

In this subsection, we construct the Bratteli diagram for the sequence of iterated wreath products of cyclic groups. Our main result is that the Hasse diagram of the poset $(\mathcal{T}^{(r)}, \leqslant)$, together with some labellings of the edges, is the Bratteli diagram of the sequence of iterated wreath products. So far we have established the correspondence of the irreducible representations with the elements of the poset. We now describe how the Hasse diagram together with the appropriate multiplicities yield the branching rules for the sequence of groups.

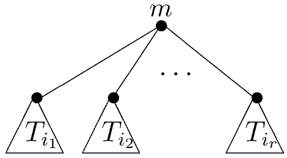
Recall that, for abelian groups, the representations and the characters are the same. We begin by ordering the irreducible representations of $W_{1,r} = \mathbb{Z}/r\mathbb{Z} = \langle z \rangle$ according to the value of the character on the generator z. We think of z as a cycle of length r. Let ω be a primitive rth root of unity. We know that the representations are given by

$$\rho_m(z) = \omega^m, \quad 1 \leqslant m \leqslant r,$$

where ρ_r is the trivial representation. Then we have the following correspondence

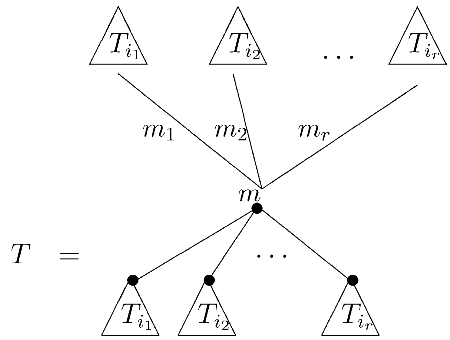
$$\rho_m \longleftrightarrow \overset{m}{\bullet}$$

for $1 \leqslant m \leqslant r$. Recall that the correspondence is given inductively. Please refer to the discussion on the representations of $W_{n,r}$ in Section 2 and the proof of Proposition 3.1. For $n \geqslant 2$, let $T_1, \ldots, T_{k_n(r)}$ denote the r-trees in $\mathcal{T}_n^{(r)}$ which index the irreducible representations of $W_{n+1,r}$. Create the labeled tree

$T =$ 

where the T_{i_j} are not necessarily all distinct r-trees and the m corresponds to a representation of the stabilizer of the maximal subtrees T_{i_j} under cyclic permutation.

In Section 3 we described an order on the r-trees given by inclusion. That is an r-tree of height n is a subtree of an r-tree T of height $n+1$ if and only if it is one of its maximal subtrees. We add the multiplicities to the edges of the Hasse diagram as follows:



where $m_j = \bar{d}_j = (d_{i_1} \cdots d_{i_r})/d_{i_j}$ and d_{i_k} is the degree of the representation indexed by T_{i_k}. By Theorem 2.1 and the above construction, we have the following theorem.

Theorem 4.1. *Let r be any positive integer. The Bratteli diagram of the sequence*

$$\mathbb{C}W_{1,r} \subset \mathbb{C}W_{2,r} \subset \cdots \subset \mathbb{C}W_{n,r} \subset \cdots$$

is the graph defined by the Hasse diagram of $(\mathcal{T}^{(r)}, \leqslant)$ where the edges are labeled by the multiplicities as described above.

Example. The Bratteli diagram of $W_{0,2} \subset W_{1,2} \subset W_{2,2}$ is given in Fig. 5. In this case all the edge weights are 1, thus we omit the labels.

If we think of the multiplicities as multiple edges, then the number of Hasse walks from the empty set at the top of the Hasse diagram to the tree T will yield the degree of the representation indexed by the r-tree T.

Corollary 4.2. *Let $T \in \mathcal{T}_n^{(r)}$ and let ρ_T denote the irreducible representation of $W_{n+1,r}$ indexed by T. Then we have the following*:

$$\operatorname{Res} \rho_T \cong \bigoplus_{\substack{h(T_i)=n-1 \\ T_i \leqslant T}} \bar{d}_i \rho_{T_i},$$

and

$$\operatorname{Ind} \rho_T \cong \bigoplus_{\substack{h(T_i)=n+1 \\ T \leqslant T_i}} \bar{d}_i \rho_{T_i}.$$

4.1.1. Degrees of the irreducible representations of $W_{n,r}$

As a consequence of Theorem 4.1 we give a purely combinatorial way to compute the degrees of the irreducible representations of $W_{n,r}$ in terms of the r-trees indexing these representations.

Recall that an r-tree, T, of height n indexes an irreducible representation of $W_{n+1,r}$. By definition of r-tree every vertex v at level j, $0 \leqslant j \leqslant n$, can be thought of as the "root" of an r-tree of height $n-j$ consisting of all vertices descending from v. And if $0 \leqslant j \leqslant n-1$ then every such r-subtree has r maximal descendant r-subtrees of height $n-j-1$.

Two maximal descendant r-subtrees of the vertex v are *distinct* if they are not equivalent as r-trees.

Let $d_i^{(j)}(T)$ denote the number of distinct maximal descendant subtrees of the ith vertex (from left to right) in the jth level in the r-tree T. Let T be an arbitrary r-tree T of height n corresponding to the irreducible representation ρ_T of $W_{n+1,r}$. We create an r-ary tree, C_T, of height n and we label the leaves by 1 and all other vertices are labeled by $d_i^{(j)}(T)$. We call C_T the *companion tree of* T.

Proposition 4.3. *The degree of the irreducible representation ρ_T of $W_{n+1,r}$ indexed by the r-ary tree T of height n is the product of the labels in the companion tree C_T, i.e.,*

$$d_T = d_{\rho_T} = \prod_{j=1}^{n} \prod_{i=1}^{r^j} d_i^{(j)}.$$

Proof. Let T be an r-tree of height n and let $T_{i_1}, T_{i_2}, \ldots, T_{i_r}$ be its maximal subtrees, which are not necessarily all distinct. By Theorem 4.1, we have that the degree d_T of the irreducible representation ρ_T of $W_{n+1,r}$ is equal to:

$$d_T = \sum_{j=1}^{l} m_j d_{i_j} = l(d_{i_1} d_{i_2} \cdots d_{i_r}), \tag{1}$$

where the sum is over the distinct T_{i_j}, l is the number of distinct maximal subtrees of T, i.e., $l = d_1^{(0)}$, and d_{i_j} is the degree of the irreducible representation of $W_{n,r}$ corresponding to T_{i_j}. The proof follows by induction, since d_{i_j} is the dimension of the irreducible representation of $W_{n,r}$ corresponding to r-tree T_{i_j} of height $n-1$. By the inductive hypothesis d_{i_j} is the

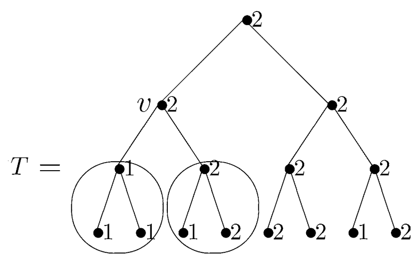


Fig. 7. Circled 2-subtrees of height 1 are maximal descendant trees with "root" at v.

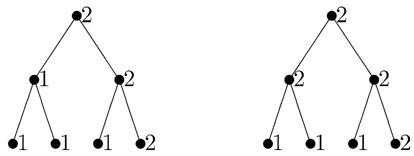


Fig. 8. Two distinct 2-trees of height 2.

product of the labels in the companion tree of T_{i_j}. Since this holds for all $j = 1, \dots, r$, we have the claim. $\square$

Corollary 4.4. *Let n_i denote the number of times that i occurs as a label of a vertex in C_T. Then*

$$d_T = 1^{n_1} 2^{n_2} \cdots r^{n_r}.$$

Proof. This is a direct consequence of Proposition 4.3 and the fact that $1 \leqslant d_i^{(j)} \leqslant r$ for all values of i and j. $\square$

Example. The representation of $W_{3,3}$ indexed by the 3-tree in Fig. 8 has degree 18.

4.2. Conjugacy classes of $W_{n,r}$

In [13], the conjugacy classes of G wr S_n are labeled by m-tuples of partitions such that the sum of the weights of all m partitions is n, where m is the number of conjugacy classes of G. Notice that for the case $r = 2$, this will yield a labeling set for the conjugacy classes of $W_{n,2}$, since $S_2 \cong \mathbb{Z}_2$. Our labeling, however, uses 2-trees and appears to be much more natural. In particular, our labeling easily generalizes to the group $W_{n,r}$.

We first describe the conjugacy classes of $W_{n,r}$. Notice that a necessary condition for two elements $(a; \pi_1), (b; \pi_2) \in W_{n,r}$ to be conjugate is that $\pi_1 = \pi_2$. Let $g^{(k)} := (e, \dots, e, g, e, \dots, e) \in (W_{n-1,r})^r$ where g is in the kth position. Straightforward computations show the following lemma:

Lemma 4.5. *Let π be a cyclic permutation not equal to the identity.*

(a) *The elements $(g_1^{(1)} g_2^{(2)} \cdots g_{r/d}^{(r/d)}; \pi)$ and $(h_1^{(1)} h_2^{(2)} \cdots h_{r/d}^{(r/d)}; \pi) \in W_{n,r}$ are conjugate if and only if g_i and h_i are conjugate in $W_{n-1,r}$.*
(b) *For any $g \in W_{n-1,r}$, $1 \leqslant k, l \leqslant r$, the elements $(g^{(l)}; \pi)$ and $(g^{(k)}; \pi)$ are conjugate.*

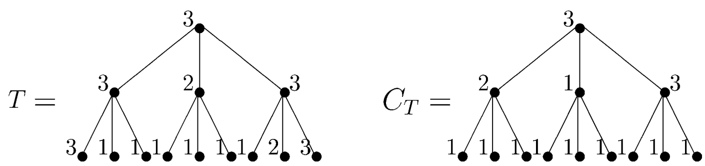


Fig. 9. A tree T and its companion tree C_T.

(c) *Every element of the form* $(g;\pi)$, *where* $g \in (W_{n-1,r})^r$ *and the order of* π *is* d, *is conjugate to some element of the form* $(h_1^{(1)}h_2^{(2)}\cdots h_{r/d}^{(r/d)};\pi)$.

(d) *Let* ι *be the identity in* $\mathbb{Z}/r\mathbb{Z}$ *and* $g \in W_{n-1,r}$. $(g;\iota)$ *is conjugate to* $(h;\iota)$ *if and only if there is a permutation* $\pi \in \mathbb{Z}/r\mathbb{Z}$ *such that* g_i *is conjugate to* $h_{\pi^{-1}(i)}$ *in* $W_{n-1,r}$.

Note. In Section 2 we computed a formula for the number of irreducible representations of $W_{n,r}$ and in Section 3 we gave a bijection between r-trees and irreducible representations. Therefore, since $W_{n,r}$ is a finite group, the number of irreducible representations is the same as the number of conjugacy classes.

Using Lemma 4.3 we now proceed to show how we can use r-trees to index the conjugacy classes of $W_{n,r}$. By Lemma 4.3, we have that if $\{c_i\}$ is a complete set of conjugacy class representatives of $W_{n-1,r}$, then a complete set of conjugacy class representatives for $W_{n,r}$ is

$$\bigcup_{\substack{\pi\in\mathbb{Z}/r\mathbb{Z}\\ \operatorname{ord}(\pi)=d}} \bigl\{\bigl(c_{i_1}^{(1)}c_{i_2}^{(2)}\cdots c_{i_{r/d}}^{(r/d)};\pi\bigr)\bigr\},$$

where c_{i_j}'s are conjugacy class representatives of $W_{n-1,r}$ (not necessarily all distinct) and $\operatorname{ord}(\pi)$ denotes the order of π.

Proposition 4.6. *There is a one-to-one correspondence between* r*-trees of height* n *and conjugacy classes of* $W_{n+1,r}$.

Proof. The proof is by induction on n. For $n=1$, $W_{1,r}=\mathbb{Z}/r\mathbb{Z}=\langle z=(1\ 2\ \cdots\ r)\rangle$, that is the group generated by the r-cycle z. For $1\leqslant k\leqslant r$ we have the following correspondence.

$$z^k \longleftrightarrow {}^{k}\bullet$$

For $n>1$, assume that the conjugacy classes of $W_{n,r}$ are labeled by r-trees of height $n-1$ in such a way that tree T_i corresponds to the conjugacy class represented by c_i. Then the conjugacy classes of $W_{n+1,r}$ are labeled by r-trees as follows:

Consider a conjugacy class $(c_{i_1}^{(1)}c_{i_2}^{(2)}\cdots c_{i_{r/d}}^{(r/d)};\pi)$, where $\pi=z^k$ for some $1\leqslant k\leqslant r$ is of order d and z is as above. The r-tree corresponding to this conjugacy class is constructed recursively. Label the root with k. On the r children of the root, attach the trees $T_{c_{i_1}}, T_{ci_2},\ldots,T_{c_{i_{r/d}}}$ from left to right on the first r/d vertices, then attach the same trees in the same order for the next r/d children of the root, and continue attaching the trees in this order a total of d times. The tree constructed is an r tree. Since we know that every conjugacy class has this form, this implies that two distinct conjugacy classes will correspond to distinct trees. Furthermore, the number of r-trees and the number of conjugacy classes are the same. This provides a bijection. □

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