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The geometric structure of unit dual quaternion with application in kinematic control

Xiangke Wang [a,b,*], Dapeng Han [c], Changbin Yu [b], Zhiqiang Zheng [a]

[a] College of Mechatronics and Automation, National University of Defense Technology, Changsha 410073, China
[b] Research School of Engineering, Australian National University, Canberra ACT 0200, Australia
[c] College of Aerospace and Material Engineering, National University of Defense Technology, Changsha 410073, China

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ABSTRACT

In this paper, the geometric structure, especially the Lie-group related properties, of unit dual quaternion is investigated. The exponential form of unit dual quaternion and its approximate logarithmic mapping are derived. Correspondingly, Lie-group and Lie-algebra on unit dual quaternions and the approximate logarithms are explored, respectively. Afterwards, error and metric based on unit dual quaternion are given, which naturally result in a new kinematic control model with unit dual quaternion descriptors. Finally, as a case study, a generalized proportional control law using unit dual quaternion is developed.

1. Introduction

A unit dual quaternion is a composition of a unit quaternion and a translational vector using Plücker coordinate. It can represent an arbitrary transformation (including rotation and translation) in 3-D (three-dimensional) space globally without singularity. In some comparative studies, see e.g. [1–3] and the references therein, it is reported that among different tools for describing transformation, such as HTM (homogeneous transformation matrix), quaternion/vector pair, Lie-algebra and alike, unit dual quaternion offers the most compact and computationally efficient screw transformation formalism. So far, the dual quaternion has turned out to be a useful tool in many research areas, such as computer-aided geometric design [4], image-based localization [5], hand-eye calibration [6], manipulators control [7] and navigation [8].

Till now, the most popular representation tool for transformation is still SE(3), which is a composition of all HTMs [9]. Many advantages of SE(3), especially in control design, lie in its geometric structure as a Lie-group. For example, a unified framework of mechanical control is established on the geometric structure of SE(3) in [10]; the generalized proportional-derivative control law is conducted in [11] based on the relationship between SE(3) and its Lie-algebra se(3). Recently, the unit dual quaternion has also been applied in control design, such as in [12–14] for transformation control. However, the geometric structure of unit dual quaternions is seldom revealed.

In this study, parallel to the work in [11], we will explore the geometric structure of unit dual quaternions; moreover, as an application, the kinematic model and control law design are also discussed. To our best knowledge, this is the first attempt to employ Lie-group related geometric structures on unit dual quaternions by providing the exponential form of unit dual quaternion and its logarithm, and deriving an approximate logarithmic mapping for unit dual quaternion. Further we prove that the set of unit dual quaternions is a Lie-group and the set of the approximate logarithmic mappings is its corresponding Lie-algebra. These results provide a new Lie-group and geometric viewpoint to understand the meanings of

* Corresponding author at: College of Mechatronics and Automation, National University of Defense Technology, Changsha 410073, China.
E-mail address: xkwang@nudt.edu.cn (X. Wang).

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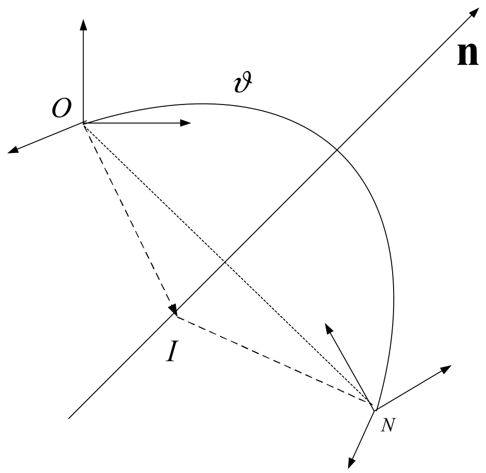

Fig. 1. Geometry of rotation: a new frame N is obtained by a frame O rotation about a unit axis $\boldsymbol{n}$ with angle $0 \leqslant \vartheta < 2\pi$.

unit dual quaternions, which are certainly non-trivial. On another hand, based on the explored properties, a classical control scheme, the proportional control law, is implemented into the unit dual quaternion Lie-group. Compared with the conventional decoupling methods on transformation control, the proposed scheme provides a unified solution without requiring decoupling to deal with the attitude and position control problem simultaneously with a global representation. Further, the control model and control method are both rooted in the derived mathematical foundation on unit dual quaternion in this paper, which makes it different to the existing results, such as the work in [12]. It is also worth pointing out that the proof of the stability for the control law is with a different method compared with that in [12].

The rest of this paper is organized as follows. Section 2 develops some geometric properties of unit dual quaternions, namely, the exponential form of unit dual quaternion and the corresponding approximate logarithmic mapping, Lie-group, Lie-algebra, left-invariant error, and metric on unit dual quaternion are all discussed. To provide a case study in control design, the kinematic control model from the left-invariant error is derived in Section 3, and then the generalized proportional control law based on dual quaternion's logarithmic mapping is designed by making use of the relationship between Lie-group and Lie-algebra. Simulations are provided in Section 4 to validate the proposed control law with the urban search and rescue simulation (USARSim) platform and quad-rotor model. The last section concludes the paper.

Notations. We employ 0 simply to denote the scalar zero. Three-dimensional vector $(0,0,0)^T$, dual number $0+\epsilon 0$ and dual vector $(0,0,0)^T+\epsilon(0,0,0)^T$ are denoted by $\mathbf{0}$, $\hat{0}$ and $\hat{\mathbf{0}}$, respectively. We denote unit quaternion $[1,0,0,0]$ and unit dual quaternion $[1,0,0,0]+\epsilon[0,0,0,0]$ by I and $\hat{\mathrm{I}}$, respectively. If not otherwise stated, a (dual) vector is denoted by the boldface, and its corresponding (dual) vector quaternion is denoted by the normal type, for example, $v=[0,\boldsymbol{v}]$ or $\hat{v}=[\hat{0},\hat{\boldsymbol{v}}]$.

2. Geometric structure

This section focuses on introducing the geometric structure of unit dual quaternions. Readers are referred to Appendix A for more details about the definitions and properties of unit dual quaternion.

2.1. Exponential and logarithm

A unit quaternion can be used to describe a rotation. For the frame rotation about a unit axis $\boldsymbol{n}$ with angle $0 \leqslant \vartheta < 2\pi$ (see Fig. 1), there is a unit quaternion

$$q=\left[\cos\left(\frac{\vartheta}{2}\right), \sin\left(\frac{\vartheta}{2}\right)\boldsymbol{n}\right] \tag{1}$$

relating a fixed vector expressed in the original frame $\boldsymbol{r}^N$ with the same vector expressed in the new frame $\boldsymbol{r}^O$ by $r^N = q^* \circ r^O \circ q$, where $r^O=[0,\boldsymbol{r}^O]$ and $r^N=[0,\boldsymbol{r}^N]$ are two vector quaternions.

The unit quaternion in (1) can be represented in exponential form, that is $q=e^{[0,\frac{\vartheta}{2}\boldsymbol{n}]}$. Correspondingly, the logarithm of a unit quaternion in (1) is $\ln q=[0,\frac{\vartheta}{2}\boldsymbol{n}]=\frac{\theta}{2}$, where $\theta=[0,\boldsymbol{\theta}]=[0,\vartheta\boldsymbol{n}]$ is a vector quaternion [15].

A unit dual quaternion can be used to represent a transformation in 3-D space. Considering a rotation q succeeded by a translation $\boldsymbol{p}^b$, according to the Chasles Theorem (refer to Theorem 2.11 in [9]), this transformation is equivalent to a screw motion, which is a rotation about a unit axis $\boldsymbol{n}$ with angle $0 \leqslant \vartheta < 2\pi$ combined with a translation d parallel to $\boldsymbol{n}$

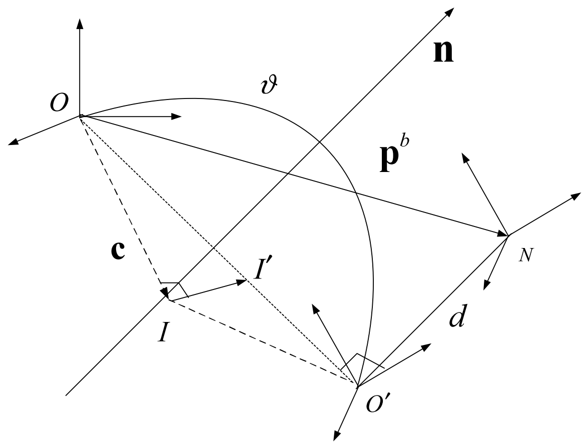

Fig. 2. Geometry of screw motion: every screw motion can be modeled as a rotation with angle $0 \leqslant \vartheta < 2\pi$ about a unit axis $\boldsymbol{n}$ and a subsequent translation d along $\boldsymbol{n}$.

illustrated in Fig. 2. Obviously, the Plucker coordinate of the rotational axis $\boldsymbol{n}$ is

$$\hat{\boldsymbol{n}} = \boldsymbol{n} + \epsilon(\boldsymbol{c} \times \boldsymbol{n}), \tag{2}$$

where $\boldsymbol{c} = \frac{1}{2}(\boldsymbol{p}^b - d\boldsymbol{n} + \cot(\frac{\vartheta}{2})\boldsymbol{n} \times \boldsymbol{p}^b)$ is the vector from the original position to the rotational center.[1] The above transformation can be represented by a unit dual quaternion

$$\hat{q} = \left[\cos\frac{\hat{\theta}}{2}, \sin\frac{\hat{\theta}}{2}\hat{\boldsymbol{n}}\right] = q + \frac{\epsilon}{2} q \circ p^b, \tag{3}$$

where $\hat{\theta} = \vartheta + \epsilon d$ is the dual angle of the screw [8]. Conversely, the screw motion can be generated by a screw $[\boldsymbol{n}, \boldsymbol{v}]$, where $\boldsymbol{v} = \boldsymbol{c} \times \boldsymbol{n} + \mu\boldsymbol{n}$ and $\mu = d/\vartheta$ [9]. Thus the Plucker coordinate of this screw motion is

$$\hat{\boldsymbol{J}} = \boldsymbol{n} + \epsilon\boldsymbol{v} = \boldsymbol{n} + \epsilon(\boldsymbol{c} \times \boldsymbol{n} + \mu\boldsymbol{n}). \tag{4}$$

Correspondingly, considering the dual quaternion representation (3) and the Plucker coordinate (4), we have

Lemma 1. *The exponential form of unit dual quaternion in* (3) *is* $\hat{q} = e^{\frac{\vartheta}{2}\hat{J}} = e^{[\hat{0}, \frac{1}{2}\hat{\theta}\hat{\boldsymbol{n}}]}$, *where* $\hat{J} = [\hat{0}, \hat{\boldsymbol{J}}]$.

Proof. In this proof, we denote $\hat{v}^n = \underbrace{\hat{v} \circ \cdots \circ \hat{v}}$, where $\hat{v}$ is a dual quaternion. Substituting (2) into (4), we obtain $\hat{\boldsymbol{J}} = (1 + \epsilon\mu)\hat{\boldsymbol{n}}$. Therefore, we have

$$\hat{J} = \left[\hat{0}, (1+\epsilon\mu)\hat{\boldsymbol{n}}\right], \tag{5}$$

$$\hat{J}^2 = \left[\hat{0}, (1+\epsilon\mu)\hat{\boldsymbol{n}}\right] \circ \left[\hat{0}, (1+\epsilon\mu)\hat{\boldsymbol{n}}\right] = \left[-(1+2\epsilon\mu), \hat{\boldsymbol{0}}\right], \tag{6}$$

$$\hat{J}^3 = \left[-(1+2\epsilon\mu), \hat{\boldsymbol{0}}\right] \circ \left[\hat{0}, (1+\epsilon\mu)\hat{\boldsymbol{n}}\right] = \left[\hat{0}, -(1+3\epsilon\mu)\hat{\boldsymbol{n}}\right], \tag{7}$$

$$\hat{J}^4 = \left[\hat{0}, -(1+3\epsilon\mu)\hat{\boldsymbol{n}}\right] \circ \left[\hat{0}, (1+\epsilon\mu)\hat{\boldsymbol{n}}\right] = \left[(1+4\epsilon\mu), \hat{\boldsymbol{0}}\right], \tag{8}$$

$$\hat{J}^5 = \left[(1+4\epsilon\mu), \hat{\boldsymbol{0}}\right] \circ \left[\hat{0}, (1+\epsilon\mu)\hat{\boldsymbol{n}}\right] = \left[\hat{0}, (1+5\epsilon\mu)\hat{\boldsymbol{n}}\right], \tag{9}$$

$$\cdots.$$

According to the Taylor's series expansion, we have

$$e^{\frac{\vartheta}{2}\hat{J}} = 1 + \frac{\vartheta}{2}\hat{J} + \left(\frac{\vartheta}{2}\hat{J}\right)^2\frac{1}{2!} + \left(\frac{\vartheta}{2}\hat{J}\right)^3\frac{1}{3!} + \left(\frac{\vartheta}{2}\hat{J}\right)^4\frac{1}{4!} + \left(\frac{\vartheta}{2}\hat{J}\right)^5\frac{1}{5!} + \cdots.$$

Using (5)–(9), we obtain

[1] Superscripts b and s relate to the body-frame (which is attached to the rigid-body) and the spatial-frame (which is relative to a fixed (inertial) coordinate frame) respectively throughout this paper. The concepts of body-frame and spatial-frame come from [9].

$$\begin{aligned}
e^{\frac{\vartheta}{2}\hat{J}} &= \left(1 - \left(\frac{\vartheta}{2}\right)^2 + \left(\frac{\vartheta}{2}\right)^4 \frac{1}{4!} - \cdots\right) + \left(\frac{\vartheta}{2} - \left(\frac{\vartheta}{2}\right)^3 \frac{1}{3!} + \left(\frac{\vartheta}{2}\right)^5 \frac{1}{5!} - \cdots\right)\hat{\boldsymbol{n}} \\
&\quad - \epsilon\mu\frac{\vartheta}{2}\left(\frac{\vartheta}{2} - \left(\frac{\vartheta}{2}\right)^3 \frac{1}{3!} + \left(\frac{\vartheta}{2}\right)^5 \frac{1}{5!} - \cdots\right) + \epsilon\mu\frac{\vartheta}{2}\left(1 - \left(\frac{\vartheta}{2}\right)^2 + \left(\frac{\vartheta}{2}\right)^4 \frac{1}{4!} - \cdots\right)\hat{\boldsymbol{n}} \\
&= \cos\frac{\vartheta}{2} + \sin\frac{\vartheta}{2}\hat{\boldsymbol{n}} + \epsilon\frac{d}{2}\left(-\sin\frac{\vartheta}{2} + \cos\frac{\vartheta}{2}\hat{\boldsymbol{n}}\right) \\
&= \cos\frac{\hat{\theta}}{2} + \sin\frac{\hat{\theta}}{2}\hat{\boldsymbol{n}} = \hat{q}.
\end{aligned}$$

Further, using (4) and $\mu\vartheta = d$, we obtain

$$\frac{\vartheta}{2}\hat{\boldsymbol{J}} = \frac{1}{2}\big(\vartheta\boldsymbol{n} + \epsilon\big((\boldsymbol{c}\times\boldsymbol{n})\vartheta + d\boldsymbol{n}\big)\big) = \frac{1}{2}(\vartheta + \epsilon d)\big(\boldsymbol{n} + \epsilon(\boldsymbol{c}\times\boldsymbol{n})\big) = \frac{1}{2}\hat{\theta}\hat{\boldsymbol{n}},$$

and therefore complete the proof. □

A similar Clifford algebra exponential is discussed in [16]. According to Lemma 1, we can obtain the logarithmic mapping of a unit dual quaternion.

Lemma 2. *Let* $\ln\hat{1} = [\hat{0}, \hat{\boldsymbol{0}}]$. *For a unit dual quaternion defined by* (3)*, its logarithmic mapping is*

$$\ln\hat{q} = \frac{\vartheta}{2}\hat{J} = \left[\hat{0}, \frac{1}{2}\hat{\theta}\hat{\boldsymbol{n}}\right]. \tag{10}$$

Proof. The result can be concluded directly by observing the fact that $\ln\hat{1} = [\hat{0}, \hat{\boldsymbol{0}}]$ and the result in Lemma 1. □

The representation of the unit dual quaternion's logarithmic mapping in (10) is not intuitive, due to the complexities of $\hat{\boldsymbol{n}}$ or $\hat{\boldsymbol{J}}$. It is not convenient to use this logarithmic mapping directly in many applications, such as control design. In many existing literature concerning the designing of controllers and observers for systems on $SO(3)$, instead of using the actual logarithm of unit quaternion, an approximate function that is quick and easy to compute algebraically is adopted [17–19].

Similarly, we give an approximate and simple representation for a unit dual quaternion's logarithmic mapping by exploring the geometric significance of (10).

Let us go back to Fig. 2. First, we make $\overrightarrow{II'} \perp \boldsymbol{c}$, where I is the rotational center of arc $\widehat{OO'}$ and I' is on the line $\overrightarrow{OO'}$. Clearly, $\overrightarrow{II'} \perp \boldsymbol{n}$ too. Denote $r = |\boldsymbol{c}|$. Then we obtain $(\boldsymbol{c}\times\boldsymbol{n}) = r\overrightarrow{II'}/|\overrightarrow{II'}|$, which implies the norm of $(\boldsymbol{c}\times\boldsymbol{n})\vartheta$ is $r\vartheta$ (or arc $\widehat{OO'}$) and vector $(\boldsymbol{c}\times\boldsymbol{n})\vartheta$ is parallel to $\overrightarrow{II'}$. When $\vartheta \to 0$, we have $r\vartheta \to |\overrightarrow{OO'}|$ and the direction of $\overrightarrow{II'}$ is to the direction of $\overrightarrow{OO'}$ too, i.e., $(\boldsymbol{c}\times\boldsymbol{n})\vartheta \to \overrightarrow{OO'}$. Further, from (4), we have

$$\hat{\boldsymbol{J}}\vartheta = \vartheta\boldsymbol{n} + \epsilon\big((\boldsymbol{c}\times\boldsymbol{n})\vartheta + d\boldsymbol{n}\big). \tag{11}$$

By replacing $(\boldsymbol{c}\times\boldsymbol{n})\vartheta$ in (11) with $\overrightarrow{OO'}$, we can get

$$\hat{\boldsymbol{J}}\vartheta \approx \big(\vartheta\boldsymbol{n} + \epsilon\big(\overrightarrow{OO'} + d\boldsymbol{n}\big)\big) = \vartheta\boldsymbol{n} + \epsilon\boldsymbol{p}^b.$$

Correspondingly, we have the following definition for unit dual quaternion's logarithmic mapping.

Definition 1. An approximate logarithmic mapping for the unit dual quaternion represented in (3) is defined by

$$\ln\hat{q} = \frac{1}{2}\big(\theta + \epsilon p^b\big), \tag{12}$$

which is a dual vector quaternion with $\theta = [0, \boldsymbol{\theta}] = [0, \vartheta\boldsymbol{n}]$ and $p^b = [0, \boldsymbol{p}^b]$.

2.2. Lie-group and Lie-algebra

Denote the sets of unit quaternions and corresponding logarithms by Q_u and V, respectively. It is proven in [20] that Q_u is a Lie-group and V is the Lie-algebra of Q_u. Note that we employ DQ_u to denote the set of unit dual quaternions in the sequel.

Theorem 3. DQ_u *is a Lie-group under dual quaternion multiplication.*

Proof. The proof is composed of two steps. First, we prove that DQ_u is a group.

1. *Identity element*. Clearly, $\hat{\mathrm{I}}$ is the identity element of DQ_u.
2. *Inverse element*. The conjugate of a unit dual quaternion is its inverse element in DQ_u.
3. *Closure*. For any $\hat{q}_1$ and $\hat{q}_2 \in DQ_u$, using $\hat{q}_1 \circ \hat{q}_1^* = \hat{q}_2 \circ \hat{q}_2^* = \hat{\mathrm{I}}$, we have $(\hat{q}_1 \circ \hat{q}_2) \circ (\hat{q}_1 \circ \hat{q}_2)^* = \hat{q}_1 \circ \hat{q}_2 \circ \hat{q}_2^* \circ \hat{q}_1^* = \hat{\mathrm{I}}$, which indicates $\hat{q}_1 \circ \hat{q}_2$ is also in DQ_u.
4. *Associate*. For any $\hat{q}_1, \hat{q}_2$ and $\hat{q}_3 \in DQ_u$, it can be verified directly that $(\hat{q}_1 \circ \hat{q}_2) \circ \hat{q}_3 = \hat{q}_1 \circ (\hat{q}_2 \circ \hat{q}_3)$, which indicates associate law holds in DQ_u.

Therefore, DQ_u is a group under '$\circ$'.

Second, we prove that DQ_u is a manifold. For any $\hat{q} = q_r + \epsilon q_d \in DQ_u$, we have $\hat{q} \circ \hat{q}^* = q_r \circ q_r^* + \epsilon(q_r \circ q_d^* + q_r^* \circ q_d) = \hat{\mathrm{I}}$, which indicates $q_r \circ q_r^* = \mathrm{I}$ and $q_r^* \circ q_d = [0, \boldsymbol{t}]$, where $[0, \boldsymbol{t}]$ is a vector quaternion. Consequently, we know q_r and q_d can be expressed by $q_r \in Q_u$ and $q_d = [0, \boldsymbol{t}] \circ q_r$. It is well known that Q_u is diffeomorphic to manifold S^3 [21], thus we conclude that Q_u is a manifold with three dimensions. So, q_r and q_d are both manifolds with three dimensions, and DQ_u is a manifold with six dimensions.

Finally, for any $\hat{q}_1$ and $\hat{q}_2 \in DQ_u$, denote $F(\hat{q}_1, \hat{q}_2) = \hat{q}_1 \circ \hat{q}_2^{-1}$. Clearly, $F(\hat{q}_1, \hat{q}_2)$ is C^∞.

Therefore, based on the above discussions, we can conclude that DQ_u is a Lie-group under dual quaternion multiplication. $\square$

The space formed by all logarithmic mappings of unit dual quaternions defined in (12) is denoted by $\hat{\mathrm{V}}$. Clearly, $\hat{\mathrm{V}}$ is equivalent to the set of all dual vector quaternions. Given any two dual vector quaternions $\hat{v}_1$ and $\hat{v}_2 \in \hat{\mathrm{V}}$, we define '[ ]' operator as

$$[\hat{v}_1, \hat{v}_2] = \hat{v}_1 \circ \hat{v}_2 - \hat{v}_2 \circ \hat{v}_1.$$

With the newly defined '[ ]' operator, we have

Theorem 4. *$\hat{\mathrm{V}}$ with operator '[ ]' is the Lie-algebra of the Lie-group DQ_u.*

Proof. We conclude this result by firstly proving that $\hat{\mathrm{V}}$ with operator '[ ]' is a Lie-algebra, and then showing that $\hat{\mathrm{V}}$ is isomorphic to the tangent space of Lie group DQ_u at identity $\hat{\mathrm{I}}$.

For any λ_1 and $\lambda_2 \in \mathbb{R}$ and for all $\hat{v}_1, \hat{v}_2$ and $\hat{v}_3 \in \hat{\mathrm{V}}$, we can obtain the following properties:

1. *Bilinear*.
$$\begin{aligned}[\lambda_1 \hat{v}_1 + \lambda_2 \hat{v}_2, \hat{v}_3] &= \lambda_1 \hat{v}_1 \circ \hat{v}_3 + \lambda_2 \hat{v}_2 \circ \hat{v}_3 - \lambda_1 \hat{v}_3 \circ \hat{v}_1 - \lambda_2 \hat{v}_3 \circ \hat{v}_2 \\ &= \lambda_1 [\hat{v}_1, \hat{v}_3] + \lambda_2 [\hat{v}_2, \hat{v}_3].\end{aligned}$$
2. *Antisymmetric*.
Let $\hat{v}_1 = v_{r1} + \epsilon v_{d1}$ and $\hat{v}_2 = v_{r2} + \epsilon v_{d2}$. According to the multiplication between dual quaternions (see (A.9) in Appendix A), we have
$$[\hat{v}_1, \hat{v}_2] = (v_{r1} \circ v_{r2} - v_{r2} \circ v_{r1}) + \epsilon(v_{d1} \circ v_{r2} - v_{r2} \circ v_{d1} + v_{r1} \circ v_{d2} - v_{d2} \circ v_{r1}).$$
Further, according to the quaternion multiplication (see (A.3) in Appendix A), we have
$$[\hat{v}_1, \hat{v}_2] = 2[0, \boldsymbol{v}_{r1} \times \boldsymbol{v}_{r2}] + 2\epsilon[0, \boldsymbol{v}_{d1} \times \boldsymbol{v}_{r2} + \boldsymbol{v}_{r1} \times \boldsymbol{v}_{d2}].$$
Clearly, we have $[\hat{v}_1, \hat{v}_2] = -[\hat{v}_2, \hat{v}_1]$.
3. *Jacobi identity*.
$$\begin{aligned}\big[\hat{v}_1, [\hat{v}_2, \hat{v}_3]\big] &= \hat{v}_1 \circ [\hat{v}_2, \hat{v}_3] - [\hat{v}_2, \hat{v}_3] \circ \hat{v}_1 \\ &= \hat{v}_1 \circ \hat{v}_2 \circ \hat{v}_3 - \hat{v}_1 \circ \hat{v}_3 \circ \hat{v}_2 - \hat{v}_2 \circ \hat{v}_3 \circ \hat{v}_1 + \hat{v}_3 \circ \hat{v}_2 \circ \hat{v}_1.\end{aligned}$$
Similarly, we obtain
$$\begin{aligned}\big[\hat{v}_2, [\hat{v}_3, \hat{v}_1]\big] &= \hat{v}_2 \circ \hat{v}_3 \circ \hat{v}_1 - \hat{v}_2 \circ \hat{v}_1 \circ \hat{v}_3 - \hat{v}_3 \circ \hat{v}_1 \circ \hat{v}_2 + \hat{v}_1 \circ \hat{v}_3 \circ \hat{v}_2, \\ \big[\hat{v}_3, [\hat{v}_1, \hat{v}_2]\big] &= \hat{v}_3 \circ \hat{v}_1 \circ \hat{v}_2 - \hat{v}_3 \circ \hat{v}_2 \circ \hat{v}_1 - \hat{v}_1 \circ \hat{v}_2 \circ \hat{v}_3 + \hat{v}_2 \circ \hat{v}_1 \circ \hat{v}_3.\end{aligned}$$
Therefore, we have $[\hat{v}_1, [\hat{v}_2, \hat{v}_3]] + [\hat{v}_2, [\hat{v}_3, \hat{v}_1]] + [\hat{v}_3, [\hat{v}_1, \hat{v}_2]] = 0$.

Therefore the '[ ]' operator is a Lie-bracket and $\hat{\mathrm{V}}$ is a Lie-algebra.

We employ $T_e(DQ_u)$ to denote the tangent space of Lie group DQ_u at identity $\hat{\mathrm{I}}$. For any $\hat{q} \in DQ_u$ and the corresponding element $T\hat{q}$ in $T_e(DQ_u)$, we have

$$T\hat{q} = \dot{\hat{q}}|_{\hat{q}=\hat{\mathrm{I}}} = \frac{1}{2}\hat{q} \circ \xi^b\bigg|_{\hat{q}=\hat{\mathrm{I}}} = \frac{1}{2}\xi^b\bigg|_{\hat{q}=\hat{\mathrm{I}}}.$$

Therefore $T_e(DQ_u)$ is the space composing of all ξ^b. Further, the twist ξ^b is represented by $[\hat{0}, \boldsymbol{\xi}^b]$ with

$$\boldsymbol{\xi}^b = \boldsymbol{\omega}^b + \epsilon\big(\dot{\boldsymbol{p}}^b + \boldsymbol{\omega}^b \times \boldsymbol{p}^b\big),$$

which is also a dual vector quaternion. Thus $T_e(DQ_u)$ is isomorphic to $\hat{\mathrm{V}}$.

It is well known that a Lie algebra is the tangent space of a Lie group at identity [22]. So, we can conclude that $\hat{\mathrm{V}}$ with operator '[ ]' is the Lie-algebra of the Lie-group DQ_u. □

Definition 2 *(Adjoint transformation).* For a unit (dual) quaternion $q \in Q_u$ or $\hat{q} \in DQ_u$, and a unit (dual) vector quaternion $v \in \mathrm{V}$ or $\hat{v} \in \hat{\mathrm{V}}$, the *adjoint transformation* is

$$Ad_q v = q \circ v \circ q^{-1} = q \circ v \circ q^* \quad \text{or} \quad Ad_{\hat{q}} \hat{v} = \hat{q} \circ \hat{v} \circ \hat{q}^{-1} = \hat{q} \circ \hat{v} \circ \hat{q}^*.$$

2.3. Norm, error, and metric

For the logarithmic mapping of a unit dual quaternion defined by (12), we give a norm definition on $\hat{\mathrm{V}}$, that is

Definition 3. The norm of the logarithmic mapping (12) is defined by

$$\|\ln \hat{q}\| = \alpha\|\boldsymbol{\theta}\| + \beta\big\|\boldsymbol{p}^b\big\|, \tag{13}$$

where $\|\cdot\|$ is the standard 2-norm, α and β are positive real numbers.

This definition is very similar to the norm definition in se(3) in [23]. It is easy to validate that (13) satisfies the norm definition.

Consider unit dual quaternions $\hat{q}$ defined by (3) and $\hat{q}_d$ defined by

$$\hat{q}_d = q_d + \frac{\epsilon}{2} q_d \circ p_d^b. \tag{14}$$

We can describe the dual quaternion error using a multiplicative dual quaternion as

$$\hat{q}_e = fl(\hat{q}, \hat{q}_d) = \hat{q}^* \circ \hat{q}_d. \tag{15}$$

Clearly, for any new unit dual quaternion $\hat{q}_a \in DQ_u$, we have $fl(\hat{q}_a \circ \hat{q}, \hat{q}_a \circ \hat{q}_d) = fl(\hat{q}, \hat{q}_d)$, which is left-invariant. Similar verification shows that $fl(\hat{q}, \hat{q}_d)$ is not right-invariant. We called such error *left-invariant error*.

Lemma 5. *For $\hat{q}$ in* (3) *and $\hat{q}_d$ in* (14)*, the left-invariant error $\hat{q}_e$ defined by* (15) *can be rewritten by*

$$\hat{q}_e = q_e + \frac{\epsilon}{2} q_e \circ p_e^b, \tag{16}$$

where $q_e = q^* \circ q_d$ and $p_e^b = p_d^b - Ad_{q_e^*} p^b$.

Proof. Substituting (3) and (14) into (15), we have

$$\hat{q}_e = \left(q + \frac{\epsilon}{2} q \circ p^b\right)^* \circ \left(q_d + \frac{\epsilon}{2} q_d \circ p_d^b\right) = q^* \circ q_d + \frac{\epsilon}{2} q^* \circ q_d \circ \big(p_d^b - Ad_{(q^* \circ q_d)^*} p^b\big).$$

Together with $q_e = q^* \circ q_d$ and $p_e^b = p_d^b - Ad_{q_e^*} p^b$, we complete the proof. □

Denote $\theta_e^b = 2\ln q_e$. According to (12) and (16), we obtain

$$\ln \hat{q}_e = \frac{1}{2}\big(\theta_e^b + \epsilon p_e^b\big). \tag{17}$$

Definition 4 *(Left-invariant metric).* For unit dual quaternions $\hat{q}$ and $\hat{q}_d$, based on left-invariant error (16), we define

$$dl(\hat{q},\hat{q}_d) = 2\|\ln\hat{q}_e\| = \alpha\left\|\boldsymbol{\theta}_e^b\right\| + \beta\left\|\boldsymbol{p}_e^b\right\|, \tag{18}$$

where α and β are positive real numbers.

Lemma 6. *Definition* 4 *is a left-invariant metric.*

Proof.

1. Definition 4 is well defined as $dl(\hat{q},\hat{q}_d) = 0$ if and only if $\hat{q}_d = \hat{q}$.
2. Using $\hat{q} = q + \frac{1}{2}q \circ p^b$ and $\hat{q}_d = q_d + \frac{\epsilon}{2}q_d \circ p_d^b$, we have $dl(\hat{q}_d,\hat{q}) = 2\alpha\|\ln(q_d^* \circ q)\| + \beta\|p^b - Ad_{q_e}p_d^b\| = 2\alpha\|\ln q_e^*\| + \beta\|-Ad_{q_e^*}p_e^b\| = 2\alpha\|\ln q_e\| + \beta\|p_e^b\| = dl(\hat{q},\hat{q}_d)$, hence Definition 4 is symmetric.
3. Definition 4 satisfies the triangle inequality as for any new $\hat{q}_a \in DQ_u$, we have

$$\begin{aligned}
dl(\hat{q},\hat{q}_d) &= 2\alpha\left\|\ln\left(q^* \circ q_d\right)\right\| + \beta\left\|p_d^b - Ad_{(q^*\circ q_d)^*}p^b\right\| \\
&= 2\alpha\left\|\ln\left(q^* \circ q_a \circ q_a^* \circ q_d\right)\right\| + \beta\left\|p_d^b - Ad_{(q_a^*\circ q_d)^*}p_a^b + Ad_{(q_a^*\circ q_d)^*}\left(p_a^b - Ad_{(q^*\circ q_a)^*}p^b\right)\right\| \\
&\leqslant 2\alpha\left\|\ln\left(q^* \circ q_a\right)\right\| + \beta\left\|p_a^b - Ad_{(q^*\circ q_a)^*}p^b\right\| + 2\alpha\left\|\ln\left(q_a^* \circ q_d\right)\right\| + \beta\left\|p_d^b - Ad_{(q_a^*\circ q_d)^*}p_a^b\right\| \\
&= dl(\hat{q},\hat{q}_a) + dl(\hat{q}_a,\hat{q}_d).
\end{aligned}$$

Moreover, it is clear that $dl(\hat{q},\hat{q}_d)$ is left-invariant as $\hat{q}_e$ is left-invariant. Therefore, we can conclude that Definition 4 is a left-invariant metric. □

Remark 1. Considering $\hat{q} = q + \frac{\epsilon}{2}p^s \circ q$ and $\hat{q}_d = q_d + \frac{\epsilon}{2}p_d^s \circ q_d$ represented in spatial-frame, we can also describe the dual quaternion error by $\hat{q}_{er} = fr(\hat{q},\hat{q}_d) = \hat{q}_d \circ \hat{q}^*$. Clearly, $\hat{q}_{er}$ is right-invariant rather than left-invariant, which is therefore called *right-invariant error*. Similarly, $\hat{q}_{er}$ can be written in the form of

$$\hat{q}_{er} = q_{er} + \frac{\epsilon}{2}p_e^s \circ q_{er}, \tag{19}$$

where $q_{er} = q_d \circ q^*$ and $p_e^s = p_d^s - Ad_{q_{er}}p^s$. Let $\theta_e^s = 2\ln q_{er}$, we have $\ln\hat{q}_{er} = \frac{1}{2}(\theta_e^s + \epsilon Ad_{q_{er}^*}p_e^s)$. Correspondingly, the right-invariant metric can be defined by $dr(\hat{q},\hat{q}_d) = \|\ln\hat{q}_{er}\| = \alpha\|\boldsymbol{\theta}_e^s\| + \beta\|\boldsymbol{p}_e^s\|$, where α and β are positive real numbers.

3. Case study: Kinematic based control

Almost all the work referring rigid-body transformation could be refreshed by employing the above discussions on unit dual quaternions. As a case study, the kinematic control problem is discussed in the following.

3.1. Kinematic control model

Physically left-invariance reflects the invariance of the metric with respect to choice of the body-frame, and the right-invariance reflects the invariance with respect to the spatial-frame [23]. Consider the following current configuration $\hat{q}$ defined in (3) with kinematic equation (A.12) and (A.14), and target configuration defined in (14) with kinematic equation

$$\dot{\hat{q}}_d = \frac{1}{2}\hat{q}_d \circ \xi_d^b, \tag{20}$$

$$\boldsymbol{\xi}_d^b = \boldsymbol{\omega}_d^b + \epsilon\left(\dot{\boldsymbol{p}}_d^b + \boldsymbol{\omega}_d^b \times \boldsymbol{p}_d^b\right). \tag{21}$$

On the basis of left-invariant error defined in (16), we can derive the following kinematic control model in body-frame:

Theorem 7 *(Kinematic control model in body-frame). Taking $\hat{q}$ and $\hat{q}_d$ as the current configuration and target configuration, respectively, the kinematic control model from* (16) *in body-frame is expressed by*

$$\dot{\hat{q}}_e = \frac{1}{2}\hat{q}_e \circ \xi_e^b, \tag{22}$$

where ξ_e^b is the error twist in body-frame in the form of

$$\xi_e^b = \xi_d^b - Ad_{\hat{q}_e^*}\xi^b = \left[0, \boldsymbol{\omega}_e^b\right] + \epsilon\left[0, \dot{\boldsymbol{p}}_e^b + \boldsymbol{\omega}_e^b \times \boldsymbol{p}_e^b\right] \tag{23}$$

with $\omega_e^b = \omega_d - Ad_{q_e^*}\omega^b$ and $p_e^b = p_d^b - Ad_{q_e^*}p^b$.

Proof. Differentiating both sides of (15), we obtain

$$\dot{\hat{q}}_e = \dot{\hat{q}}^* \circ \hat{q}_d + \hat{q}^* \circ \dot{\hat{q}}_d. \tag{24}$$

Substituting (A.12) and (20) into (24), we have $\dot{\hat{q}}_e = -\frac{1}{2}\xi^b \circ \hat{q}^* \circ \hat{q}_d + \frac{1}{2}\hat{q}^* \circ \hat{q}_d \circ \xi_d^b = \frac{1}{2}\hat{q}_e \circ (\xi_d^b - Ad_{\hat{q}_e^*}\xi^b)$. From (23), we have $\dot{\hat{q}}_e = \frac{1}{2}\hat{q}_e \circ \xi_e^b$ and

$$\begin{aligned}\xi_e^b &= \xi_d^b - \hat{q}_e^* \circ \xi \circ \hat{q}_e \\ &= \omega_d^b - Ad_{q_e^*}\omega^b + \epsilon\big(\big(\dot{p}_d^b + \omega_d^b \times p_d^b\big) - Ad_{q_e^*}\big(\dot{p}^b + \omega^b \times p^b\big) - Ad_{q_e^*}\omega^b \times p_e^b\big) \\ &= \omega_e^b + \epsilon\big(\big(\dot{p}_d^b - Ad_{q_e^*}p^b \times \omega_e^b - Ad_{q_e^*}\dot{p}^b\big) + \big(\omega_d^b \times p_d^b - \omega_e^b \times Ad_{q_e^*}p^b - Ad_{q_e^*}\big(\omega^b \times p^b\big) - Ad_{q_e^*}\omega^b \times p_e^b\big)\big).\end{aligned}$$

Using $p_e^b = p_d^b - Ad_{q_e^*}p^b$ yields

$$\xi_e^b = \omega_e^b + \epsilon\big(\big(\dot{p}_d^b - Ad_{q_e^*}p^b \times \omega_e^b - Ad_{q_e^*}\dot{p}^b\big) + \big(\omega_d^b \times p_d^b - Ad_{q_e^*}\omega^b \times p_d^b - \omega_e^b \times Ad_{q_e^*}p^b\big)\big).$$

By direct computations, we can verified that $\dot{p}_d^b - Ad_{q_e^*}p^b \times \omega_e^b - Ad_{q_e^*}\dot{p}^b = \dot{p}_e^b$ and $\omega_d^b \times p_d^b - Ad_{q_e^*}\omega^b \times p_d^b - \omega_e^b \times Ad_{q_e^*}p^b = \omega_e^b \times p_e^b$. Thus, it is obtained

$$\boldsymbol{\xi}_e^b = \boldsymbol{\omega}_e^b + \epsilon\big(\dot{\boldsymbol{p}}_e^b + \boldsymbol{\omega}_e^b \times \boldsymbol{p}_e^b\big).$$

Hence, we complete the proof. □

3.2. Generalized proportional control law on DQu

The control input of kinematic control model (22)–(23) is $\boldsymbol{\xi}_e^b$. We assume $\boldsymbol{\xi}_d^b$ and $\hat{q}_e$ are known in prior, then the actual control input is $\boldsymbol{\xi}^b$, which consists of angular velocity $\boldsymbol{\omega}^b$ and linear velocity $\dot{\boldsymbol{p}}^b$. Note that we have proven that DQ_u is a Lie-group and $\hat{\mathrm{V}}$ is its Lie-group in Section 2. By using these Lie-group and Lie-algebra, the generalized proportional control law is proposed as follows.

Theorem 8 *(Generalized proportional control law). For kinematic control model* (22)–(23)*, the generalized proportional control law*

$$\xi_e^b = -2\hat{k} \cdot \ln \hat{q}_e \tag{25}$$

exponentially stabilizes configuration $\hat{q}$ to configuration $\hat{q}_d$ globally, where $\hat{k} = k_r + \epsilon k_d$ is a dual vector quaternion with each nonzero component greater than zero, and symbol '·' is the dot production between two dual vector quaternions.

Proof. For $q_e = q^* \circ q_d$ given in (16), it is easy to obtain that $\dot{q}_e = \frac{1}{2}q_e \circ \omega_e^b$ by direct computations. Further, with the acid of (1), it is obtain $\omega_e^b = 2q_e^* \circ \dot{q}_e = [0, \dot{\vartheta}_e\boldsymbol{n}_e + \sin(\vartheta_e)\dot{\boldsymbol{n}}_e - 2\sin^2(\frac{\vartheta_e}{2})\boldsymbol{n}_e \times \dot{\boldsymbol{n}}_e]$. Thus, we obtain

$$\big(\boldsymbol{\theta}_e^b\big)^T\boldsymbol{\omega}_e^b = \big(\boldsymbol{\theta}_e^b\big)^T\dot{\vartheta}_e\boldsymbol{n}_e = \big(\boldsymbol{\theta}_e^b\big)^T\dot{\boldsymbol{\theta}}_e^b. \tag{26}$$

Substituting (23) and (17) into (25), we obtain

$$\omega_e^b + \epsilon\big(\dot{p}_e^b + \omega_e^b \times p_e^b\big) = -(k_r + \epsilon k_d)\cdot\big(\theta_e^b + \epsilon p_e^b\big). \tag{27}$$

Let $\boldsymbol{k}_r = (k_{r1}, k_{r2}, k_{r3})^T$ and $\boldsymbol{k}_d = (k_{d1}, k_{d2}, k_{d3})^T$. Considering the definition of dot production in (A.10), from (27), we can obtain

$$\begin{cases}\boldsymbol{\omega}_e^b = -K_r\boldsymbol{\theta}_e^b, \\ \dot{\boldsymbol{p}}_e^b = -K_d\boldsymbol{p}_e^b - \boldsymbol{\omega}_e^b \times \boldsymbol{p}_e^b,\end{cases} \tag{28}$$

where $K_r = diag(k_{r1}, k_{r2}, k_{r3})$ and $K_d = diag(k_{d1}, k_{d2}, k_{d3})$, respectively.

Consider the left-invariant metric defined in (18) as the Lyapunov function candidate V with states $\boldsymbol{\theta}_e^b$ and $\boldsymbol{p}_e^b$. Clearly, V is positive definite, and when $\|\boldsymbol{\theta}_e^b\| \to \infty$ and $\|\boldsymbol{p}_e^b\| \to \infty$, we have $V \to \infty$ too.

Differentiating (18) and using (26) and (28) yields

$$\begin{aligned}\dot{V} &= 2\alpha\big(\boldsymbol{\theta}_e^b\big)^T\dot{\boldsymbol{\theta}}_e^b + 2\beta\big(\boldsymbol{p}_e^b\big)^T\dot{\boldsymbol{p}}_e^b = -2\alpha\big(\boldsymbol{\theta}_e^b\big)^T K_r\boldsymbol{\theta}_e^b - 2\beta\big(\boldsymbol{p}_e^b\big)^T K_d\boldsymbol{p}_e^b - 2\beta\big(\boldsymbol{p}_e^b\big)^T\big(\boldsymbol{\omega}_e^b \times \boldsymbol{p}_e^b\big) \\ &= -2\alpha\big(\boldsymbol{\theta}_e^b\big)^T K_r\boldsymbol{\theta}_e^b - 2\beta\big(\boldsymbol{p}_e^b\big)^T K_d\boldsymbol{p}_e^b,\end{aligned}$$

which is negative definite.

Finally, let $k_{\min} = \min(k_{r1}, k_{r2}, k_{r3}, k_{d1}, k_{d2}, k_{d3})$, we have $\dot{V} \leqslant -2k_{\min}(\alpha\|\boldsymbol{\theta}_e^b\| + \beta\|\boldsymbol{p}_e^b\|) = -2k_{\min}V$. Thus, control law (25) guarantees configuration $\hat{q}$ exponentially converging to configuration $\hat{q}_d$ globally with converging rate $e^{-2k_{\min}}$. □

Remark 2. From right-invariant error (19), the kinematic control model in spatial-frame can be derived as

$$\dot{\hat{q}}_{er} = \frac{1}{2}\xi_e^s \circ \hat{q}_{er}, \tag{29}$$

$$\xi_e^s = \xi_d^s - Ad_{\hat{q}_{er}}\xi^s = \omega_e^s + \epsilon\left(\dot{p}_e^s + p_e^s \times \omega_e^s\right), \tag{30}$$

where $\omega_e^s = \omega_d^s - Ad_{q_{er}}\omega^s$ and $p_e^s = p_d^s - Ad_{q_{er}}p^s$. Based on (29) and (30), the similar generalized proportional control law in spatial-frame can also be conducted.

Remark 3. Control law (25) is a unified one. When $\hat{q}_d$ is a constant, it serves as a regulation law. When $\hat{q}_d$ is moving, a tracking law is achieved. Moreover, unit dual quaternion based control laws provide a harmony between rotation and translation, which can control attitude and position globally without singularity and with concise notions. A similar logarithmic feedback based control law on SE(3) is proposed in [11], where the control law is conducted on the basis of 4×4 HTM. In this study, an approximate logarithmic mapping of unit dual quaternion, rather than matrix, is used to design control law, which leads to control design being more easy and accessible.

4. Simulations on USARSim

Simulations are provided to verify the proposed control law (25) on the Urban Search And Rescue Simulation (USARSim) platform with a quad-rotor model (refer to http://usarsim.sourceforge.net/wiki/index.php/Main_Page for details of USARSim and quad-rotor model).

The workspace of quad-rotor is isomorphic to $SO(2) \otimes \mathbb{R}^3$. We assume that the rotating axis is z-axis, thus the current configuration and the target configuration can be represented by

$$\hat{q} = \left[\cos\frac{\gamma}{2}, 0, 0, \sin\frac{\gamma}{2}\right] + \frac{\epsilon}{2}\left[\cos\frac{\gamma}{2}, 0, 0, \sin\frac{\gamma}{2}\right] \circ [0, x, y, z],$$

and

$$\hat{q}_d = \left[\cos\frac{\gamma_d}{2}, 0, 0, \sin\frac{\gamma_d}{2}\right] + \frac{\epsilon}{2}\left[\cos\frac{\gamma_d}{2}, 0, 0, \sin\frac{\gamma_d}{2}\right] \circ [0, x_d, y_d, z_d],$$

respectively, where γ and γ_d are related to $SO(2)$, (x, y, z) and (x_d, y_d, z_d) are related to $\mathbb{R}^3$. For simplicity, $\hat{q}$ and $\hat{q}_d$ are described by (γ, x, y, z) and $(\gamma_d, x_d, y_d, z_d)$, respectively. We can control the angular velocity $\dot{\gamma}$ and the linear velocity $(\dot{x}, \dot{y}, \dot{z})^T$ of quad-rotor in body-frame, i.e., the control input is its twist in body-frame.

The target configuration is designed by

$$\begin{cases} \gamma_d = 0.1t + \dfrac{\pi}{3}, \\ x_d = -0.1t + 8, \\ y_d = 0.2t - 46.67, \\ z_d = 25. \end{cases} \tag{31}$$

The original configuration of quad-rotor is set to be $(0, 9.75m, -46.67m, 20m)$. We employ control law (25) to control a quad-rotor to track target configuration (31). The parameter $\hat{\boldsymbol{k}}$ in (25) is set to be $(0, 0, 0.5) + \epsilon(1, 1, 1)$ in simulations. Then the simulation results are shown in Figs. 3 to 5.

Fig. 3 shows the evolutions of the current trajectories in (γ, x, y, z) with respect to the target trajectories. Fig. 4 shows the current configuration with respect to the target configuration. Fig. 5 shows the evolution of errors, defined by (18) with $\alpha = 1$ and $\beta = 2$, versus time. In these figures, all current trajectories in (γ, x, y, z) converge to target trajectories asymptotically, and the current configuration converges to the target configuration too. Thus, control law (25) tracks the target configuration well.

5. Conclusion

A new type of rigid-body transformation group, unit dual quaternion Lie-group, is investigated in this study. From the derived exponential form of a unit dual quaternion, the properties of Lie-group DQ_u and its Lie-group $\hat{\mathrm{V}}$ are revealed, which provide a new way of research on rigid-body's transformation. Correspondingly, by utilizing the established Lie-group structure, how kinematic model and control design can be implemented for general rigid-body motion are developed as well.

Appendix A. Mathematical preliminaries

Basic notions about quaternion, dual number, and dual quaternion are reviewed. More details can be found in, for example, [5,8,12].

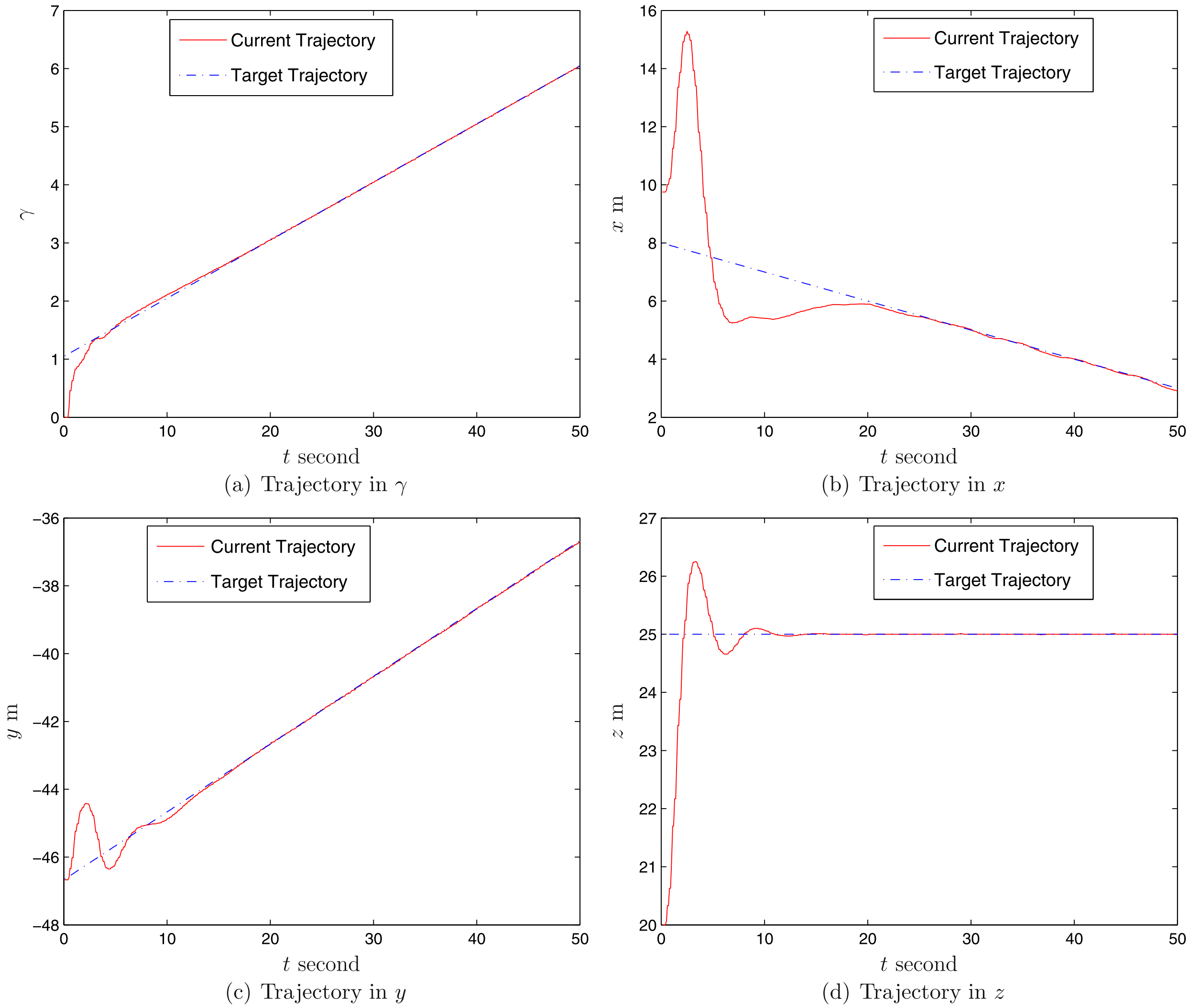

Fig. 3. Trajectories of control law (25) in (γ, x, y, z) directions versus time.

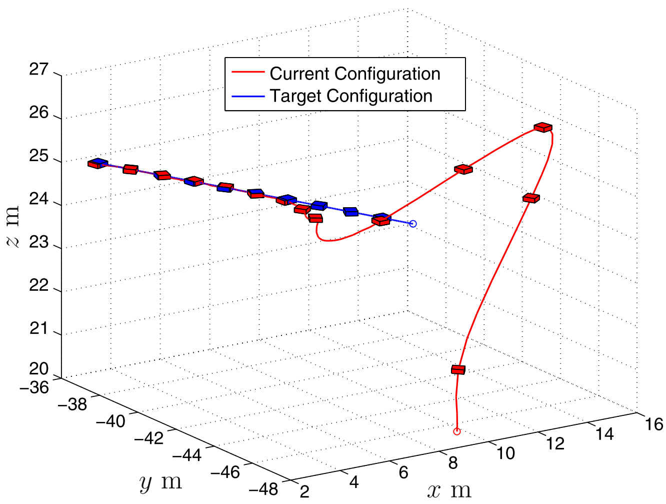

Fig. 4. Current configuration with respect to target configuration.

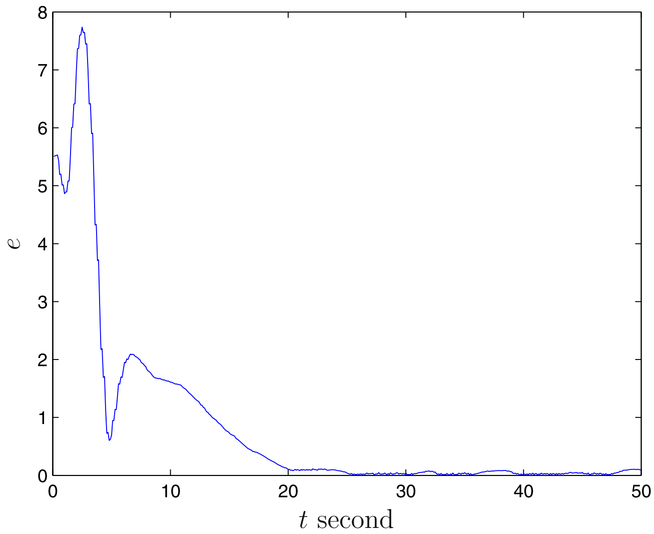

Fig. 5. Tracking errors versus time.

A.1. Quaternion

A quaternion was invented by Hamilton as an extension of a complex number to $\mathbb{R}^4$ in 1860. Formally, a quaternion q can be defined by

$$q = [s, \boldsymbol{v}], \tag{A.1}$$

where s is a scalar (called the *scalar part*), and $\boldsymbol{v}$ is a three-dimensional vector (called the *vector part*). Obviously, a three-dimensional vector can be treated equivalently as a quaternion with vanishing real part, called the *vector quaternion*.

The *conjugate* of a quaternion represented by (A.1) is $q^* = [s, -\boldsymbol{v}]$. Let λ be a scalar, the scalar production of a quaternion (A.1) is $\lambda q = [\lambda s, \lambda \boldsymbol{v}]$. For two quaternions $q_1 = [s_1, \boldsymbol{v}_1]$ and $q_2 = [s_2, \boldsymbol{v}_2]$, the *addition* and the *multiplication* operations are, respectively, defined by

$$q_1 + q_2 = [s_1 + s_2, \boldsymbol{v}_1 + \boldsymbol{v}_2], \tag{A.2}$$

$$q_1 \circ q_2 = \left[s_1 s_2 - \boldsymbol{v}_1^T \boldsymbol{v}_2, s_1 \boldsymbol{v}_2 + s_2 \boldsymbol{v}_1 + \boldsymbol{v}_1 \times \boldsymbol{v}_2\right]. \tag{A.3}$$

It is should be noted that the quaternion multiplication is associative and distributive but not commutative.

The multiplicative inverse element of a quaternion q is $q^{-1} = 1/(q \circ q^*) \circ q^*$. If $q \circ q^* = \mathrm{I}$, then q is called *unit quaternion*. For a unit quaternion, $q^{-1} = q^*$.

The kinematic equations of a unit quaternion are

$$\dot{q} = \frac{1}{2}\omega^s \circ q \quad \text{or} \quad \dot{q} = \frac{1}{2} q \circ \omega^b, \tag{A.4}$$

where ω^s and ω^b are the *angular velocity in spatial-frame* and the *angular velocity in body-frame*, respectively.

A.2. Dual number

Dual numbers were invented by Clifford in 1873 and further developed by Study in 1891. A *dual number* is defined by

$$\hat{z} = a + \epsilon b \quad \text{with } \epsilon^2 = 0, \text{ but } \epsilon \neq 0, \tag{A.5}$$

where a and b are real numbers, called the *real part* and the *dual part*, respectively, and ϵ is nilpotent such as $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. The *conjugate* of a dual number (A.5) is $\hat{z}^* = a - \epsilon b$. The scalar production of a dual number (A.5) is $\lambda \hat{z} = \lambda a + \epsilon \lambda b$, where λ is a scalar.

By definition, for two dual numbers $\hat{z}_1 = a_1 + \epsilon b_1$ and $\hat{z}_2 = a_2 + \epsilon b_2$, the following operations hold

$$\hat{z}_1 + \hat{z}_2 = (a_1 + a_2) + \epsilon(b_1 + b_2) \quad \text{and} \quad \hat{z}_1 \hat{z}_2 = (a_1 a_2) + \epsilon(a_1 b_2 + b_1 a_2).$$

It should be noted that the production of a dual number and its conjugate is $\hat{z}\hat{z}^* = (a + \epsilon b)(a - \epsilon b) = a^2$.

Dual vectors are a generalization of dual numbers whose real and dual parts are both three-dimensional vectors. By definition, for two dual vectors $\hat{\boldsymbol{v}}_1 = \boldsymbol{v}_{r1} + \epsilon \boldsymbol{v}_{d1}$ and $\hat{\boldsymbol{v}}_2 = \boldsymbol{v}_{r2} + \epsilon \boldsymbol{v}_{d2}$, we have

$$\hat{\boldsymbol{v}}_1\hat{\boldsymbol{v}}_2 = \boldsymbol{v}_{r1}^T\boldsymbol{v}_{r2} + \epsilon\big(\boldsymbol{v}_{d1}^T\boldsymbol{v}_{r2} + \boldsymbol{v}_{r1}^T\boldsymbol{v}_{d2}\big),$$
$$\hat{\boldsymbol{v}}_1 \times \hat{\boldsymbol{v}}_2 = \boldsymbol{v}_{r1} \times \boldsymbol{v}_{r2} + \epsilon(\boldsymbol{v}_{d1} \times \boldsymbol{v}_{r2} + \boldsymbol{v}_{r1} \times \boldsymbol{v}_{d2}).$$

A.3. Dual quaternion

A dual quaternion is a quaternion with dual number components, i.e., $\hat{q} = [\hat{s}, \hat{\boldsymbol{v}}]$, where $\hat{s}$ is a dual number and $\hat{\boldsymbol{v}}$ is a dual vector. A dual quaternion also can be treated as a dual number with the quaternion components, which is

$$\hat{q} = q_r + \epsilon q_d, \tag{A.6}$$

where q_r and q_d are both quaternions. The *conjugate* of a dual quaternion $\hat{q}$ in (A.6) is

$$\hat{q}^* = q_r^* + \epsilon q_d^*. \tag{A.7}$$

For two dual quaternions $\hat{q}_1 = q_{r1} + \epsilon q_{d1}$ and $\hat{q}_2 = q_{r2} + \epsilon q_{d2}$, the *addition* and the *multiplication* are

$$\hat{q}_1 + \hat{q}_2 = q_{r1} + q_{r2} + \epsilon(q_{d1} + q_{d2}), \tag{A.8}$$
$$\hat{q}_1 \circ \hat{q}_2 = q_{r1} \circ q_{r2} + \epsilon(q_{r1} \circ q_{d2} + q_{d1} \circ q_{r2}), \tag{A.9}$$

respectively. According to (A.7) and (A.9), it is obtained $(\hat{q}_1 \circ \hat{q}_2)^* = \hat{q}_2^* \circ \hat{q}_1^*$.

The multiplicative inverse element of dual quaternion is $\hat{q}^{-1} = 1/(\hat{q} \circ \hat{q}^*) \circ \hat{q}^*$. If $\hat{q} \circ \hat{q}^* = \hat{\mathrm{I}}$, then the dual quaternion $\hat{q}$ is called *unit dual quaternion*. For a unit dual quaternion, we have $\hat{q}^{-1} = \hat{q}^*$.

Obviously, a dual vector can be treated equivalently as a dual quaternion with vanishing real part, called *dual vector quaternion*. For two dual vector quaternions $\hat{k} = k_r + \epsilon k_d = [0, k_{r1}, k_{r2}, k_{r3}] + \epsilon[0, k_{d1}, k_{d2}, k_{d3}]$ and $\hat{v} = [0, \boldsymbol{v}_r] + \epsilon[0, \boldsymbol{v}_d]$, the dot production of $\hat{k}$ and $\hat{v}$ is defined by

$$\hat{k} \cdot \hat{v} = [0, K_r\boldsymbol{v}_r] + \epsilon[0, K_d\boldsymbol{v}_d], \tag{A.10}$$

where $K_r = diag(k_{r1}, k_{r2}, k_{r3})$ and $K_d = diag(k_{d1}, k_{d2}, k_{d3})$, which are both 3×3 diagonal matrices with diagonal entries k_{r1}, k_{r2}, k_{r3} and k_{d1}, k_{d2}, k_{d3}, respectively.

The kinematic equations of a unit dual quaternion are

$$\dot{\hat{q}} = \frac{1}{2}\xi^s \circ \hat{q}, \tag{A.11}$$
$$\dot{\hat{q}} = \frac{1}{2}\hat{q} \circ \xi^b, \tag{A.12}$$

where

$$\boldsymbol{\xi}^s = \boldsymbol{\omega}^s + \epsilon\big(\dot{\boldsymbol{p}}^s + \boldsymbol{p}^s \times \boldsymbol{\omega}^s\big), \tag{A.13}$$
$$\boldsymbol{\xi}^b = \boldsymbol{\omega}^b + \epsilon\big(\dot{\boldsymbol{p}}^b + \boldsymbol{\omega}^b \times \boldsymbol{p}^b\big), \tag{A.14}$$

in which ξ^s and ξ^b are called *twists*, specially, ξ^s is called twist-in-spatial-frame, and ξ^b is called twist-in-body-frame.

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