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First and second order sufficient conditions for strict minimality in nonsmooth vector optimization

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Abstract

In this paper we present first and second order sufficient conditions for strict local minima of orders 1 and 2 to vector optimization problems with an arbitrary feasible set and a twice directionally differentiable objective function. With this aim, the notion of support function to a vector problem is introduced, in such a way that the scalar case and the multiobjective case, in particular, are contained. The obtained results extend the multiobjective ones to this case. Moreover, specializing to a feasible set defined by equality, inequality, and set constraints, first and second order sufficient conditions by means of Lagrange multiplier rules are established.

Keywords: Vector optimization; Strict local minimum; First and second order sufficient optimality conditions; Support function

1. Introduction

Classical optimality conditions for differentiable programming problems with constraints are basic results in many fields, such as optimization theory, control theory, the study of stability and sensitivity in mathematical programming, the convergence of algorithms, the best approximation problem... They are also a basic support for practical applications in numerical computation, operations research, engineering, etc.

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First and second order necessary optimality conditions for programs in abstract spaces, with $\mathbb{R}$-valued or vector-valued functions, have been provided by many authors. Among those we may refer to Hoffmann and Kornstaedt [11], Ben-Tal and Zowe [1], Linnemann [16], Tang [19], Cominetti [5], Maruyama [17]. Of these, only in [1,19] second order sufficient conditions are established for differentiable programs. Borwein [4] also establishes a sufficient condition for twice Fréchet differentiable programs with equality and set constraints. In differentiable multiobjective programming (all the spaces are finite-dimensional) we refer to [3,15] and references therein. Ward [20] and Studniarski [18] study necessary and sufficient conditions for strict minima of order m in nondifferentiable scalar programs.

In [15], the authors, extending some of Hestenes's ideas to multiobjective ($f : \mathbb{R}^n \to \mathbb{R}^p$) programs, have developed a theory of first and second order sufficient conditions for strict local Pareto (the ordering cone is $\mathbb{R}^p_+$) minimality of orders 1 and 2 for f twice Fréchet differentiable. Following in this line, we try to generalize the results obtained there to vector optimization problems of the type (2.1), where f is twice directionally differentiable (according to Definition 2.2), not necessarily twice Fréchet differentiable, and the partial order in Y is given by a convex cone. With this purpose, the concept of support function to a vector problem is introduced, containing, in particular, the multiobjective and scalar cases. The obtained results generalize the classical ones for the scalar case (for example, the results collected in [9, Sections 4.6 and 4.7]), some results contained in [15], and some results provided by other authors, such as Borwein [4] or Ben-Tal and Zowe [1,2], for vector or multiobjective problems. Furthermore, specializing to a feasible set defined by equality, inequality, and set constraints, first and second order sufficient conditions are established. Finally, we remark that these sufficient conditions are close to different existing necessary conditions.

2. Notations and preliminaries

Let X be a normed space and M a subset of X. As usual we denote by $B(x_0, \delta)$ the open ball centered at x_0 and radius δ, by $\operatorname{int} M$ ($\operatorname{cl} M$) the interior (closure) of the set M and by $\operatorname{cone} M$ the cone generated by M. The topological dual space to X is X^*. If $\lambda \in X^*$ and $x \in X$, we will use λx instead of $\lambda(x)$ or the also usual $\langle \lambda, x \rangle$. If L_1 and L_2 are mappings, we will write $L_2 L_1$ for the composition $L_2 \circ L_1$.

The positive polar cone to M is $M^+ = \{\lambda \in X^*\colon\ \lambda x \geqslant 0,\ \forall x \in M\}$ and the strictly positive polar cone is $M^{s+} = \{\lambda \in X^*\colon\ \lambda x > 0,\ \forall x \in M \setminus \{0\}\}$.

We are interested in the following general vector optimization problem:

$$\min\bigl\{ f(x)\colon\ x \in M \bigr\}, \tag{2.1}$$

where $f : X \to Y$, Y is a normed space, and $M \subset X$ is arbitrary.

Throughout this paper, $D \subset Y$ is a convex closed pointed cone with nonempty interior, which defines the partial order in Y.

Let us recall that the point $x_0 \in M$ is a local minimum for problem (2.1), denoted $x_0 \in \operatorname{lmin}(f, M)$, if there exists a neighborhood U of x_0 such that

$$M \cap U \cap N_f = \emptyset, \tag{2.2}$$

where $N_f = \{x \in X\colon f(x) - f(x_0) \in -D \setminus \{0\}\}$. If, in particular, $Y = \mathbb{R}$ and $D = \mathbb{R}_+$, the usual notion of local minimum is obtained.

Checking (2.2) is not easy and, consequently, approximations to M and to N_f at the point x_0 are usually employed. One of the most used approximations is the tangent cone (Definition 2.1). In its turn, if the sets are determined by constraint functions (as N_f) we usually use the linearized cones, which are defined by directional derivatives (Definition 2.2) of the involved functions.

Definition 2.1. Let $M \subset X$ and $x_0 \in \operatorname{cl} M$. The tangent (contingent) cone to M at x_0 is

$$T(M, x_0) = \left\{v \in X\colon \exists t_n \to 0^+,\ \exists x_n \in M \text{ such that } \lim_{n\to\infty} \frac{x_n - x_0}{t_n} = v\right\}.$$

It is well known that $v \in T(M, x_0)$, with $\|v\| = 1$, if and only if there exists a sequence $x_n \to x_0$, with $x_n \in M \setminus \{x_0\}$, such that $(x_n - x_0)/\|x_n - x_0\| \to v$.

Definition 2.2. Let $f : X \to Y$ and $x_0, v \in X$.

(a) The directional derivative of f at x_0 in the direction v is

$$df(x_0, v) = \lim_{(t,u)\to(0^+,v)} \frac{f(x_0 + tu) - f(x_0)}{t}.$$

(b) The Dini derivative of f at x_0 in the direction v is

$$Df(x_0, v) = \lim_{t\to 0^+} \frac{f(x_0 + tv) - f(x_0)}{t}.$$

(c) The second order directional derivative of f at x_0 in the direction v is

$$d^2 f(x_0, v) = \lim_{(t,u)\to(0^+,v)} \frac{f(x_0 + tu) - f(x_0) - t\, df(x_0, u)}{t^2/2}.$$

(d) The second order Dini derivative of f at x_0 in the direction v is

$$D^2 f(x_0, v) = \lim_{t\to 0^+} \frac{f(x_0 + tv) - f(x_0) - t\, Df(x_0, v)}{t^2/2}.$$

(e) We say that f is directionally differentiable at x_0 if $df(x_0, v)$ exists for all $v \in X$, and f is twice directionally differentiable at x_0 if $d^2 f(x_0, v)$ exists for all $v \in X$.

Derivative (d) has been used by Hiriart-Urruty and Seeger [10], (c) has been used by Ben-Tal and Zowe [2], and (d), considering "lim inf" instead of "lim," by Studniarski [18], in these two last cases for stating sufficient optimality conditions.

If f is Fréchet differentiable at x_0, its Fréchet derivative is denoted by $\nabla f(x_0)$. If it exists, we have $\nabla f(x_0)v = df(x_0, v)$. If f is twice Fréchet differentiable at x_0, its second Fréchet derivative is denoted by $\nabla^2 f(x_0)$ that we consider as a continuous bilinear application from $X \times X$ into Y. Thus, for $v \in X$, $\nabla^2 f(x_0)(v, v)$ is a vector in Y.

It is well known that if a function f is directionally differentiable at x_0, then $df(x_0, \cdot)$ is continuous on X [7, Theorem 3.2] and f is continuous at x_0 [7, p. 28]. In the next proposition, the continuity of the second derivative is proved.

Proposition 2.3. *Let us suppose that for $f : X \to Y$ its second directional derivative $d^2 f(x_0, v)$ exists for all $v \in X$. Then the function $d^2 f(x_0, \cdot)$ is continuous on X.*

Proof. Suppose that $d^2 f(x_0, \cdot)$ is not continuous at some $v \in X$. Then there exist $\varepsilon > 0$ and $v_n \to v$ such that

$$\left\| d^2 f(x_0, v_n) - d^2 f(x_0, v) \right\| \geqslant \varepsilon, \quad \forall n \in \mathbb{N}.$$

Since

$$\lim_{k\to\infty} \frac{f(x_0 + t_k v_n) - f(x_0) - t_k\, df(x_0, v_n)}{t_k^2/2} = d^2 f(x_0, v_n)$$

for each $n \in \mathbb{N}$ and for each sequence $t_k \to 0^+$, we can find a sequence $\alpha_n \to 0^+$ such that

$$\left\| \frac{f(x_0 + \alpha_n v_n) - f(x_0) - \alpha_n\, df(x_0, v_n)}{\alpha_n^2/2} - d^2 f(x_0, v) \right\| \geqslant \frac{\varepsilon}{2}$$

for all n. But this is a contradiction, because the expression in the left side of the inequality tends to 0 when $n \to \infty$. □

It is also known that if f is Lipschitzian on a neighborhood of x_0 and the Dini derivative $Df(x_0, v)$ exists, then the directional derivative $df(x_0, v)$ also exists and they are the same. This statement is not valid for second order derivatives, as one can show with $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = |y - x^2|$ (which is Lipschitzian), $x_0 = (0, 0)$ and $v = (1, 0)$, because $D^2 f(x_0, v) = 2$ and $d^2 f(x_0, v)$ does not exist. On the other hand, obviously, if $d^2 f(x_0, v)$ exists, then $D^2 f(x_0, v)$ also exists and they are the same. Another result on equality of the second derivatives is provided in the next proposition.

Proposition 2.4. (i) *Let $f : X \to Y$ be Fréchet differentiable on a neighborhood of x_0 and suppose that $\nabla f(\cdot)$ is stable at x_0, i.e., there are $k > 0$ and $\delta > 0$ such that*

$$\left\| \nabla f(x) - \nabla f(x_0) \right\| \leqslant k\|x - x_0\|, \quad \forall x \in B(x_0, \delta). \tag{2.3}$$

If $D^2 f(x_0, v)$ exists, then $d^2 f(x_0, v)$ also exists and they are equal.

(ii) *If $f : X \to Y$ is twice Fréchet differentiable at x_0, then*

$$d^2 f(x_0, v) = \nabla^2 f(x_0)(v, v), \quad \forall v \in X.$$

Proof. (i) Taking into account the definitions of $D^2 f(x_0, v)$ and $d^2 f(x_0, v)$, it is enough to prove that

$$\lim_{(t,u)\to(0^+,v)} \frac{f(x_0 + tu) - f(x_0 + tv) - t\nabla f(x_0)(u - v)}{t^2/2} = 0. \tag{2.4}$$

The mean value theorem establishes that

$$\left\| f(z) - f(y) - \nabla f(x_0)(z - y) \right\| \leqslant \|z - y\| \sup_{x\in[y,z]} \left\| \nabla f(x) - \nabla f(x_0) \right\|.$$

Applying this inequality to $z = x_0 + tu$ and $y = x_0 + tv$ and taking into account (2.3), we deduce

$$\begin{aligned}&\big\| f(x_0+tu)-f(x_0+tv)-t\nabla f(x_0)(u-v)\big\|\\&\quad\leqslant t\|u-v\|\sup_{x\in[y,z]}\big\|\nabla f(x)-\nabla f(x_0)\big\|\leqslant kt\|u-v\|\sup_{x\in[y,z]}\|x-x_0\|.\end{aligned}$$

Now, $x=y+\theta(z-y)=x_0+tv+\theta t(u-v)$ with $\theta\in[0,1]$, and as $u\to v$ we can assume that $u\in B(v,\varepsilon)$ for some $\varepsilon>0$. Therefore,

$$\sup_{x\in[y,z]}\|x-x_0\|\leqslant t\sup_{\theta\in[0,1]}\big\|v+\theta(u-v)\big\|\leqslant t\big(\|v\|+\varepsilon\big).$$

Consequently,

$$\big\| f(x_0+tu)-f(x_0+tv)-t\nabla f(x_0)(u-v)\big\|\leqslant kt^2\|u-v\|\big(\|v\|+\varepsilon\big).$$

From here, we get (2.4).

(ii) It follows from Proposition 1.1 of Studniarski [18]. □

Let us consider the vector optimization problem (2.1). The following notion introduced in [12, Definition 3.1] is basic for the development of this paper.

Definition 2.5. Let $m\geqslant 1$ be an integer. We say that the point $x_0\in M$ is a strict local minimum of order m for problem (2.1), denoted $x_0\in \operatorname{strl}(m,f,M)$, if there exist $\alpha>0$ and a neighborhood U of x_0 such that

$$\big(f(x)+D\big)\cap B\big(f(x_0),\alpha\|x-x_0\|^m\big)=\emptyset,\quad \forall x\in M\cap U\setminus\{x_0\}.$$

We have that every strict local minimum of order m is also of order j, for all $j\geqslant m$, and every strict local minimum of order m is a local minimum, that is, $\operatorname{strl}(m,f,M)\subset \operatorname{lmin}(f,M)$ (see [12]).

This notion extends the usual notion of strict minimizer of order m [20, Definition 1.1] in scalar programming.

The next lemma provides a characterization for a point that is not a strict local minimum of order m, which will be very useful in arguments by reduction to the absurd. Its proof follows immediately from Definition 2.5. Proposition 2.7 establishes a property of the strict minima related to the composition with a continuous linear application.

Lemma 2.6. *Consider problem* (2.1). $x_0\notin \operatorname{strl}(m,f,M)$ *if and only if there exist sequences* $x_n\in M\cap B(x_0,1/n)\setminus\{x_0\}$ *and* $d_n\in D$ *such that*

$$b_n:=f(x_n)-f(x_0)+d_n\in B\left(0,\frac{1}{n}\|x_n-x_0\|^m\right).$$

Proposition 2.7. *Let* $\bar Y$ *be a normed space,* $\bar D\subset\bar Y$ *the convex cone that provides to* $\bar Y$ *a partial order,* $f:X\to Y$ *a function, and* $\psi:Y\to\bar Y$ *a positive* ($\psi(D)\subset\bar D$) *continuous linear application. If* $x_0\in\operatorname{strl}(m,\psi f,M)$*, then* $x_0\in\operatorname{strl}(m,f,M)$.

Proof. By assumption, there exist a neighborhood U of x_0 and $\alpha>0$ such that

$$\big(\psi f(x)+\bar D\big)\cap B\big(\psi f(x_0),\alpha\|x-x_0\|^m\big)=\emptyset,\quad \forall x\in M\cap U\setminus\{x_0\}.\tag{2.5}$$

Since ψ is linear and continuous, there exists $\beta > 0$ such that $\psi(B_Y(0,1)) \subset B_{\bar{Y}}(0,\beta)$ and, consequently,

$$\psi\big(B_Y(0,r)\big) \subset B_{\bar{Y}}(0,r\beta), \quad \forall r > 0, \tag{2.6}$$

where B_Y and $B_{\bar{Y}}$ denote balls in Y and $\bar{Y}$, respectively.

Let us prove that

$$\big(f(x)+D\big) \cap B\left(f(x_0), \frac{\alpha}{\beta}\|x-x_0\|^m\right) = \emptyset, \quad \forall x \in M \cap U \setminus \{x_0\}.$$

Suppose that there exist $x \in M \cap U \setminus \{x_0\}$ and $d \in D$ such that

$$f(x)+d-f(x_0) \in B\left(0, \frac{\alpha}{\beta}\|x-x_0\|^m\right).$$

Then, from (2.6) we deduce that $\psi f(x) + \psi(d) - \psi f(x_0) \in B_{\bar{Y}}(0, \alpha\|x-x_0\|^m)$ with $\psi(d) \in \bar{D}$, contradicting (2.5). □

We are going to introduce the remaining necessary notation.

Let W and Z be normed spaces, $g: X \to W$ and $h: X \to Z$ two functions, and $Q \subset X$ and $K \subset W$ two arbitrary sets. Usually, when one is trying to state necessary conditions, Q and K are convex and K, furthermore, a cone, but now we do not need these requirements. Let S be the set defined by the constraints

$$S = \big\{x \in X\colon\ g(x) \in -K,\ h(x) = 0\big\}. \tag{2.7}$$

In many instances we can provide more precise information on optimality conditions when the feasible set M of problem (2.1) has a special form. It is very common to consider that $M = S \cap Q$, and so we have three types of constraints: inequality, equality, and set constraints. We will suppose that f, g, and h are directionally differentiable at x_0.

In finite-dimensional spaces, the linearized cone is defined using the active components of g at x_0. Now it is not possible to define it this way, and instead, the linearized cone to S at x_0 is defined by

$$C(S,x_0) = \big\{v \in X\colon\ dg(x_0,v) \in \operatorname{cl}\operatorname{cone}\big(-K - g(x_0)\big),\ dh(x_0,v) = 0\big\}.$$

Obviously, it is a closed cone not necessarily convex.

For the function f we can define two linearized cones, the first one open and the second one closed, as follows:

$$C_0(f,x_0) = \big\{v \in X\colon\ df(x_0,v) \in -\operatorname{int} D\big\}$$

and

$$C(f,x_0) = \big\{v \in X\colon\ df(x_0,v) \in -D\big\}.$$

Lastly, we enunciate two lemmas for subsequent reference. The second one is an extension of Result 4.2 of Corley [6] that can be seen in [14].

Lemma 2.8. *Let $M \subset X$ be a set with nonempty interior and $\lambda \in M^+ \setminus \{0\}$. If $x \in \operatorname{int} M$, then $\lambda x > 0$.*

Proof. Though it is a well-known result (see, for example, [19, Lemma 3.3]), we provide a proof based on optimality conditions. If $\lambda x = 0$, then x is a minimum of the (differentiable) function λ on the set M, and since $x \in \operatorname{int} M$ we have that its Fréchet derivative at x is zero, $\nabla\lambda(x) = 0$. But since the Fréchet derivative of a continuous linear application is equal to itself, we deduce that $\nabla\lambda(x) = \lambda = 0$, in contradiction to the assumption. □

Lemma 2.9. *Let $f : X \to Y$ be directionally differentiable at $x_0 \in M \subset X$. If $x_0 \in \operatorname{lmin}(f, M)$ then $T(M, x_0) \cap C_0(f, x_0) = \emptyset$. In particular, if $Y = \mathbb{R}$, we have $df(x_0, v) \geqslant 0$ for all $v \in T(M, x_0)$.*

3. Support functions

In the next definition the notion of support function to a general vector problem is introduced.

Definition 3.1. Let $f : X \to Y$, $M \subset X$, $x_0 \in M$, $F : X \to \mathbb{R}$ be directionally differentiable at x_0 and $\lambda \in D^+$. We will say that the pair (λ, F) is a local support for f at x_0 on M if the following conditions hold:

(1) $F(x) \leqslant \lambda f(x)$, $\forall x \in M \cap B(x_0, \delta)$ for some $\delta > 0$;
(2) $F(x_0) = \lambda f(x_0)$;
(3) $dF(x_0, v) \geqslant 0$, $\forall v \in T(M, x_0)$;
(4) $\lambda \neq 0$.

We will say that (λ, F) is a (global) support if condition (1) is satisfied for all $x \in M$, and will say that it is a weak local support if conditions (1)–(3) are satisfied.

This definition obviously contains a scalarization process.

Remark 3.2. (1) If $X = \mathbb{R}^n$, $Y = \mathbb{R}^p$, $D = \mathbb{R}^p_+$, F is Fréchet differentiable and we replace (3) by (3′) $\nabla F(x_0) = 0$, then Definition 3.1 becomes Definition 3.1 of [15]. If, in particular, $p = 1$ (i.e., $Y = \mathbb{R}$), this definition is equivalent to stating that $\lambda^{-1}F$ is a support (in the Hestenes sense [9, p. 217]) for f.

(2) If the Fritz John conditions for the set $M = S \cap Q$ are satisfied, where Q is convex and S is given by (2.7) (assuming that f, g, and h are directionally differentiable with convex derivative), that is, there exist $\lambda \in Y^*$, $\mu \in W^*$, $\nu \in Z^*$ all nonzero such that

$$\lambda \in D^+, \quad \mu \in K^+, \quad \mu g(x_0) = 0, \tag{3.1}$$

$$\lambda\, df(x_0, v) + \mu\, dg(x_0, v) + \nu\, dh(x_0, v) \geqslant 0, \quad \forall v \in T(Q, x_0), \tag{3.2}$$

then, letting F be the Lagrangian function,

$$F = \lambda f + \mu g + \nu h, \tag{3.3}$$

we have that (λ, F) is a weak support for f at x_0 on $S \cap Q$ and the proof is easy. Obviously, if conditions (3.1) and (3.2) hold with $\lambda \neq 0$ (Kuhn–Tucker conditions), (λ, F) is a support.

The next proposition states basic properties satisfied if a support exists. Note that the first property is the first order necessary optimality condition (Lemma 2.9).

Proposition 3.3. *Let $f : X \to Y$ be directionally differentiable at $x_0 \in M \subset X$.*

(a) *If (λ, F) is a local support for f at x_0 on M, then*

$$T(M, x_0) \cap C_0(f, x_0) = \emptyset.$$

(b) *If (λ, F) is a weak local support for f at x_0 on M, F is twice directionally differentiable at x_0, $0 \in \operatorname{lmin}(dF(x_0, \cdot), M - x_0)$ and there exists $v \in T(M, x_0)$ such that $d^2F(x_0, v) > 0$, then $\lambda \neq 0$, that is, (λ, F) is a local support.*

Proof. (a) Let $\varphi(x) = \lambda f(x) - F(x)$. Conditions (1)–(3) of Definition 3.1 are equivalent to the following:

(1) $\varphi(x) \geqslant 0$, $\forall x \in M \cap B(x_0, \delta)$;
(2) $\varphi(x_0) = 0$;
(3) $dF(x_0, v) = \lambda\, df(x_0, v) - d\varphi(x_0, v) \geqslant 0$ for all $v \in T(M, x_0)$.

Conditions (1) and (2) imply that $x_0 \in \operatorname{lmin}(\varphi, M)$. Applying Lemma 2.9 it follows that $d\varphi(x_0, v) \geqslant 0$, $\forall v \in T(M, x_0)$. Taking into account condition (3), we deduce that

$$\lambda\, df(x_0, v) \geqslant 0, \quad \forall v \in T(M, x_0). \tag{3.4}$$

Reasoning “ad absurdum,” suppose that there exists $v \in T(M, x_0) \cap C_0(f, x_0)$. Then, $df(x_0, v) \in -\operatorname{int} D$. Since $\lambda \in D^+$ and $\lambda \neq 0$, by Lemma 2.8 we have $\lambda\, df(x_0, v) < 0$, contradicting (3.4).

(b) We have that $v = \lim_{n\to\infty} v_n$ for some sequences $v_n \in X$ and $t_n \to 0^+$ such that $x_n := x_0 + t_n v_n \in M$. Suppose that $\lambda = 0$. With the notation of part (a), now $\varphi(x) = -F(x)$. Hence, $d\varphi(x_0, \cdot) = -dF(x_0, \cdot)$ and $d^2\varphi(x_0, v) = -d^2F(x_0, v) < 0$. Furthermore, since $\varphi(x) \geqslant 0$, $\forall x \in M \cap B(x_0, \delta)$, $\varphi(x_0) = 0$, and $-d\varphi(x_0, x_n - x_0) = dF(x_0, x_n - x_0) \geqslant 0$, $\forall n \in \mathbb{N}$ (by assumption), it follows that

$$d^2\varphi(x_0, v) = \lim_{n\to\infty} \frac{\varphi(x_0 + t_n v_n) - \varphi(x_0) - d\varphi(x_0, x_n - x_0)}{t_n^2/2} \geqslant 0,$$

which is a contradiction. □

4. First order sufficient conditions

Theorem 4.1 and Corollary 4.3 below provide first order sufficient conditions for strict local minimality of order 1 (the first one also necessary conditions). To prove the second one we need a lemma. In the remainder of the work, we assume that the space X is finite-dimensional.

Theorem 4.1. *Let us suppose that $x_0 \in M \subset X$ and f is directionally differentiable at x_0. Then, $T(M, x_0) \cap C(f, x_0) = \{0\}$ if and only if $x_0 \in \operatorname{strl}(1, f, M)$.*

Proof. Let us suppose that $x_0 \notin \mathrm{strl}(1, f, M)$, then, by Lemma 2.6, there exist sequences $x_n \in M \cap B(x_0, 1/n) \setminus \{x_0\}$ and $d_n \in D$ such that

$$f(x_n) - f(x_0) + d_n = b_n \in B\left(0, \frac{1}{n}\|x_n - x_0\|\right). \tag{4.1}$$

Without loss of generality, since X is finite-dimensional, we can assume that

$$\lim_{n\to\infty} \frac{x_n - x_0}{\|x_n - x_0\|} = v$$

for some $v \in T(M, x_0)$ with $\|v\| = 1$. Dividing in (4.1) by $\|x_n - x_0\|$ and taking the limit, we have

$$\lim_{n\to\infty}\left(\frac{f(x_n) - f(x_0)}{\|x_n - x_0\|} + \frac{d_n}{\|x_n - x_0\|}\right) = 0.$$

Since the first term within the limit converges to $df(x_0, v)$, we have that the second term also converges to a certain vector $d \in D$ because D is closed. Therefore, $df(x_0, v) = -d \in -D$, and consequently $v \in T(M, x_0) \cap C(f, x_0) = \{0\}$, which is a contradiction.

Now let us see the converse. Let $x_0 \in \mathrm{strl}(1, f, M)$. By definition there exist $\alpha > 0$ and a neighborhood U of x_0 such that

$$\big(f(x) + D\big) \cap B\big(f(x_0), \alpha\|x - x_0\|\big) = \emptyset, \quad \forall x \in M \cap U \setminus \{x_0\}. \tag{4.2}$$

Suppose that there exists $v \in T(M, x_0) \cap C(f, x_0)$, $v \neq 0$. We can suppose that $\|v\| = 1$. Since v belongs to the tangent cone, there exists a sequence $x_n \in M \setminus \{x_0\}$ converging to x_0 such that $\lim_{n\to\infty}((x_n - x_0)/t_n) = v$, being $t_n = \|x_n - x_0\|$. Since f is directionally differentiable, we deduce that $\lim_{n\to\infty}((f(x_n) - f(x_0))/t_n) = df(x_0, v) \in -D$ because $v \in C(f, x_0)$. Set $df(x_0, v) = -d_0 \in -D$. For the previous $\alpha > 0$, there exists $n_0 \in \mathbb{N}$ such that $(f(x_n) - f(x_0))/t_n \in -d_0 + B(0, \alpha)$ for all $n \geqslant n_0$. Hence, $f(x_n) + t_n d_0 \in f(x_0) + B(0, \alpha t_n)$, which is in contradiction to (4.2). □

This theorem generalizes Theorem 4.6.3 of Hestenes [9] and, partially, Corollary 3.2 in [13], in which it is assumed $Y = \mathbb{R}^p$ and $D = \mathbb{R}^p_+$. Notice that in the converse the finite-dimensionality of X is not used.

If the cone D has a compact base, from this theorem we deduce, taking into account Theorem 4.5 in [14], that every strict local minimum of order 1 is a local proper Borwein efficient solution of type 2 (see [14]).

Lemma 4.2. *Let S be given by* (2.7)*, $Q \subset X$, $x_0 \in S \cap Q$, and g and h directionally differentiable at x_0. Then*

$$T(S \cap Q, x_0) \subset C(S, x_0) \cap T(Q, x_0).$$

Proof. Since $T(S \cap Q, x_0) \subset T(S, x_0) \cap T(Q, x_0)$, it is enough to prove that

$$T(S, x_0) \subset C(S, x_0).$$

By definition of the tangent cone, given $v \in T(S, x_0)$ there exist sequences $t_n \to 0^+$ and $x_n \in S$ such that $\lim_{n\to\infty}((x_n - x_0)/t_n) = v$. Since g is directionally differentiable it follows that $\lim_{n\to\infty}((g(x_n)-g(x_0))/t_n) = dg(x_0,v)$. Now, $(g(x_n)-g(x_0))/t_n \in \operatorname{cone}(-K-g(x_0))$, and therefore, $dg(x_0,v) \in \operatorname{cl}\operatorname{cone}(-K-g(x_0))$.

In a similar way, $dh(x_0,v)=0$ is proved since in this case $h(x_n)=h(x_0)=0$ for all $n\in\mathbb{N}$. Consequently, $v\in C(S,x_0)$. □

The next result follows immediately from Theorem 4.1 and the lemma above.

Corollary 4.3. *Let S be given by* (2.7), *$Q\subset X$, $x_0\in S\cap Q$, and f, g, and h directionally differentiable at x_0. If $C(S,x_0)\cap T(Q,x_0)\cap C(f,x_0)=\{0\}$, then $x_0\in \operatorname{strl}(1,f,S\cap Q)$.*

In Theorem 4.4 and Corollary 4.5, sufficient conditions for strict minimality based on the notion of support function are provided, the first one for an arbitrary set and the second one for a set defined by the three kinds of constraints.

Theorem 4.4. *Let f be directionally differentiable at $x_0\in M\subset X$. If*

(a) *(λ,F) is a local support for f at x_0 on M and*
(b) *$T(M,x_0)\cap[C(f,x_0)\setminus C_0(f,x_0)]=\{0\}$,*

then $x_0\in\operatorname{strl}(1,f,M)$.

Proof. Condition (b) is equivalent to

$$T(M,x_0)\cap C(f,x_0)\cap C_0(f,x_0)^c=\{0\}. \tag{4.3}$$

By Proposition 3.3, $T(M,x_0)\cap C_0(f,x_0)=\emptyset$, hence, $T(M,x_0)\cap C_0(f,x_0)^c=T(M,x_0)$. Therefore, taking into account (4.3), it follows that $T(M,x_0)\cap C(f,x_0)=\{0\}$. By Theorem 4.1, $x_0\in\operatorname{strl}(1,f,M)$. □

Corollary 4.5. *Let S be given by* (2.7), *$Q\subset X$, $x_0\in S\cap Q$, and f, g, and h directionally differentiable at x_0. If* (*Kuhn–Tucker*) *conditions* (3.1) *and* (3.2), *$\lambda\neq 0$ hold and $C(S,x_0)\cap T(Q,x_0)\cap[C(f,x_0)\setminus C_0(f,x_0)]=\{0\}$, then $x_0\in\operatorname{strl}(1,f,S\cap Q)$.*

Proof. By Remark 3.2(2), if F is the Lagrangian function given by (3.3), (λ,F) is a support for f at x_0 on $S\cap Q$, and then it suffices to apply Lemma 4.2 and Theorem 4.4. □

Let us remark that this corollary generalizes Theorem 7.2 of Hestenes [9, Chapter 4] and, partially (there a superstrict minimum is obtained), Corollary 4.1 in [15]. Notice that convexity for Q or for the derivatives is not needed.

5. Second order sufficient conditions

In this section, different second order sufficient conditions for strict local minimality of order 2 are provided.

The following theorem establishes a sufficient condition for a strict local minimum of order 2 in a problem with an arbitrary feasible set.

Theorem 5.1. *Let $M \subset X$, $x_0 \in M$, and $f : X \to Y$ directionally differentiable at x_0. If for every $v \in T(M, x_0) \cap C(f, x_0) \setminus \{0\}$ there exists (λ, F), weak local support for f at x_0 on M, with F twice directionally differentiable at x_0 such that*

$$0 \in \operatorname{lmin}\big(dF(x_0, \cdot), M - x_0\big), \tag{5.1}$$

and $d^2F(x_0, v) > 0$, then $x_0 \in \operatorname{strl}(2, f, M)$.

Proof. Suppose that $x_0 \notin \operatorname{strl}(2, f, M)$. Then, by Lemma 2.6, there exist sequences $x_n \in M \cap B(x_0, 1/n) \setminus \{x_0\}$ and $d_n \in D$ such that

$$f(x_n) - f(x_0) + d_n = b_n \in B\left(0, \frac{1}{n} t_n^2\right), \tag{5.2}$$

where $t_n = \|x_n - x_0\|$. Choosing a subsequence, if necessary, we can assume that

$$\lim_{n\to\infty} \frac{x_n - x_0}{t_n} = v \in T(M, x_0) \quad \text{with } \|v\| = 1.$$

Dividing in (5.2) by t_n and taking the limit we obtain (as in the proof of Theorem 4.1) that $df(x_0, v) \in -D$. Therefore, $v \in T(M, x_0) \cap C(f, x_0) \setminus \{0\}$. By assumption, there exists a weak local support (λ, F) such that (5.1) holds and $d^2F(x_0, v) > 0$.

Now, applying to (5.2) the continuous linear function λ we get

$$\lambda f(x_n) - \lambda f(x_0) + \lambda d_n = \lambda b_n,$$

which can be written

$$F(x_0 + t_n v_n) - F(x_0) - t_n\, dF(x_0, v_n) + \varphi(x_n) + dF(x_0, x_n - x_0) + \lambda d_n = \lambda b_n,$$

where $v_n = (x_n - x_0)/t_n$ and $\varphi = \lambda f - F$, the function defined in the proof of Proposition 3.3. Dividing by $t_n^2/2$ and taking the limit we obtain

$$\lim_{n\to\infty} \frac{F(x_n) - F(x_0) - t_n\, dF(x_0, v_n)}{t_n^2/2} + \lim_{n\to\infty} \frac{\varphi(x_n) + dF(x_0, x_n - x_0) + \lambda d_n}{t_n^2/2} = 0.$$

As the first limit exists and is equal to $d^2F(x_0, v)$, then the second one exists and is nonnegative since $\varphi(x) \geqslant 0$ for all $x \in M \cap B(x_0, \delta)$, $dF(x_0, x_n - x_0) \geqslant 0$ by (5.1), and $\lambda d_n \geqslant 0$ because $\lambda \in D^+$. It follows that $d^2F(x_0, v) \leqslant 0$, which is a contradiction. □

Remark 5.2. (1) If $T(M, x_0) \cap C(f, x_0) = \{0\}$, by Theorem 4.1, $x_0 \in \operatorname{strl}(1, f, M)$, and, therefore, also $x_0 \in \operatorname{strl}(2, f, M)$.

(2) Notice that f is not required to be twice directionally differentiable.

(3) By virtue of Proposition 3.3, λ has to be different from 0. This applies to results from now on.

This theorem generalizes Theorem 4.6.4 of Hestenes [9] and Theorem 5.1 in [15].

In the following proposition, which is evident, we provide two conditions, each of them implying (5.1).

Proposition 5.3. *Two sufficient conditions for* (5.1) *to hold are the following*:

(i) F *is Fréchet differentiable with* $\nabla F(x_0) = 0$;
(ii) $dF(x_0, u) \geqslant 0$, $\forall u \in T(M, x_0)$, *and* $(M - x_0) \cap B(0, \delta) \subset T(M, x_0)$ *for some* $\delta > 0$.

Notice that if F is directionally differentiable at x_0 and $dF(x_0, v) = 0$ for all $v \in X$, then F is Fréchet differentiable [8, p. 266].

As a consequence of Theorem 5.1 we obtain the next corollary, in which the existence of a support function is reduced to finding a multiplier.

Corollary 5.4. *Let* $M \subset X$, $x_0 \in M$, *and* $f : X \to Y$ *twice directionally differentiable at* x_0. *If for every* $v \in T(M, x_0) \cap C(f, x_0) \setminus \{0\}$ *there exists* $\lambda \in D^+$ *such that*

$$0 \in \operatorname{lmin}\big(\lambda\, df(x_0, \cdot), M - x_0\big) \tag{5.3}$$

and $\lambda\, d^2 f(x_0, v) > 0$, *then* $x_0 \in \operatorname{strl}(2, f, M)$.

Proof. We define $F(x) = \lambda f(x) - \lambda\, df(x_0, x - x_0)$ for all $x \in X$. Let us see that (λ, F) is a weak local support for f at x_0 on M satisfying (5.1).

In fact, the condition $F(x) \leqslant \lambda f(x)$ for $x \in M \cap B(x_0, \delta)$ is clear because $\lambda f(x) - F(x) = \lambda\, df(x_0, x - x_0) \geqslant 0$ by (5.3). The condition $F(x_0) = \lambda f(x_0)$ is also clear. Find the directional derivative of F,

$$\begin{aligned} dF(x_0, w) &= \lim_{(t,u)\to(0^+,w)} \frac{F(x_0 + tu) - F(x_0)}{t} \\ &= \lim_{(t,u)\to(0^+,w)} \frac{\lambda f(x_0 + tu) - \lambda f(x_0) - \lambda\, df(x_0, tu)}{t} = 0 \end{aligned}$$

because λ is linear and continuous and $df(x_0, \cdot)$ is positively homogeneous and continuous. With this, (5.1) and condition (3) of Definition 3.1 are satisfied. Finally, it is also easy to verify that $d^2F(x_0, v) = \lambda\, d^2 f(x_0, v) > 0$, and so we can apply Theorem 5.1 to conclude. □

Although Corollary 5.4 is simple to apply, Theorem 5.1 is more general, as it can be shown with the following data: $f(x, y) = (x + 2y^2, y - y^2)$, $M = \{(x, y)\colon -x - y^2 \leqslant 0\}$, $D = \mathbb{R}^2_+$, and $x_0 = (0, 0)$. We have that $F(x, y) = y^2$, with $\lambda = (1, 0)$, is a support satisfying the conditions of Theorem 5.1 on the vectors of $T(M, x_0) \cap C(f, x_0) \setminus \{0\} = \{(0, y)\colon y < 0\}$, so $x_0 \in \operatorname{strl}(2, f, M)$. But, there is no λ satisfying the hypotheses of Corollary 5.4.

In the following results we study other sufficient conditions in which the support function does not change with the vector.

Proposition 5.5. *Let* $f : X \to Y$ *twice directionally differentiable at* $x_0 \in M \subset X$. *Suppose that one of the following conditions is satisfied*:

(i) *There exists* $\lambda \in D^+$ *such that* (5.3) *holds and*

$$\lambda\, d^2 f(x_0, v) > 0, \quad \forall v \in T(M, x_0) \cap C(f, x_0) \setminus \{0\};$$

(ii) *There exists* $\lambda \in D^{s+}$ *such that* (5.3) *holds and*

$$\lambda\, d^2 f(x_0, v) > 0, \quad \forall v \in T(M, x_0) \cap \operatorname{Ker} df(x_0, \cdot) \setminus \{0\}.$$

Then $x_0 \in \operatorname{strl}(2, f, M)$.

Proof. Case (i) follows from Corollary 5.4.

(ii) Let us prove that $T(M, x_0) \cap C(f, x_0) = T(M, x_0) \cap \operatorname{Ker} df(x_0, \cdot)$.

Choose $v \in T(M, x_0) \cap C(f, x_0)$, then $df(x_0, v) \in -D$. Suppose that $df(x_0, v) \neq 0$; then since $\lambda \in D^{s+}$ one has $\lambda\, df(x_0, v) < 0$. On the other hand, if we define $\varphi(x) = \lambda\, df(x_0, x - x_0)$ we have $d\varphi(0, u) = \lambda\, df(x_0, u)$, $\forall u \in X$, and as $0 \in \operatorname{lmin}(\varphi, M - x_0)$ by (5.3), it follows that $d\varphi(0, u) \geqslant 0$ for all $u \in T(M - x_0, 0) = T(M, x_0)$ by Lemma 2.9. In particular, $\lambda\, df(x_0, v) = d\varphi(0, v) \geqslant 0$, and we have a contradiction. Accordingly, $v \in \operatorname{Ker} df(x_0, \cdot)$.

Now, part (i) applies and we obtain the result. □

In the following proposition another possibility with $\lambda \in D^+$ is considered.

Proposition 5.6. *Let* $f : X \to Y$ *be twice directionally differentiable at* $x_0 \in M \subset X$, $\bar{Y}$ *a normed space equipped with the order induced by the convex cone* $\bar{D} \subset \bar{Y}$ *and* $\lambda \in D^+$. *Suppose that there exist a positive continuous linear application* $\psi : Y \to \bar{Y}$ *and* $\bar{\lambda} \in \bar{D}^{s+}$ *satisfying* $\lambda = \bar{\lambda}\psi$ *and such that* (5.3) *holds and* $\lambda\, d^2 f(x_0, v) > 0$, $\forall v \in T(M, x_0) \cap \operatorname{Ker} \psi\, df(x_0, \cdot) \setminus \{0\}$. *Then* $x_0 \in \operatorname{strl}(2, f, M)$.

Proof. Setting $f_0 = \psi f$, then, by assumption, we have $0 \in \operatorname{lmin}(\bar{\lambda}\, df_0(x_0, \cdot), M - x_0)$ and $\bar{\lambda}\, d^2 f_0(x_0, v) > 0$, $\forall v \in T(M, x_0) \cap \operatorname{Ker} f_0(x_0) \setminus \{0\}$. By Proposition 5.5, $x_0 \in \operatorname{strl}(2, f_0, M)$, and by Proposition 2.7, $x_0 \in \operatorname{strl}(2, f, M)$. □

This proposition is especially interesting if $Y = \mathbb{R}^p$ and the cone D is polyhedral, $D = \{y \in \mathbb{R}^p\colon\ Ay \geqslant 0\}$ being $A : \mathbb{R}^p \to \mathbb{R}^k$ linear, because $\psi = A$, with $\bar{D} = \mathbb{R}^k_+$, satisfies the hypotheses in a natural way.

Corollary 5.7. *Let* $X = \mathbb{R}^n$, $Y = \mathbb{R}^p$, $D = \mathbb{R}^p_+$, *and* $f : \mathbb{R}^n \to \mathbb{R}^p$ *be twice directionally differentiable at* $x_0 \in M \subset \mathbb{R}^n$. *If there exists* $\lambda \in \mathbb{R}^p_+$ *such that* (5.3) *holds and*

$$\lambda\, d^2 f(x_0, v) > 0,$$
$$\forall v \in T(M, x_0) \cap \left\{v \in \mathbb{R}^n\colon\ \lambda_i\, df_i(x_0, v) = 0,\ i = 1, \ldots, p\right\},\ v \neq 0,$$

then $x_0 \in \operatorname{strl}(2, f, M)$.

Proof. Rearranging, we can suppose, without loss of generality, that

$$\lambda = (\lambda_1, \ldots, \lambda_k, 0, \ldots, 0) \quad \text{with } k \geqslant 1 \text{ and } \lambda_1 > 0, \ldots, \lambda_k > 0.$$

In Proposition 5.6 we choose $\psi : \mathbb{R}^p \to \mathbb{R}^k$ given by $\psi(y_1, \ldots, y_p) = (y_1, \ldots, y_k)$, $\bar{D} = \mathbb{R}^k_+$, and $\bar{\lambda} = (\lambda_1, \ldots, \lambda_k)$, which allow us to conclude. □

If in particular, $p = 1$, we deduce the following corollary for scalar optimization, that in spite of its simplicity (especially if f is twice Fréchet differentiable) we have not found in the literature.

Corollary 5.8. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be twice directionally differentiable at $x_0 \in M \subset \mathbb{R}^n$. If $0 \in \operatorname{lmin}(df(x_0,\cdot), M - x_0)$ and $d^2 f(x_0, v) > 0$, $\forall v \in T(M, x_0) \cap \operatorname{Ker} df(x_0,\cdot) \setminus \{0\}$, then $x_0 \in \operatorname{strl}(2, f, M)$.*

As an illustrative example, consider $f(x, y) = y + x^2 - y^2$, $M = \{(x, y) \in \mathbb{R}^2\colon\ y \geqslant \sin^2(1/x)$ if $x \neq 0$, $y \geqslant 0$ if $x = 0\}$ and $x_0 = (0, 0)$. Obviously, Corollary 5.8 applies.

If $M = \mathbb{R}^n$, Theorem 3.2 of Ben-Tal and Zowe [2] follows from this corollary taking into account Proposition 2.4.

Next the general result, Theorem 5.1, is applied to the case in which $M = S \cap Q$ comes defined by inequality, equality and set constraints.

Theorem 5.9. *Let S be given by* (2.7), *$Q \subset X$, and f, g, h twice directionally differentiable at $x_0 \in S \cap Q$. If for every $v \in C(S, x_0) \cap T(Q, x_0) \cap C(f, x_0) \setminus \{0\}$ there exist $(\lambda, \mu, \nu) \in D^+ \times K^+ \times Z^*$ such that calling $L = \lambda f + \mu g + \nu h$ the following conditions hold*:

(a) $\mu g(x_0) = 0$;
(b) $0 \in \operatorname{lmin}(dL(x_0,\cdot), S \cap Q - x_0)$;
(c) $d^2 L(x_0, v) > 0$.

Then $x_0 \in \operatorname{strl}(2, f, S \cap Q)$.

Proof. Let $F(x) = L(x) - dL(x_0, x - x_0)$, $\forall x \in X$. It is proved, as in another occasions, that (λ, F) is a weak local support for f at x_0 on $S \cap Q$ with $dF(x_0,\cdot) = 0$ (condition (a) is needed to verify that $F(x_0) = \lambda f(x_0)$). On the other hand, $d^2 F(x_0, v) = d^2 L(x_0, v) > 0$, so Theorem 5.1 allows us to conclude because $T(S \cap Q, x_0) \subset C(S, x_0) \cap T(Q, x_0)$. □

If g is not considered, $Y = \mathbb{R}$, $D = \mathbb{R}_+$, and f and h are twice Fréchet differentiable, Theorem 9.2 of Borwein [4] follows from the previous theorem.

If f, g, and h are of C^1 class on a neighborhood of x_0, this theorem is close to Corollary 3.1 of Maruyama [17], in which second order necessary conditions are stated for scalar programs. Notice that for this class of functions it can be proved that $d^2 f(x_0, v) = 2 f^{(2)}(x_0, v, 0)$, this last derivative being the (parabolic) derivative used by Maruyama [17, Definition 2.2].

If in particular $Q = X$, we deduce the following corollary.

Corollary 5.10. *Let S be given by* (2.7), *$x_0 \in S$, and f, g, h twice Fréchet differentiable at x_0. If for every $v \in C(S, x_0) \cap C(f, x_0) \setminus \{0\}$ there exists a Lagrangian function $L = \lambda f + \mu g + \nu h$ such that $(\lambda, \mu, \nu) \in D^+ \times K^+ \times Z^*$, $\mu g(x_0) = 0$, $\nabla L(x_0) = 0$, and $\nabla^2 L(x_0)(v, v) > 0$, then $x_0 \in \operatorname{strl}(2, f, S)$.*

If we take into account that every strict local minimum of order 2 is a strict minimum, this corollary together with Corollary 4.3 (for differentiable functions) become Theorem 11.1 of Ben-Tal and Zowe [1]. Notice, as these authors point out in Example 1, that if X is not finite-dimensional, the result is not valid.

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