Absolute Flatness of the Full Transformation Semigroups

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The purpose of the present paper is to prove the following.
Let $\mathcal{F}_X$ be the full transformation semigroup on a set $X$, that is, the semigroup of all mappings $X$ into itself, with composition being from left to right.

THEOREM A. $\mathcal{F}_X$ is left absolutely flat if and only if $X$ is finite.

In [5], T. E. Hall showed that every semigroup with the strong representation extension property is an amalgamation base in the class of all semigroups and that a semigroup $S$ has the strong representation extension property if and only if $S$ has both the representation extension property and the free representation extension property. J. M. Howie [7] and S. Bulman-Fleming and K. McDowell [3] made the observation that a semigroup $S$ is left absolutely flat if and only if $S$ has the free representation extension property. T. E. Hall [5, Remark 1], N. M. Khan (unpublished), and K. Shoji [8] showed that the full transformation semigroups have the representation extension property. Consequently, we have

THEOREM B. $\mathcal{F}_X$ has the strong representation extension property if and only if $|X| < \infty$.

Also we obtain

COROLLARY C. $\mathcal{F}_X$ is an amalgamation base in the class of all semigroups if $|X| < \infty$. 

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Remark 1. In the case where \( X \) is infinite, it remains open whether or not \( \mathcal{F}_X \) is an amalgamation base for all semigroups. However, by using a result of C. J. Ash [1, Theorem 2.1(i)] and Corollary C, we know that there are many subsemigroups of \( \mathcal{F}_X \) which are amalgamation bases for semigroups, for example, the subsemigroup \( \{ s \in \mathcal{F}_X \mid s \text{ is finite subset of } X \} \). Moreover, let \( \mathcal{F}_X = \{ s \in \mathcal{F}_X \mid |X| < \infty \} \), a subsemigroup of \( \mathcal{F}_X \). Our proof of Theorem 1 is valid for the monoid \( \mathcal{U} \) and hence \( \mathcal{U} \) has the free representation extension property, while a minor modification of the proof of [8, Theorem 4.2] shows that \( \mathcal{U} \) has the representation extension property. Consequently, it follows that \( \mathcal{U} \) has the strong representation extension property.

Remark 2. The author has also shown that the results above hold true for the partial transformation semigroup on any finite set.

1. Preliminaries

Throughout this paper, let \( S \) denote a monoid (with an identity \( 1_S \)). As far as we can, we use notations and definitions from Clifford and Preston [4] for semigroup theory.

We recall some definitions and results, which will be used in the sequel, from Bulman-Fleming and McDowell [2].

**Definition 1.1.** A monoid \( S \) is called left absolutely flat if every left \( S \)-set is flat.

**Result 1.2** (Cf. [2, Lemma 2.2]). A left \( S \)-set \( B \) is flat if and only if, for every right \( S \)-set \( A \), and every \( a, a' \in A, b, b' \in B, a \otimes b = a' \otimes b' \) in \( A \otimes B \) implies \( a \otimes b = a' \otimes b' \) in \( (aS \cup a'S) \otimes B \), where \( \otimes \) means tensor product (over \( S \)).

**Result 1.3** [2, Lemma 1.2]. In the notations above, \( a \otimes b = a' \otimes b' \) in \( A \otimes B \) if and only if there exist \( a_1, ..., a_n \in A, b_1, ..., b_n \in B, s_1, ..., s_n \) and \( t_1, ..., t_n \in S \) such that

\[
\begin{align*}
a &= a_1 s_1, & s_1 b &= t_1 b_2 \\
a_1 t_1 &= a_2 s_2, & s_2 b_2 &= t_2 b_3 \\
&\vdots & \vdots \\
a_{n-1} t_{n-1} &= a_n s_n, & s_n b_n &= t_n b' \\
a_n t_n &= a'.
\end{align*}
\]
We call the system of Eqs. (1.1) a scheme of length \( n \) over \( A \) and \( B \) joining \( (a, b) \) to \( (a', b') \).

In the remainder of this section, we will be concerned only with the full transformation semigroups. We will often use the following notations.

\( \mathcal{T}_X \) is the full transformation semigroup on a set \( X \) (with composition being from left to right). For any \( s \in \mathcal{T}_X \), (i) \( X_s \) is the range of \( s \), the rank of \( s \) is the cardinal number of \( X_s \) and is denoted by \( \text{rank}(s) \), and (ii) the partition \( \pi_s \) is the equivalence relation defined by \( x \sim_y \) \( (x, y) \in X \) if and only if \( xs = ys \). Also, \( s^* \) is an inverse of \( s \) in \( \mathcal{T}_X \), that is, \( s = ss^*s \), \( s^* = s^*ss^* \). Green's \( \mathcal{L} [\mathcal{R}] \)-relation on \( \mathcal{T}_X \) is denoted by \( \mathcal{L} [\mathcal{R}] \).

**Result 1.4** [4, p. 52, Lemma 2.5]. Given \( s, t \in \mathcal{T}_X \), there exists \( u \in \mathcal{T}_X \) such that \( us = t \) if and only if \( Xu \subseteq Xs \). In this case, we write \( t \leq_X s \). Thus \( s \leq \mathcal{R} t \) if and only if \( Xs = Xu \).

**Result 1.5** [4, p. 52, Lemma 2.6]. Given \( s, t \in \mathcal{T}_X \), there exists \( v \in \mathcal{T}_X \) such that \( xv = t \) if and only if \( \pi_x \subseteq \pi_t \). In this case, we write \( t \leq \mathcal{R} s \). Thus \( s \leq \mathcal{R} t \) if and only if \( \pi_s \subseteq \pi_t \).

**Lemma 1.6.** Let \( X \) be a set. For any \( s, t \in \mathcal{T}_X \) with \( s \leq \mathcal{R} t \), there exist inverses \( s^*, t^* \) of \( s, t \) in \( \mathcal{T}_X \) such that \( ss^*tt^* < \mathcal{R} s \), where \( s \leq \mathcal{R} t \) means \( s \leq \mathcal{R} t \), \((s, t) \notin \mathcal{R} \).

**Proof.** Suppose that \( s \leq \mathcal{R} t \). Then by Result 1.5, there exist \( x, y \in X \) such that \( xt = yt \) but \( xs \neq ys \). Then there exists an inverse \( t^* \) of \( t \) such that \( x(t(t^*)) = y(t(t^*)) = x \), while there exists an inverse \( s^* \) of \( s \) such that \( x(s(s^*)) = x \), \( y(s(s^*)) = y \). Set \( s' = ss^*(tt^*)s \). Then \( s' \leq \mathcal{R} s \), \( xs' = ys' \), but \( xs \neq ys \). Hence \( s' \leq \mathcal{R} s \) as required.

**Lemma 1.7.** Let \( X \) be a finite set and \( S = \mathcal{T}_X \). Take any scheme (1.1) of length \( n \). There exists a scheme over \( aS \cup a_1S \) and \( B \) joining \( (a, b) \) to \( (a_1, t_1b_2) \) as follows: there exist \( c_1, ..., c_m \in aS, s'_1, u_1, ..., u_m, v_1, ..., v_m \in S \) such that

\[
\begin{align*}
    a &= c_1u_1, & u_1b &= v_1b \\
    c_1v_1 &= c_2u_2, & u_2b &= v_2b \\
    & \vdots & & \vdots \\
    c_{m-1}v_{m-1} &= c_mu_m, & u_mb &= v_mb \\
    c_mv_m &= a_1s'_1, & s'_1b &= t_1b_2 \\
    s'_1 &\leq \mathcal{R} t_1.
\end{align*}
\]
Proof: If \( s_1 \leq t_1 \), then the result of the lemma is trivial. Suppose that \( s_1 \not\leq t_1 \). By Lemma 1.6, there exist inverses \( s_1^\#, t_1^\# \) of \( s_1, t_1 \) such that \( s_1^\# t_1^\# s_1 < a s_1 \). Set \( s_1 = s_1 s_1^\# (t_1 t_1^\#) s_1 \). Then

\[
\begin{align*}
a &= a(s_1^\# s_1), \\
(s_1^\# s_1) b &= (s_1^\# t_1 t_1^\# s_1) b \\
a(s_1^\# t_1^\# s_1) &= (a s_1) s_1^\# t_1^\# s_1 = a_1 s_1', \\
s_1' b &= (s_1 s_1^\# t_1^\# s_1 b - s_1 s_1^\# s_1 b - s_1 b) = t_1 b_2.
\end{align*}
\]

By repeating the above argument, we can get the result.

**Definition 1.8.** Let \( S = \mathcal{P}_X \), where \( X \) is a finite set. The rank of a scheme (1.1) is the sum of the ranks of all \( s_i \)'s and \( t_i \)'s.

Note that the rank of a scheme of length \( n \) is at least \( 2n \).

2. The Proof of the "If" Part of Theorem A

Let \( X \) be a finite set. For simplicity, let \( S = \mathcal{P}_X \). We shall show that \( S \) is left absolutely flat. For this purpose, we shall prove that every left \( S \)-set \( B \) is flat. So, we take a scheme (1.1) of length \( n \) over \( A \) and \( B \) joining \((a, b)\) to \((a', b')\) and we shall prove that

there exists a scheme over \((aS \cup a'S)\) and \( B \) joining \((a, b)\) to \((a', b')\). \quad (2.1)

We use induction on \( n \) (the length of the scheme). The first \( m \) pairs of the equations in Lemma 1.7 form a scheme over \( aS \) and \( B \) joining \((a, b)\) to \((c_m v_m, b)\) (in particular, \( c_m v_m \in aS \)); and by replacing the first pair of equations in scheme (1.1) by the pair

\[
c_m v_m = a_1 s_1', \quad s_1' b = t_1 b_2
\]

we obtain a scheme over \( A \) and \( B \) joining \((c_m v_m, b)\) to \((a', b')\), also of length \( n \), but such that \( s_1' \leq t_1 \). It follows that we may assume without loss of generality that in (1.1)

\[
s_1 \leq t_1. \quad (2.2)
\]

**Case I: \( n = 1 \).** Then \( a = a_1 s_1, s_1 b = t_1 b', a_1 t_1 = a' \). By Result 1.5 and (2.2), there exists \( u \in S \) with \( s_1 = t_1 u \). Hence

\[
\begin{align*}
a &= a_1 s_1 = a_1 t_1 u = a_1 (t_1 t_1^\# t_1) u = a'(t_1 t_1^\# t_1 u), \\
(t_1^\# t_1 u) b &= t_1^\# s_1 b = (t_1^\# t_1) b', \quad a'(t_1 t_1^\#) = a',
\end{align*}
\]

as required. Now we assume that Statement (2.1) holds whenever \( 1 \leq n \leq k \).
Case II: $n = k + 1$. We use induction on $m$ (the rank of the scheme).

Subcase II.1: $m = 2(k + 1)$. Then all the elements $s_i$, $t_i$ are constant mappings. Then it follows from the left side of (1.1) that $a = (a_1 s_1) s_1$, $(a_1 s_i) t_i = (a_1 s_i + 1) s_i + 1$ $(1 \leq i \leq n - 1)$, $(a_n s_n) t_n = a'$, and $aS = (a_1 s_1) S = (a_1 t_1) S = \cdots = (a_n s_n) S = (a_n t_n) S = a'S$. Therefore (2.1) holds.

We assume, for any scheme (1.1) of length $k + 1$ and rank $m$, that Statement (2.1) holds whenever $2(k + 1) \leq m \leq p$.

Subcase II.2: $m = p + 1$. We shall prove first preliminary lemmas.

**Lemma 2.1.** Suppose that there exists $1 \leq i \leq n - 1$ such that $s_i \not\asymp t_i$, $s_{i+1} \not\asymp t_{i+1}$. Then (2.1) holds.

**Proof.** Note that $a - t_i t_i s_i = t_i s_{i+1} t_{i+1} - t_{i+1} s_{i+1}$. Thus we have

$$a_{i-1} t_i s_i = a_i t_i s_i = a_i (t_i t_i s_i) s_i = (a_{i+1} s_{i+1}) t_i s_i = a_{i+1} (s_{i+1} t_i s_i),$$

$$(s_{i+1} t_i s_i) t_{i+1} = s_{i+1} t_{i+1} = t_{i+1} b_{i+1}$$

(where $a_{i-1} t_i s_i = a_i t_i s_i = a_i (t_i t_i s_i) s_i = (a_{i+1} s_{i+1}) t_i s_i = a_{i+1} (s_{i+1} t_i s_i)$, $s_{i+1} t_i s_i = s_{i+1} t_{i+1}$). Hence we get a scheme of length $k$ joining $(a, b)$ to $(a', b')$. By the inductive assumption (on $n$), (2.1) holds.

**Lemma 2.2.** Suppose that there exists $1 < i < n - 1$ such that $s_i \not\asymp t_i$, $s_{i+1} \not\asymp t_{i+1}$. Then (2.1) holds.

**Proof.** Since $t_i \not\asymp s_{i+1}$, there exists $x \in X$ with $xt_i \in X \setminus X s_{i+1}$. Note that rank($t_i$) $\geq 2$. So there exists $y \in X$ with $yt_i \neq xt_i$. Let $u \in S$ which maps the elements of $x t_i$ to $y$ and leaves all other elements fixed. Let $v \in S$ which maps $xt_i$ to $yt_i$ and leaves all other elements fixed. Then $ut_i = t_i v$ and $a_i(ut_i) = a_i(t_i v) = (a_{i+1} s_{i+1}) v = a_{i+1} s_{i+1} = a_i t_i$. Since $s_i \not\asymp t_i$, there exists $w \in S$ with $t_i w = s_i$. Thus we have

$$a_{i-1} t_i s_i = a_i (t_i w) = a_i (ut_i) w = a_i (us_i),$$

$$(us_i) b_i [(us_i) b, if i = 1] = (ut_i) b_{i+1},$$

$$a_i (ut_i) = a_i (t_i) = a_{i+1} s_{i+1}, \quad \text{rank}(ut_i) < \text{rank}(t_i).$$

By taking $us_i$, $ut_i$ instead of $s_i$, $t_i$, we get a scheme of length $k + 1$, but of rank less than $p + 1$ over $A$ and $B$ joining $(a, b)$ to $(a', b')$. By the inductive assumption (on $m$), (2.1) holds.

**Lemma 2.3.** Suppose that there exists $1 \leq i \leq n - 1$ such that $t_i \not\asymp s_{i+1}$, $t_{i+1} \not\asymp s_{i+1}$. Then (2.1) holds.
Proof. Note that \( t_i s_{i+1}^* s_{i+1} = t_i \) and \( s_{i+1}^* t_{i+1} = t_{i+1} \). Thus we have

\[
s_i b_i = (t_i b_i + 1) = (t_i s_{i+1}^* s_{i+1}) b_i = (t_i s_{i+1}^* t_{i+1}) b_{i+2},
\]

\[
a_i(t_i s_{i+1}^* t_{i+1}) = (a_i s_{i+1}^* t_{i+1} = a_i + t_{i+1}^*) = a_i + s_{i+1} t_{i+1},
\]

where \( b = b_i \) if \( i = 1 \), \( b' = b_{i+2} \), \( a' = a_{i+2} s_{i+2} \) if \( i = n - 1 \). Hence we get a scheme of length \( k \) joining \( (a, b) \) to \( (a', b') \). By the inductive assumption (on \( n \)), (2.1) holds.

**Lemma 2.4.** Suppose that there exists \( 1 \leq i \leq n - 1 \) such that 
\[
 t_i s_{i+1} < s_{i+1}^* t_{i+1} < t_{i+1}.
\]

Then (2.1) holds.

Proof. Since \( s_{i+1} < t_{i+1} \), by Lemma 1.6 there exist inverses \( s_{i+1}^*, t_{i+1}^* \) of \( s_{i+1}, t_{i+1} \) such that

\[
s_{i+1} > t_{i+1} s_{i+1}^* (t_{i+1} t_{i+1}^*) s_{i+1} = (u, \text{say}).
\]

Set \( v = t_i s_{i+1}^*(t_{i+1} t_{i+1}^*) s_{i+1} \). Then we shall show that

\[
s_i b_i = v b_{i+1}, \quad a_i v = a_{i+1} u, \quad u b_{i+1} = t_{i+1} b_{i+2},
\]

where \( b_{i+2} = b' \) if \( i = n - 1 \).

Proof for (2.4). First,

\[
s_i b_i = t_i b_i + 1 = t_i(s_{i+1}^* s_{i+1}) b_i + 1 \quad \text{(since \( t_i \leq s_{i+1} \))}
\]

\[
= t_i s_{i+1}^* (t_{i+1} b_{i+2}) = t_i s_{i+1}^* (t_{i+1} t_{i+1}^* t_{i+1}) b_i + 2
\]

\[
= t_i s_{i+1}^* t_{i+1} t_{i+1}^* (s_{i+1} b_{i+1}) = v b_{i+1}.
\]

Second,

\[
a_i v = a_i(t_i s_{i+1}^* t_{i+1} t_{i+1}^* s_{i+1})
\]

\[
= (a_i s_{i+1}^* t_{i+1} t_{i+1}^* s_{i+1}) = a_i + t_{i+1}^* u.
\]

Third,

\[
ub_{i+1} = (s_{i+1} s_{i+1}^* t_{i+1} t_{i+1}^* s_{i+1}) b_{i+1}
\]

\[
= s_{i+1}^* s_{i+1}^* t_{i+1} t_{i+1}^* (t_{i+1} b_{i+2}) = (s_{i+1} s_{i+1}^* t_{i+1} t_{i+1}) b_{i+2}
\]

\[
= s_{i+1} s_{i+1}^* (s_{i+1} b_{i+1}) = s_{i+1} b_{i+1} = t_{i+1} b_{i+2}.
\]

Note that \( \text{rank}(v) \leq \text{rank}(t_i) \), \( \text{rank}(u) \leq \text{rank}(s_{i+1}) \) by (2.3). By replacing \( t_i, s_{i+1} \) by \( v, u \), we get, from the scheme (1.1), a scheme of length \( k + 1 \), but of rank less than \( p + 1 \) joining \( (a, b) \) to \( (a', b') \). By the inductive assumption (on \( m \)), (2.1) holds.
We can assume that
\[ s_i \prec_{\mathcal{S}} t_1. \quad (2.5) \]

Reason for (2.5). By (2.2), \( s_1 \preceq_{\mathcal{S}} t_1 \). If \( s_1 \not\preceq_{\mathcal{S}} t_1 \), then \( a_1 t_1 \in a_1 s_1 S = aS \).

Hence Statement (2.1) follows from the inductive assumption (on \( n \)).

By Lemma 2.1, Lemma 2.2, and (2.5), we can assume that \( t_1 \prec_{\mathcal{S}} s_2 \). Then, by Lemma 2.3 and Lemma 2.4, we can assume that \( s_2 \prec_{\mathcal{S}} t_2 \). By using these arguments, repeatedly, we can assume that \( s_n \prec_{\mathcal{S}} t_n \). So, by applying Lemma 1.7 to the scheme (1.1), regarded as joining \( (a', b') \) to \( (a, b) \), we can find \( t'_n \in S \) such that
\[
\begin{align*}
t'_n \preceq_{\mathcal{S}} s_n, \\
a' &= c_1 u_1, \\
   u_1 b' &= v_1 b', \\
c_1 v_1 &= c_2 u_2, \\
   u_2 b' &= v_2 b', \\
   &\vdots \\
c_{q-1} v_{q-1} &= c_q v_q, \\
   u_q b' &= v_q b', \\
c_q v_q &= a_n t'_n, \\
   t'_n b' &= s_n b_n
\end{align*}
\]

where \( c_i \in a'S, u_i, v_i \in S \). By replacing \( t_n, a' \) by \( t'_n, a_n t'_n \), we get, from the scheme (1.1), a scheme of length \( k + 1 \) but of rank less than \( p + 1 \) (since \( t'_n \preceq_{\mathcal{S}} s_n \preceq_{\mathcal{S}} t_n \)), joining \( (a, b) \) to \( (a_n t'_n, b') \). By the inductive assumption, there exists a scheme over \( aS \cup a_n t'_n S \) (= \( aS \cup c_q v_q S \subseteq aS \cup a'S' \)) and \( B \) joining \( (a, b) \) to \( (a_n t'_n, b) \). In reverse, scheme (2.6) is a scheme over \( a'S \) and \( \{ b' \} (\subseteq B) \) joining \( (a_n t'_n, b') \) to \( (a', b') \). Combining these last two schemes gives a scheme over \( aS \cup a'S \) and \( B \) joining \( (a, b) \) to \( (a', b') \), as required.

3. The Proof of the "Only If" Part of Theorem A

To prove that \( \mathcal{F}_X (|X| = \infty) \) is not left absolutely flat, we shall construct a right \( \mathcal{F}_X \)-set \( A \), a left \( \mathcal{F}_X \)-set \( B \), and elements \( a, a', b, b' \) such that \( a \otimes b = a' \otimes b' \) in \( A \otimes B \) but \( a \otimes b \neq a' \otimes b' \) in \( (a \mathcal{F}_X \cup b \mathcal{F}_X) \otimes B \). Hereafter, let \( X \) be an infinite set. For simplicity, let \( S = \mathcal{F}_X \).

Step 1. Construct a right \( S \)-set \( A \), and elements \( a, a' \).

Since \( |X| = \infty \), there exist two partitions \( \{X_i | i \in \mathbb{Z}\} \), \( \{Y_i | i \in \mathbb{Z}\} \) of \( X \) such that
\[
\begin{align*}
every X_i \uplus Y_i \text{ is distinct from any } Y_i \uplus X_i, \\
   X_i \subseteq Y_{i-1} \cup Y_i, \\
Y_i \subseteq X_i \cup X_{i+1} \text{ for all } i \in \mathbb{Z}, \quad (3.1)
\end{align*}
\]

where \( \mathbb{Z} \) is the set of all integers.
Let \( \{ x_i | i \in \mathbb{Z} \} \), \( \{ y_i | i \in \mathbb{Z} \} \) be sets of the representatives of the partitions \( \{ X_i | i \in \mathbb{Z} \} \), \( \{ Y_i | i \in \mathbb{Z} \} \), respectively. We define mappings \( e, f \in S \) as follows: for any \( x \in X \), \( xe = x_i \) if \( x \in X_i \), \( xf = y_i \) if \( x \in Y_i \).

We set
\[
A = S, \quad a = e, \quad a' = f.
\]

Since, by Result 1.5, \( \pi_e \cup \pi_f \subseteq \pi_s \) for all \( s \in eS \cap fS \), it follows easily from (3.1) that
\[
aS \cap a'S = \{ s \in S | |X_s| = 1 \}, \quad (3.2)
\]

**Step 2.** Construct a left \( S \)-set \( B \) and elements \( b, b' \).

Let \( \theta = \{ (u, v) \in S, S | u\epsilon = \epsilon v \text{ or } uf = vf \} \), where \( e, f(\in S) \) are as in Step 1. The relation \( \theta \) on \( S \) is reflexive, symmetric, and left compatible, so that the transitive closure \( \theta^l \) of \( \theta \) is a left congruence on \( S \).

We define two relations \( \xi, \eta \) on \( S \) as
\[
\xi = \{ (ue, ve) | (u, v) \in \theta^l \} \cup I_S,
\]
\[
\eta = \{ (uf, vf) | (u, v) \in \theta^l \} \cup I_S,
\]

where \( I_S \) is the identity relation on \( S \). Clearly, \( \xi, \eta \) are left congruences on \( S \).

Define a mapping \( \varphi : Se/(\xi | Se) \to Sf/(\eta | Sf) \) by \( (ue) \xi \varphi = (uf) \eta \) for all \( u \in S \). To see that \( \varphi \) is well-defined, suppose that \( u\epsilon = \epsilon v \in S \). By the definition of \( \xi \), there exist \( p, q \in S \) such that \((p, q) \in \theta^l, u\epsilon = p\epsilon, q\epsilon = v\epsilon \). Then \((u, p) \in \theta, (q, v) \in \theta \) and so, \((u, v) \in \theta^l \) and \( uf \eta vf \). Hence \( \varphi \) is well-defined. Now it is clear that \( \varphi \) is an \( S \)-isomorphism, if we regard \((Se)/(\xi | Se)\) and \((Sf)/(\eta | Sf)\) as left \( S \)-sets.

Let \( M = (S/\xi) \cup (S/\eta) \) (disjoint union). We define the relation \( A \) on \( M \) as
\[
A = \{ (m, m\rho), (m\rho, m) | m \in (Se)/(\xi | Se) \} \cup I_M.
\]

Then \( A \) is a left congruence on the left \( S \)-set \( M \). So we get the factor left \( S \)-set \( M/A \). We can identify each element \( s\xi \in S/\xi [s\eta \in S/\eta] \) with the element \((s\xi)A \in M/A [(s\eta)A] \). We set
\[
B = M/A, \quad b = 1_S\xi, \quad b' = 1_S\eta.
\]

Since \((e\xi) \varphi = f\eta \), we note that
\[
eb = fb' \quad \text{(in } B\text{).} \quad (3.3)
\]

For a pair of partitions \( \Pi, \Omega \) of \( X \), if each \( \Pi \)-class is a finite union of \( \Omega \)-classes, then we say that \( \Pi \) is a finite expansion of \( \Omega \).
Lemma 3.1. Let \( u, v \in S_\epsilon \) with \( u \subseteq v \). If \( \Pi_{eu} \) is a finite expansion of \( \Pi_\epsilon \), then so is \( \Pi_{ev} \).

Proof. We shall show first that

for any \( s \in S_\epsilon \), \( \Pi_{(ese)} \) is a finite expansion of \( \Pi_\epsilon \) if and only if \( \Pi_{(esf)} \) is a finite expansion of \( \Pi_\epsilon \). (3.4)

Suppose that \( \Pi_{ese} \) is a finite expansion of \( \Pi_\epsilon \) and take any \( x \in X \). By the definition of \( f \), there exists an integer \( k \) such that \( x(esf) = y_k \). Put \( U = \{ y \in X \mid y(es) \in X_k \} \), \( V = \{ z \in X \mid z(es) \in X_{k+1} \} \). By (3.1), we have

\[
\Pi_{(esf)} = \{ y \in X \mid y(es) \in X_k \} \subseteq U \cup V,
\]

\[
U = \{ y \in X \mid y(ese) = x_k \},
\]

\[
V = \{ z \in X \mid z(ese) = x_{k+1} \}.
\]

So, by the assumption, each of \( U \) and \( V \) is a finite union of \( X_i \)'s. Hence \( x\Pi_{(esf)} \) is contained in a finite union of \( X_i \)'s. Since \( \Pi_{(esf)} \supseteq \Pi_\epsilon \), it follows that \( x\Pi_{(esf)} \) is a finite union of \( X_i \)'s. Thus we conclude that \( \Pi_{(esf)} \) is a finite expansion of \( \Pi_\epsilon \). Similarly, we can prove the "if" part of (3.4).

Next, let \( u, v \) be as in the statement of the lemma and suppose that \( \Pi_{(eu)} \) is a finite expansion of \( \Pi_\epsilon \). Since \( u \subseteq v \), there exist \( u_1, u_2, \ldots, u_r \in S \) such that \( u = u_1e, v = u_re, u_1 \theta u_2 \theta \cdots u_{r-1} \theta u_r \). We shall show here that for each \( 1 \leq i \leq r, \Pi_{(eu_i)} \) is a finite expansion of \( \Pi_\epsilon \). The case if \( i = 1 \) is similar to that of \( i = 1 \) above. So suppose that for some \( 1 \leq i \leq r, \Pi_{(eu_i)} \) is a finite expansion of \( \Pi_\epsilon \). Since either \( u_i e = u_{i+1} e \) or \( u_i f = u_{i+1} f \), it follows easily from (3.4) that \( \Pi_{(eu_{i+1})} \) is a finite expansion of \( \Pi_\epsilon \). Eventually, we obtain that \( \Pi_{(eu_r)} = \Pi_\epsilon \) is a finite expansion of \( \Pi_\epsilon \). The proof of the lemma is complete.

Step 3. Show that

\[ a \otimes b = a' \otimes b' \quad \text{in} \quad A \otimes B. \]  

(3.5)

Proof for (3.5). Set \( a_1 = 1_S, s_1 = e, t_1 = f \). Then

\[ a (= e = 1_{Se}) = a_1 s_1, \quad s_1 b (= eb = fb' \text{ (by (3.3))}) \]

\[ = t_1 b'. \quad a_1 t_1 (= 1_{Se} f = f) = a'. \]

Hence by Result 1.3, \( a \otimes b = a' \otimes b' \), as required.

Step 4.

\[ a \otimes b \neq a' \otimes b' \quad \text{in} \quad (aS \cup a'S) \otimes B. \]  

(3.6)

To see this, suppose, conversely, that there exists a scheme (1.1) over \( (aS \cup a'S) \) and \( B \) joining \( (a, b) \) to \( (a', b') \). From \( a_n \in aS \cup a'S \) we have that \( a_n \in a'S \). Since \( a_n \in aS \) we have \( a_n = au \) for some \( u \in S \), whence
\[ \text{aut}_e = a, t_n = a', \] that is, \[ \text{eut}_e = f, \] whence \[ \Pi_e \subseteq \Pi_f, \] a contradiction. Similarly, we obtain that \( a_i \in \alpha S \). It follows that there exists \( 1 \leq i \leq n - 1 \) such that \( a_i \in \alpha S, a_{i-1} \in \alpha' S \). Then \( a_i t_i = a_{i+1} s_{i+1} \in \alpha S \cap \alpha' S \). By (3.2), we have

\[ a_i t_i \] is a constant mapping of \( X \) into itself. \hspace{1cm} (3.7)

On the other hand, \( b_{i+1} \) is \( u \xi \) (or \( u \eta \)) (for some \( u \in S \)). By (1.1) and the fact that all \( a_i s_i \)'s and \( a_i t_i \)'s are in \( S \), it follows that \( e = (e_{1S}) \xi = e(1S \xi) = (a_1 s_1) b = a_1 (t_1 b_2) = \cdots = a_{i-1} (t_{i-1} b_i) = (a_i s_i) b_i = a_i (t_i b_{i+1}) = (a_i t_i) \xi \) (or \( (a_i t_i) u \)). Whence \( e = (a_i t_i) u \xi \) or \( (a_i t_i) u \eta \). In the former case, by the definition of \( \xi \), \( a_i t_i u \in S e \) and \( e \xi (a_i t_i) u e \). In the latter case, by the definition of \( \eta \) and \( \Delta \), \( a_i t_i u \in S \eta \) and \( (a_i t_i) u \eta = (a_i t_i) u \eta \Delta = (a_i t_i u \xi) \Delta = (a_i t_i) u e \xi \). So, in any case, \( e \xi (a_i t_i) u e \). Then, by Lemma 3.1, \( \Pi_{(a_i t_i \xi) u} \) is a finite expansion of \( \Pi_e \) (since \( ea_i = a_i \)). Hence, by Result 1.5, \( \Pi_{(a_i t_i)} \) is a finite expansion of \( \Pi_e \), since \( (a_i t_i) u \xi \leq \sigma (a_i t_i) \leq \sigma e \). This contradicts (3.7). Therefore, we obtain (3.6). Thus we have shown that the left \( \Sigma \)-set \( B \) is not flat. This completes the proof of the "only if" part of Theorem A.

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