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THE FUNDAMENTAL GROUP AT INFINITY

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LET G be a finitely presented infinite group which is semistable at infinity, let X be a finite complex whose fundamental group is G, and let ω be a base ray in the universal covering space $\tilde{X}$. The *fundamental group at* ∞ of G is the topological group $\pi_1^e(\tilde{X},\omega) \equiv \varprojlim \{\pi_1(\tilde{X} - L) \mid L \subset \tilde{X}$ is compact$\}$. We prove the following analogue of Hopf's theorem on ends: $\pi_1^e(\tilde{X},\omega)$ *is trivial, or is infinite cyclic, or is freely generated by a non-discrete pointed compact metric space; or else the natural representation of* G *in the outer automorphisms of* $\pi_1^e(\tilde{X},\omega)$ *has torsion kernel*. A related manifold result is: *Let* G *be torsion free* (*not necessarily finitely presented*) *and act as covering transformations on a connected manifold* M *so that the quotient of* M *by any infinite cyclic subgroup is non-compact; if* M *is semistable at* ∞ *then the natural representation of* G *in the mapping class group of* M *is faithful*. The latter theorem has applications in 3-manifold topology. Copyright © 1996 Elsevier Science Ltd

0. INTRODUCTION

It is a classical theorem of Hopf [13] that if G is a finitely generated infinite group then the number of ends of G is 1, 2 or ∞; equivalently, the abelian group $H^1(G,\mathbb{Z}G)$ is 0, $\mathbb{Z}$ or $\bigoplus_1^\infty \mathbb{Z}$. More recently, Farrell [6] proved that if G is finitely presented and contains an element of infinite order then the abelian group $H^2(G,\mathbb{Z}G)$ is 0, $\mathbb{Z}$ or is infinitely generated; indeed, a result of ours, Addendum 4.11, allows the strengthened conclusion 0, $\mathbb{Z}$ or $\bigoplus_1^\infty \mathbb{Z}$ provided G is semistable at each end.[1] This conclusion can be reinterpreted as a "higher end theorem": let X be a finite connected CW complex with $\pi_1(X,v) \cong G$; then Hopf's theorem says that the universal cover, $\tilde{X}$, has one, two or infinitely many "components at infinity", and Farrell's theorem says that the first cohomology of $\tilde{X}$ "at infinity" is 0 or $\mathbb{Z}$ or "large" ($\bigoplus_1^\infty \mathbb{Z}$ in the semistable case).

In this paper we prove a π_1-version of the latter theorem. Picking a base ray ω in $\tilde{X}$, we consider the fundamental group at the end determined by ω, $\pi_1^e(\tilde{X},\omega)$, i.e. $\varprojlim_n \pi_1(\tilde{X} - L_n, \omega(n))$, where $\{L_n\}$ is an exhausting sequence of finite subcomplexes. This is an inverse limit of discrete groups but it has a natural topology which makes it a separable metrizable totally disconnected topological group. Recalling that the set, $E(Y)$, of ends of a suitable locally compact space Y is, in a natural way, a compact totally disconnected metrizable space, one should not be surprised that $\pi_1^e(\tilde{X},\omega)$ is to be taken with its topology.

The flavor of our theorems is that $\pi_1^e(\tilde{X},\omega)$ is trivial or $\mathbb{Z}$ or free of infinite rank or satisfies a fourth condition. But "free of infinite rank" must be understood in the topological sense: "freely generated" by an infinite compact metrizable space. This is made precise in Section 1.

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[1] Terminology in this introduction is defined in subsequent sections.

We give topological and group theoretic versions of our results.

I. **Topology.** The *mapping class group* of $\tilde{X}$, denoted by $\mathcal{M}(\tilde{X})$, is the (discrete) group of ambient isotopy classes of self-homeomorphisms of $\tilde{X}$. The *weak mapping class group*, $\mathcal{WM}(\tilde{X})$ is the (discrete) group of proper homotopy classes of self-homeomorphisms; it is a quotient of $\mathcal{M}(\tilde{X})$. We say $\tilde{X}$ is *strongly connected* at ∞ if any two proper rays in $\tilde{X}$ are properly homotopic; this implies $\tilde{X}$ has one end. The space $\tilde{X}$ is *simply connected at* ∞ if $\tilde{X}$ is strongly connected at ∞ and $\pi_1^e(\tilde{X}, \omega)$ is trivial.

THEOREM A. *Let the infinite group $G = \pi_1(X, v)$, where X is a finite connected complex. If G has two ends, $\pi_1^e(\tilde{X}, \omega)$ is trivial for any base ray ω. Otherwise, one of the following holds*:

(i) *$\tilde{X}$ is simply connected at ∞;*

(ii) *$\tilde{X}$ is strongly connected at ∞, and $\pi_1^e(\tilde{X}, \omega)$ is discrete and infinite cyclic;*

(iii) *$\tilde{X}$ is strongly connected at ∞, and $\pi_1^e(\tilde{X}, \omega)$ is freely generated by an infinite (pointed) compact metrizable space;*

(iv) *letting $\rho: G \to \mathcal{WM}(\tilde{X})$ denote the natural representation of G by covering translations, every element of* ker ρ *has finite order in G.*

Note that when G is torsion free, (iv) simplifies to:

(iv)′ *the representation ρ is faithful.*

Theorem A is proved by assuming (iv) does not hold and showing that for any infinite cyclic subgroup $J \leqslant \ker \rho$, $\pi_1^e(\tilde{X}, \omega)$ is freely generated by the pointed space of ends $(E(\tilde{X}/J), z)$ where $\tilde{X}/J$ has either one, two or infinitely many ends; see Theorem 4.5. For the two-ended part, see Remark 4.6.

Examples. For (i) take $X =$ the 3-torus; for (ii) take $X =$ the 2-torus; for (iii) take $X = (S^1 \vee S^1) \times S^1$; for (iv) take $X =$ a Davis manifold (see Remark 4.7), which must satisfy (iv) since, although $\tilde{X}$ is strongly connected at ∞, it does not satisfy (i)–(iii).

We say that G is *semistable at* ∞ if $\tilde{X}$ is strongly connected at ∞; we recall in Section 1 why this property only depends on G. It is unknown whether a finitely presented group with one end can fail to be semistable at ∞. We note:

COROLLARY A′. *If, in Theorem A, G is torsion free and has one end but is not semistable at ∞ then the representation $\rho: G \to \mathcal{WM}(\tilde{X})$ is faithful.*

Obviously, $\mathcal{WM}(\tilde{X})$ can be replaced by $\mathcal{M}(\tilde{X})$ in these theorems.

Theorem A and Corollary A′ follow from a more general result, Theorem 3.1, about the existence of free and properly discontinuous actions of the group of integers on suitable locally compact spaces. Another corollary of Theorem 3.1 is:

THEOREM B. *Let the torsion free group G act as a group of covering transformations on a connected manifold M so that, for every $g \in G$, $M/\langle g \rangle$ is not compact. If M is not strongly connected at ∞, then the natural representation $\rho: G \to \mathcal{M}(M)$ is faithful.*

Myers applies this is in [20] to get examples of contractible open 3-manifolds which non-trivially cover other open 3-manifolds but which do not cover closed 3-manifolds. He constructs an irreducible contractible open 3-manifold M which is not strongly connected at ∞, and whose mapping class group has the property that the only torsion-free subgroups which are isomorphic to closed irreducible 3-manifold groups are groups for which it is known that the universal cover must be $\mathbb{R}^3$. If M were to cover a closed 3-manifold with fundamental group G, one could use Theorem B to embed G in $\mathscr{M}(M)$, thereby getting a contradiction.

II. **Group theory.** Let $\mathrm{Aut}(\pi_1^e(\tilde{X},\omega))$ be the group of automorphisms of this topological group, and $\mathrm{Inn}(\pi_1^e(\tilde{X},\omega))$ the subgroup of inner automorphisms. As usual, write $\mathrm{Out}(\pi_1^e(\tilde{X},\omega))$ for the (discrete) group of outer automorphisms, i.e. Out $\equiv$ Aut/Inn.

Here is our group theoretic result:

THEOREM C. *Let $G = \pi_1(X, v)$ where X is a finite connected complex. Assume G is infinite and semistable at ∞. Then the isomorphism class of $\pi_1^e(\tilde{X},\omega)$ (as a topological group) depends only on G, and one of the following holds:*

(i) *$\pi_1^e(\tilde{X},\omega)$ is trivial;*
(ii) *$\pi_1^e(\tilde{X},\omega)$ is discrete and infinite cyclic;*
(iii) *$\pi_1^e(\tilde{X},\omega)$ is freely generated by an infinite (pointed) compact metrizable space;*
(iv) *letting $\bar{\rho}: G \to \mathrm{Out}(\pi_1^e(\tilde{X},\omega))$ denote the natural representation of G induced by ρ, every element of* ker $\bar{\rho}$ *has finite order in G.*

Again, when G is torsion free, (iv) simplifies to:

(iv)′ *the representation $\bar{\rho}$ is faithful.*

Theorem C follows from Theorem 4.8. The examples given after Theorem A all apply here too.

COROLLARY C′. *If, in Theorem C, G has a subgroup of finite index whose center contains an element of infinite order then* (iv) *does not hold. In fact, $\pi_1^e(\tilde{X},\omega)$ is either trivial or infinite cyclic or freely generated by a (pointed) Cantor set.*

This is proved at the end of Section 4.

Examples. Let $X_1 = (S^1 \vee S^1) \times S^1$, and let X_2 be the mapping torus of a map $S^1 \to S^1$ of degree $n \geqslant 2$. The corresponding fundamental groups are $G_1 = (\mathbb{Z} * \mathbb{Z}) \times \mathbb{Z}$ and $G_2 = \langle x, y \mid x^{-1}yx = y^n \rangle$, and X_i is a finite $K(G_i, 1)$. Now, G_1 has center and is torsion free so, by Corollary C′, G_1 satisfies (iii) but not (iv). On the other hand, as the reader can check, G_2 satisfies (iv), but also (iii).

So we see that (iii) and (iv) in Theorem C may overlap: it would be interesting to understand the extent of this. But no other overlap can occur among the four conditions in Theorem C. In particular, using [24] we obtain:

COROLLARY C″. *If, in Theorem C, G contains an element of infinite order and is stable at ∞ (i.e. $\pi_1^e(\tilde{X},\omega)$ is discrete), neither* (iii) *nor* (iv) *holds, so $\pi_1^e(\tilde{X},\omega)$ is either trivial or infinite cyclic.*

This is a restatement of Theorem 5.3.

All these theorems follow from our basic results, Theorems 3.1, 3.3 and 5.2, concerning free $\mathbb{Z}$-actions. We will not repeat their statements here but we believe they are of interest in their own right.

1. ENDS AND THE FUNDAMENTAL GROUP AT INFINITY

We begin by recalling the theory of ends. Throughout this paper Y denotes a separable, locally compact, non-compact, metrizable, connected, locally connected topological space: the examples we have in mind are countable, connected, strongly[2] locally finite, infinite CW complexes. Let $L_1 \subset L_2 \subset \cdots$ be compact subsets of Y, where $Y = \bigcup_n L_n$, $L_n \subset \operatorname{int} L_{n+1}$, and every (path) component of $Y - L_n$ is unbounded (i.e. has non-compact closure). Then the resulting inverse sequence $\{\pi_0(Y - L_n)\}$ of discrete finite spaces has a compact totally disconnected metrizable inverse limit space, $E(Y)$, called the *space of ends* of Y. An *end* of Y is a point of this space.

A *proper ray* in Y is a proper[3] map $\omega: [0, \infty) \to Y$. Two proper rays in Y *determine the same end* if their restrictions to $\mathbb{N} \subset [0, \infty)$ are properly homotopic. There is an obvious canonical bijection between the equivalence classes of proper rays so formed and the points of $E(Y)$. We will call Y *strongly connected at each end* if any two proper rays which determine the same end are properly homotopic, and *strongly connected at* ∞ if in addition Y has only one end. There are many well-known examples of one-ended spaces Y which are not strongly connected at ∞ (e.g. Whitehead's contractible open 3-manifolds).

Pick a proper ray ω as base ray. By reparametrizing ω if necessary we can (and will always) assume $\omega([n, \infty)) \subset Y - L_n$ for all n. We obtain an inverse sequence $\mathscr{G}(Y, \omega) \equiv \{\pi_1(Y - L_n, \omega(n))\}$ of fundamental groups, where the bonding morphism $\pi_1(Y - L_{n+1}, \omega(n+1)) \to \pi_1(Y - L_n, \omega(n))$ is induced by inclusion using $\omega|[n, n+1]$ as base path. Each of these fundamental groups is to be regarded as a discrete topological group and the inverse limit, $\pi_1^e(Y, \omega)$, is to be given the natural (possibly non-discrete) topology; i.e. take $\varprojlim$ in the category of separable metrizable totally disconnected topological groups. Thus topologized, $\pi_1^e(Y, \omega)$ is the *fundamental group of* Y *at* ∞ *based at* ω.

Up to canonical isomorphism $\pi_1^e(Y, \omega)$ is independent of the choice of finite complexes $\{L_n\}$. However, dependence on ω is more delicate. Certainly, if ω_1 is properly homotopic to ω_2 then $\pi_1^e(Y, \omega_1)$ is isomorphic to $\pi_1^e(Y, \omega_2)$; one constructs an isomorphism using a proper homotopy between ω_1 and ω_2 in the same way as one shows that the fundamental groups of a space based at different base points in the same path component are isomorphic, using a path between the base points. But if ω_1 and ω_2 are not properly homotopic, it is not always true that $\pi_1^e(Y, \omega_1) \cong \pi_1^e(Y, \omega_2)$, even when ω_1 and ω_2 determine the same end. For a counterexample one can adapt a shape theoretic example of Borsuk (see [16, p. 132] and [8]).

The preceding paragraph implies that if Y is strongly connected at each end then $\pi_1^e(Y, \omega)$ only depends (up to isomorphism) on the end determined by ω. Only in this case is $\pi_1^e(Y, \omega)$ a useful invariant. Otherwise, important fundamental group information at that end might not be detected by π_1^e; for example, in the case of a Whitehead manifold, π_1^e is trivial. Indeed, more complicated machinery exists to capture the relevant information in

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[2] The *carrier* of a cell in a CW complex in the smallest subcomplex (necessarily finite) containing that cell. A CW complex is *strongly locally finite* if the carriers of cells form a locally finite cover. The universal cover of a finite complex has this property, as does every locally finite simplicial complex. See [8] for more on this.

[3] A map is *proper* if the pre-image of every compact set is compact.

the general case (see Appendix). However, in this paper we will be dealing mostly with spaces which are strongly connected at each end, so π_1^e is the right invariant for us.

We say Y is *simply connected at* ∞ if Y is strongly connected at ∞ and $\pi_1^e(Y, \omega)$ is trivial: this is equivalent to the usual definition.

An inverse sequence of groups $\mathscr{G} \equiv G_1 \xleftarrow{\phi_1} G_2 \xleftarrow{\phi_2} \cdots$ is *semistable* (or *essentially epimorphic* or *Mittag–Leffler*) if $\forall m \exists n \geqslant m$ such that image $(G_k \to G_m)$ is independent of $k \geqslant n$; $\mathscr{G}$ is *essentially monomorphic* if $\exists m$ such that $\forall n \geqslant m$ $\exists k \geqslant n$ such that kernel $(G_k \to G_m) = \text{kernel}(G_k \to G_n)$; $\mathscr{G}$ is *stable* (or *essentially isomorphic*) if both of these properties hold. If every ϕ_i is epic or monic or both then the corresponding property of $\mathscr{G}$ holds. Conversely, if $\mathscr{G}$ is semistable then $\mathscr{G}$ is pro-isomorphic (see Appendix) to a sequence of epimorphisms, and corresponding statements hold for the other two cases.

PROPOSITION 1.1. *$\mathscr{G}(Y, \omega)$ is semistable if and only if Y is strongly connected at the end determined by ω. $\mathscr{G}(Y, \omega)$ is stable if and only if Y is strongly connected at the end determined by ω and $\pi_1^e(Y, \omega)$ is discrete.*

Proof. This is well-known to experts in shape theory. For the first part see [17, Theorem 2.1] or [8]. For the second part see [4].

We say that Y has *stable* fundamental group at ∞ based at ω if Y is strongly connected at the end determined by ω and $\pi_1^e(Y, \omega)$ is discrete; cf. [22].

Our main theorems say that under appropriate hypotheses the topological group $\pi_1^e(Y, \omega)$ is "freely generated" by a given compact metric space. Here is the appropriate notion of freeness in topological groups. Let $\mathscr{C}_0$ be the category of pointed totally disconnected complete separable metrizable spaces and let $\mathscr{M}$ be the category of complete separable metrizable totally disconnected topological groups. *The free object of $\mathscr{M}$ generated by the object (E, z) of $\mathscr{C}_0$* is a continuous function $f:(E, z) \to (\Gamma, 1)$, where Γ is an object of $\mathscr{M}$, satisfying the usual universal property. The force of "pointedness" is the suppression of one potential generator: if E is one point, Γ is trivial; if E is two points Γ is discrete infinite cyclic, etc.

We end this section with some remarks about 1-dimensional homology at ∞. We use integer coefficients. Define $\mathscr{H}_1(Y) = \{H_1(Y - L_n)\}$. Then $H_1^e(Y) \equiv \varprojlim \mathscr{H}_1(Y)$ is a topological abelian group. There is a related direct sequence $\mathscr{H}^1(Y) = \{H^1(Y - L_n)\}$ whose direct limit is $H_e^1(Y)$.

The following can be derived easily from [10, Section 3] and [11]. It is also proved in [8].

PROPOSITION 1.2. *Assume $H_1(Y)$ is finitely generated. Then each $H_1(Y - L_n)$ and $H^1(Y - L_n)$ is finitely generated. Moreover,* (i) *$\mathscr{H}_1(Y)$ is semistable if and only if $H_e^1(Y)$ is free abelian;* (ii) *$\mathscr{H}_1(Y)$ is stable if and only if $H_e^1(Y)$ is finitely generated free abelian.* (iii) *$\mathscr{H}_1(Y)$ is stable and $H_1^e(Y)$ has rank k if and only if $H_e^1(Y)$ is free abelian of rank k.*

2. THE FUNDAMENTAL GROUP AT INFINITY OF $Y \times \mathbb{R}$

Pick a tree T in Y so that $T \hookrightarrow Y$ induces a homeomorphism $E(T) \to E(Y)$. Pick a proper base ray ω in T. As before, we assume that $\omega([n, \infty)) \subset Y - L_n$. Let the path components of $Y - L_n$ be $C_{n,1}, \ldots, C_{n,k_n}$ where the indexing is chosen so that $\omega([n, \infty)) \subset C_{n,1}$. Thus $C_{n+1,1} \subset C_{n,1}$ for all n. We also adopt the convention $C_{n+1,2} \subset C_{n,2}$ whenever $k_n \geqslant 2$. We wish to compute $\pi_1^e(Y \times \mathbb{R}, \omega)$, first in general and then when $\pi_1(Y) \cong \mathbb{Z}$.

Let $U_n = (Y \times \mathbb{R}) - (L_n \times [-n, n])$. Then U_n is the union of $Y_n^+ \equiv Y \times (n, \infty)$, $Y_n^- = Y \times (-\infty, -n)$ and $\bigcup_{i=1}^{k_n} C_{n,i} \times (-n-1, n+1)$. Using T for base points and base paths, the inclusion $C_{n,i} \hookrightarrow Y$ induces $\phi_{n,i}: \pi_1(C_{n,i}) \to \pi_1(Y)$. Write $P_{n,i} =$ image $(\theta_{n,i}) \leqslant \pi_1(Y)$.

Write $y_{n,1} = \omega(n)$ and take $T'_n = T_n^+ \cup T_n^- \cup (\{y_{n,1}\} \times [-n-1, n+1])$ as the tree[4] of base points and base paths in U_n where $T_n^{\pm}$ is the copy of T in $Y \times \{\pm(n+1)\} \subset Y_n^{\pm}$. Pick $y_{n,i} \in T \cap C_{n,i}$ for $2 \leqslant i \leqslant k_n$. Let $\tau_{n,i}$ be the path $\{y_{n,i}\} \times [-n-1, n+1]$, oriented positively, $1 \leqslant i \leqslant k_n$. Then $\tau_{n,i}$ determines an element $t_{n,i}$ of $\pi_1(U_n, T'_n)$; $t_{n,1}$ is trivial. By the generalized van Kampen theorem [23, pp. 138–139], $\pi_1(U_n, T'_n)$ can be presented as the fundamental group of a graph of groups, namely

$$((\pi_1(Y) *_{P_{n,1}} \pi_1(Y)) *_{P_{n,2}}) \cdots *_{P_{n,k_n}}. \tag{2.1}$$

That is, amalgamate across $P_{n,1}$, then take HNN extensions where the stable letters $t_{n,2}, \ldots, t_{n,k_n}$ identify the (successive images of the) subgroups $P_{n,2}, \ldots, P_{n,k_n}$. Write $Q_n = \pi_1 Y * \pi_1 Y * \langle t_{n,2}, \ldots, t_{n,k_n} \rangle$, and let $\alpha_n: Q_n \twoheadrightarrow \pi_1(U_n)$ be the obvious epimorphism. Then the following diagram commutes:

$$\begin{array}{ccc} Q_n & \xleftarrow{\beta_{n+1}} & Q_{n+1} \\ \downarrow{\scriptstyle\alpha_n} & & \downarrow{\scriptstyle\alpha_{n+1}} \\ \pi_1(U_n, T'_n) & \xleftarrow[i_\#]{} & \pi_1(U_{n+1}, T'_{n+1}) \end{array}$$

Here, $\beta_{n+1}(t_{n+1,j}) = t_{n,i}$ whenever $C_{n+1,j} \subset C_{n,i}$ and $i \geqslant 2$; $\beta_{n+1}(t_{n+1,j}) = 1$ if $C_{n+1,j} \subset C_{n,1}$; and β_{n+1} is the "identity" on $\pi_1(Y) * \pi_1(Y)$. The bonding homomorphism $i_\#$ is induced by inclusion where base trees are matched up in the obvious way.

Proposition 2.2. *The space $Y \times \mathbb{R}$ is strongly connected at infinity.*

Proof. Again, this is well-known: since Y is non-compact it is easy to see that $Y \times \mathbb{R}$ has one end, and semistability is clear from the commutativity of the above diagram. □

We will be interested in the special case where $\pi_1(Y) \cong \mathbb{Z}$. Then each $P_{n,i}$ is generated by a non-negative integer $m(n, i)$, and $\pi_1(U_n, T'_n)$ is presented by

$$\langle a, b, t_{n,2}, \ldots, t_{n,k_n} \mid a^{m(n,1)} = b^{m(n,1)}, a^{m(n,r)} = t_{n,r} b^{m(n,r)} t_{n,r}^{-1} \text{ for } r \geqslant 2 \rangle. \tag{2.3}$$

In our situation we will know that this group has non-trivial center; as we shall see, this imposes severe restrictions on the exponents $m(n, r)$.

3. FREE $\mathbb{Z}$-ACTIONS

Our purpose is to prove Theorems 3.1 and 3.3.

Theorem 3.1. *Let the infinite cyclic group $J \equiv \langle j \rangle$ act as a group of covering transformations on Y, and let Y/J be non-compact. If j is properly homotopic to id_Y then Y is strongly connected at ∞. Pick a base point $z \in E(Y/J)$ and a proper base ray ω in Y. If Y is simply connected then $\pi_1^e(Y, \omega)$ is freely generated by $(E(Y/J), z)$; in particular if Y/J has $k(< \infty)$ ends then $\pi_1^e(Y, \omega)$ is stable and is free of rank $k - 1$.*

[4]There is a tacit assumption, here, that T has been chosen so that $T \cap C_{n,i}$ is connected for all n and i. This can easily be arranged.

Let $f: Y \to Y$ be a proper homotopy equivalence where Y is strongly connected at ∞. Then for any proper ray ω in Y, $f \circ \omega$ is properly homotopic to ω. Choose a proper homotopy $F:[0, \infty) \times I \to Y$ realizing this. Then, as indicated in Section 1 (see also Appendix), F defines an isomorphism $c_F: \pi_1^e(Y, f \circ \omega) \to \pi_1^e(Y, \omega)$. Write $\phi = c_F \circ f_\#$ for the resulting automorphism of $\pi_1^e(Y, \omega)$. Obviously we have:

PROPOSITION 3.2. *When Y is strongly connected at ∞, the property that ϕ be an inner automorphism of $\pi_1^e(Y, \omega)$ is independent of ω and of F.*

THEOREM 3.3. *Let Y be simply connected and strongly connected at ∞. Let the infinite cyclic group $J \equiv \langle j \rangle$ act as a group of covering transformations on Y with Y/J non-compact. Pick $z \in E(Y/J)$. If, for some ω, j induces an inner automorphism of $\pi_1^e(Y, \omega)$ then $\pi_1^e(Y, \omega)$ is freely generated by $(E(Y/J), z)$. In particular, if Y/J has $k(<\infty)$ ends then $\pi_1^e(Y, \omega)$ is stable and is free of rank $k-1$.*

We begin with the proof of Theorem 3.1 and we assume its hypotheses until it is proved (after Proposition 3.13). However, Y will not be assumed simply connected until after Corollary 3.7.

Following [6] we apply the "Borel trick". Let Z be the quotient of the diagonal action of J on $Y \times \mathbb{R}$. We have a commutative diagram

$$\begin{array}{ccccc} Y & \longleftarrow & Y \times \mathbb{R} & \longrightarrow & \mathbb{R} \\ \downarrow & & \downarrow & & \downarrow \\ Y/J & \longleftarrow & Z & \longrightarrow & S^1 \end{array}$$

where the verticals are quotient maps (indeed, covering projections) and the upper horizontals are projections. This gives two different ways of looking at Z:

(i) The principal bundle $\mathbb{R} \to Z \to Y/J$ has a section and therefore gives Z homeomorphic to $Y/J \times \mathbb{R}$ (see [21, Theorems 8.3 and 12.2]).

(ii) The bundle $Y \to Z \to S^1$ gives Z as the mapping torus of j.

Since j is properly homotopic to id_Y, we conclude:

PROPOSITION 3.4. *$(Y/J) \times \mathbb{R}$ is proper homotopy equivalent to $Y \times S^1$.*

COROLLARY 3.5. *Y is strongly connected at ∞.*

Proof. By hypothesis, Y/J is non-compact, so Propositions 3.4 and 2.2 imply that $Y \times S^1$ is strongly connected at ∞, hence also Y.

The next proposition is a folk theorem: a proof is sketched in Appendix.

PROPOSITION 3.6. *Let $f: Y_1 \to Y_2$ be a proper homotopy equivalence (where Y_1 and Y_2 satisfy the hypotheses of Y in Section 1.). Then f induces an isomorphism $\pi_1^e(Y_1, \omega) \to \pi_1^e(Y_2, f \circ \omega)$.*

COROLLARY 3.7. *The topological groups $\pi_1^e(Y/J \times \mathbb{R}, \omega')$ and $\pi_1^e(Y, \omega) \times \mathbb{Z}$ are isomorphic for any base rays ω and ω', where $\mathbb{Z}$ is discrete.*

For the rest of this section we assume Y is simply connected. Then $\pi_1(Y/J) \cong \mathbb{Z}$, and (2.3) applies. To use Corollary 3.7, we must compute the center of the group presented by (2.3). This is the group E_{k_n} in a sequence defined inductively by

$$E_1 = \langle a, b \mid a^{m(n,1)} = b^{m(n,1)} \rangle$$

and, assuming E_{r-1} defined,

$$E_r = \langle E_{r-1}, t_{n,r} \mid a^{m(n,r)} = t_{n,r} b^{m(n,r)} t_{n,r}^{-1} \rangle.$$

The next two lemmas are proved by standard normal form arguments (see [15]):

LEMMA 3.8. *Let Γ be the free product with amalgamation $\Gamma_1 *_\phi \Gamma_2$ where Δ_i is a proper subgroup of Γ_i and $\phi: \Delta_1 \to \Delta_2$ is an isomorphism. Then $Z(\Gamma) = Z(\Gamma_1) \cap \Delta_1 \cap Z(\Gamma_2)$ (i.e. amalgamated elements in both centers).*

LEMMA 3.9. *Let Γ be the HNN extension Γ_{1*_ϕ} where Δ_1 and Δ_2 are proper subgroups of Γ_1 and $\phi: \Delta_1 \to \Delta_2$ is an isomorphism. Then $Z(\Gamma) = Z(\Gamma_1) \cap \mathrm{Fix}(\phi)$.*

We compute the centers of the groups E_r by successively applying these lemmas, but we must be careful to ensure that the relevant subgroups are proper, a condition which could fail in E_1 when $m(n,1) = 1$ or in E_2 when $m(n,1) = m(n,2) = 1$. Call these "exceptional cases".

PROPOSITION 3.10. *If E_{k_n} has non-trivial center, then all the integers $m(n,r)$ are positive.*

Proof. If $k_n \geqslant 2$ the lemmas give $Z(E_{k_n}) \leqslant Z(E_{k_n - 1}) \leqslant \cdots \leqslant Z(E_2)$. Suppose $m(n,r) = 0$, where $k_n \geqslant r \geqslant 2$. Then $E_r \cong E_{r-1} * \langle t_{n,r} \rangle$ and since E_{r-1} is non-trivial it follows that $Z(E_r) = \{1\}$, implying $Z(E_{k_n}) = \{1\}$. Next, suppose $m(n,1) = 0$ where $k_n \geqslant 1$. Then $E_1 = \langle a, b \rangle$ and since this is not an exceptional case we can add $Z(E_2) \leqslant Z(E_1) = \{1\}$ to the above containments, giving $Z(E_{k_n}) = \{1\}$. □

PROPOSITION 3.11. *Assume E_{k_n} has non-trivial center. Let N be the least common multiple of the positive integers $m(n,r)$. Then $Z(E_{k_n})$ is generated by $a^N = b^N$ unless $k_n = 2$ and $m(n,1) = m(n,2) = 1$. In the latter case $Z(E_2)$ is the free abelian group generated by a and $t_{n,2}$.*

Proof. Again this comes from successively applying Lemma 3.9 and Proposition 3.10, handling the exceptional cases separately. The first exception happens to fit into the general formula, and the second has been noted in the statement of the proposition. □

Obviously, we have:

PROPOSITION 3.12. *If every $m(n,r) = 1$, then $E_{k_n} \cong \mathbb{Z} \times \langle t_{n,2}, \ldots, t_{n,k_n} \rangle$ the product of $\mathbb{Z}$ with a free group of rank $k_n - 1$.*

Now we can complete the proof of Theorem 3.1. Using the notation of Section 2 for Y/J in place of Y, we have $\pi_1(U_n) \cong E_{k_n}$ where the latter is presented by (2.3). The bond $\pi_1(U_{n+1}) \to \pi_1(U_n)$ maps a to a and b to b, maps $t_{n+1,j}$ to $t_{n,i}$ when $C_{n+1,j} \subset C_{n,i}$ and $i \geqslant 2$, and maps $t_{n+1,j}$ to 1 when $C_{n+1,j} \subset C_{n,1}$. We will prove in a moment that for n sufficiently large every $m(n,r) = 1$. In that case the structure of E_{k_n} is given by Proposition 3.12. Moreover, the bond identifies the $\mathbb{Z}$-factors (given by Proposition 3.12), so we conclude:

PROPOSITION 3.13. *Let $f: (E(Y/J), z) \to F$ be the free object of $\mathcal{M}$ generated by $(E(Y/J), z)$. If, for large n, every $m(n,r) = 1$ then $\pi_1^e(Y/J \times \mathbb{R}, \omega')$ is isomorphic to $F \times \mathbb{Z}$.*

Let t generate the $\mathbb{Z}$-factor of $\pi_1^e(Y,\omega) \times \mathbb{Z}$. Using Corollary 3.7, we identify t with an element of $\pi_1^e(Y/J \times \mathbb{R}, \omega') \equiv \varprojlim_n E_{k_n}$. Then t is central and is not a proper power. We write $N(n)$ in place of N above in order to deal with varying n. We have $k_n \leqslant k_{n+1}$ and $m(n,i) \leqslant m(n+1,j)$ whenever $C_{n+1,j} \subset C_{n,i}$. In particular, $m(n,i) \leqslant m(n+1,i)$ when $i = 1$ or 2. There are two cases.

Case 1: Assume $\lim_{n\to\infty} k_n \neq 2$ or $\lim_{n\to\infty} m(n,1) \neq 1$ or $\lim_{n\to\infty} m(n,2) \neq 1$. Since t is central and the bonds are onto, $Z(E_{k_n}) \neq \{1\}$ for large n. By Proposition 3.11, t projects to $a^{s(n)N(n)}$ for some $s(n) \in \mathbb{Z} - \{0\}$ and since "a maps to a", $s(n)N(n)$ is independent of (large) n. But $N(n) | N(n+1)$ by definition of the integers $m(n,r)$. So $|s(n+1)| \leqslant |s(n)|$. So $s_0 \equiv \lim_{n\to\infty} s(n)$ and $N_0 \equiv \lim_{n\to\infty} N(n)$ make sense and are finite. We conclude that, in the inverse limit, t is represented by $(a^{s_0N_0}) = (a)^{s_0N_0}$ where (a) abbreviates the sequence $(a,a,a,\cdots)$. Since t is not a proper power, $s_0N_0 = \pm 1$; so $N_0 = 1$. So for all large n, every $m(n,r) = 1$. In particular, the assumption is reduced to $\lim_{n\to\infty} k_n \neq 2$. By Proposition 3.13 and Corollary 3.7, $\pi_1^e(Y,\omega) \times \mathbb{Z}$ is isomorphic to $F \times \mathbb{Z}$. Since $\lim k_n \neq 2$, F has trivial center. So t generates the center of $\pi_1^e(Y,\omega) \times \mathbb{Z}$. Factoring out centers we get $\pi_1^e(Y,\omega) \cong F$.

Case 2. Assume $\lim_{n\to\infty} k_n = 2$ and $\lim_{n\to\infty} m(n,1) = \lim_{n\to\infty} m(n,2) = 1$. Then $E_{k_n} \cong \mathbb{Z} \times \mathbb{Z}$ for large n. But also $E_{k_n} \cong \pi_1^e(Y,\omega) \times \mathbb{Z}$ for large n, by Corollary 3.7. Thus $\pi_1^e(Y,\omega)$ is abelian and is isomorphic to $\mathbb{Z}$, which in this case is F.

With Corollary 3.5, the proof of Theorem 3.1 is complete. □

We now turn to Theorem 3.3, assuming its hypotheses. In the previous proof we used the hypothesis on j to conclude that the mapping torus $T(j)$ had the proper homotopy type of $Y \times S^1$ and hence (Proposition 3.4) that $Y \times S^1$ was proper homotopy equivalent to $(Y/J) \times \mathbb{R}$. But in fact we only needed the weaker Corollary 3.7. Thus in the present case it is enough to show that $\pi_1^e(T(j),\omega)$ and $\pi_1^e(Y,\omega) \times \mathbb{Z}$ are isomorphic.

Write $U_n = Y - L_n$ and $j_n: U_{n+1} \to U_n$ for the restriction of j, where without loss of generality we assume $j(U_{n+1}) \subset U_n$. Then a typical neighborhood[5] of ∞ in $T(j)$ is $V_n \equiv U_n \cup T(j_n)$; here $T(j_n) = (U_{n+1} \times I)/\sim$ where $(x,1) \sim (j(x),0)$ for all $x \in U_{n+1}$. Write $i_n: U_{n+1} \hookrightarrow U_n$ for the inclusion. The generalized van Kampen theorem [23, pp. 138–139] gives (suppressing base points):

$$\pi_1(V_n) \cong \langle \pi_1(U_n), t \,|\, t \cdot i_{n\#}(g) \cdot t^{-1} = j_{n\#}(g),\ \forall g \in \pi_1(U_{n+1}) \rangle.$$

The present hypothesis on j ensures that there exists $u_n \in \pi_1(U_n)$ with $j_{n\#}(g) = u_n^{-1} \cdot i_{n\#}(g) \cdot u_n$, and $i_{n\#}(u_{n+1}) = u_n$ for all n; to see this use the functors $\mathcal{P}$ and $\mathcal{L}$ described in Appendix. Write $A_{n+1} = i_{n\#}(\pi_1(U_{n+1}))$. Then we have an HNN decomposition

$$\pi_1(V_n) = \langle \pi_1(U_n), u_n t \,|\, u_n t a (u_n t)^{-1} = a,\ \forall a \in A_{n+1} \rangle.$$

Taking inverse limits we get $\pi_1^e(T(j),\omega) \cong \pi_1^e(Y,\omega) \times \mathbb{Z}$ as claimed. □

4. FREE COCOMPACT ACTIONS

In this section, we prove convenient restatements of Theorems A and C, namely Theorems 4.5 and 4.8.

[5] V_n is not open in $T(j)$, so, strictly, V_n should be slightly fattened.

Let X be a finite connected CW complex with base vertex v, and let $G = \pi_1(X, v)$. Then G is finitely presented. Assume G contains an element j of infinite order.[6] Then Theorems 3.1 and 3.3 can be applied to the universal cover $\tilde{X}$. These theorems impose hypotheses on j and conclude, among other things, that $\tilde{X}$ is strongly connected at infinity. We pause for three background remarks on this point.

Remark 4.1. Hopf [13] proved that $E(\tilde{X})$ is either a one-point space, a two-point space or a Cantor set (i.e. a non-empty totally disconnected compact metrizable space in which every point is a limit point). Moreover, the topological type of $E(\tilde{X})$ only depends on G. This holds even if the infinite group G does not have an element of infinite order.

Remark 4.2. It is unknown whether $\tilde{X}$ must be strongly connected at each end. It is known that this property of $\tilde{X}$ only depends on G; see the proof of [14, Theorem 3]—the property is invariant under Tietze transformations. In view of this, and Proposition 1.1, one says that G is *semistable at each end* [resp. *semistable at* ∞] if $\tilde{X}$ is strongly connected at each end [resp. at ∞]. Again, the existence of an element of infinite order is irrelevant.

Remark 4.3. Many groups are known to be semistable at each end; see [19] and the references cited there. Moreover, if every one-ended finitely presented infinite group is semistable at infinity then every finitely presented infinite group is semistable at each end; see [18].

The conclusions of Theorems 3.1 and 3.3 (with $Y = \tilde{X}$) consist of properties of $\pi_1^e(\tilde{X}, \omega)$. So we need a strengthened version of Remark 4.2.

PROPOSITION 4.4. *If $\tilde{X}$ is strongly connected at ∞ then the topological group $\pi_1^e(\tilde{X}, \omega)$ is independent of ω and depends only on G.*

Proof. For independence of ω see Section 1. For the other part, use the background material outlined in Appendix together with the proof of [14, Theorem 3]. □

Now we can state the main results of this section, Theorems 4.5 and 4.8; proofs are given later in this section.

THEOREM 4.5. *Let X be a finite connected complex, let $G \equiv \pi_1(X, v)$ contain an infinite cyclic subgroup J generated by a covering transformation j which is properly homotopic to $\mathrm{id}_{\tilde{X}}$, and let $\tilde{X}/J$ be non-compact. Then $\tilde{X}$ is strongly connected at ∞. The space $E(\tilde{X}/J)$ is a one-point space, or a two-point space, or is non-discrete. Pick a base point $z \in E(\tilde{X}/J)$. Then $\pi_1^e(\tilde{X}, \omega)$ is freely generated by $(E(\tilde{X}/J), z)$ in the above sense. In particular, $\pi_1^e(\tilde{X}, \omega)$ is trivial, or discrete infinite cyclic, or freely generated by a non-discrete totally disconnected compact metrizable space.*

Remark 4.6. If $\tilde{X}/J$ is compact for some J then it is compact for all J. In that case, $\tilde{X}$ has two ends, is strongly connected at both ends and, for any ω, $\pi_1^e(\tilde{X}, \omega)$ is trivial. This is easily deduced from Theorem 5.12 of [23].

Remark 4.7. In a well-known example due to Davis [3], $\tilde{X}$ is strongly connected at ∞ and $\pi_1^e(\tilde{X}, \omega)$ is the inverse limit of n-fold free products of a group H where $n \to \infty$, each

[6] It is unknown whether every finitely presented infinite group contains an element of infinite order.

bond kills the last free factor, and H is the fundamental group of a homology 3-sphere which bounds a contractible 4-manifold. Thus H can be chosen finite, giving torsion in $\pi_1^e(\tilde{X}, \omega)$. In these examples, G is torsion free. One concludes from Theorem 4.5 that no non-trivial element of G is properly homotopic to $\mathrm{id}_{\tilde{X}}$.

One would like a purely group theoretic version of Theorem 4.5 in which everything depends on G rather than X. The offending hypothesis is that j be properly homotopic to $\mathrm{id}_{\tilde{X}}$; higher homotopy invariants of a particular X might obstruct this. The desired result is:

THEOREM 4.8. *Let G and X be as in Theorem 4.5. Assume G is semistable at ∞. Let G contain an infinite cyclic subgroup J generated by a covering transformation j which induces an inner automorphism of $\pi_1^e(\tilde{X}, \omega)$. Pick $z \in E(\tilde{X}/J)$. Then $\pi_1^e(\tilde{X}, \omega)$ is freely generated by $(E(\tilde{X}/J), z)$ and is either trivial, or discrete infinite cyclic, or freely generated by a non-discrete totally disconnected compact metrizable space.*

The conclusions of Theorems 4.5 and 4.8 are strengthenings of the conclusions of Theorems 3.1 and 3.3, respectively. To get the stronger conclusions, we need some homological results. Recall the homological notation introduced in Section 1.

THEOREM 4.9. [11]. *If G is finitely presented $H^2(G, \mathbb{Z}G)$ is isomorphic to $H_e^1(\tilde{X})$.*

Proof. This follows from the Lemma in [11]; see also [8]. □

THEOREM 4.10. [6, Corollary 5.2]. *Let the finitely presented group G contain an element of infinite order. Then the abelian group $H^2(G, \mathbb{Z}G)$ is either 0, infinite cyclic or infinitely generated.*

ADDENDUM 4.11. *With G as in Theorem 4.10, if G is semistable at each end then $H^2(G, \mathbb{Z}G)$ is free abelian of rank 0, 1 or ∞.*

Proof. First assume G has one end. By Proposition 1.1, $\mathscr{G}(\tilde{X}, \omega)$ is semistable, hence (abelianizing) $\mathscr{H}_1(\tilde{X})$ is semistable, hence, by Proposition 1.2(i), $H_e^1(\tilde{X})$ is free abelian. Apply Theorems 4.9 and 4.10.

If G has two ends, it is simply connected at both ends (see 4.6) so $H^2(G, \mathbb{Z}G) = 0$ by [11].

For the infinite ended case, some care is needed. Suppose the conclusion were false. Then, by Theorems 4.9 and 4.10, $H_e^1(\tilde{X}) \equiv \varinjlim_n H^1(\tilde{X} - L_n)$ would not be free abelian. So $\mathscr{H}_1(\tilde{X}) \equiv \{H_1(\tilde{X} - L_n)\}$ would not be semistable. It is straightforward to show (see [8] for details) that in this case there would be a component Z_n of $\tilde{X} - L_n$, with $Z_n \supset Z_{n+1}$, such that the inverse sequence $\{H_1(Z_n)\}$ is not semistable. But if ω is the proper ray which determines the end defined by $\{Z_n\}$ then $\mathscr{G}(\tilde{X}, \omega)$ abelianizes to $\{H_1(Z_n)\}$. Since $\mathscr{G}(\tilde{X}, \omega)$ is semistable by hypothesis, so is $\{H_1(Z_n)\}$. Contradiction. □

Proof of Theorem 4.5. By Theorem 3.1, $\tilde{X}$ is strongly connected at ∞ and $\pi_1^e(\tilde{X}, \omega)$ is freely generated by $(E(\tilde{X}/J), z)$. Assume $E(\tilde{X}/J)$ has $k < \infty$ elements. Then $\mathscr{G}(\tilde{X}, \omega)$ is stable, so the abelianization of the finitely generated free group $\pi_1^e(\tilde{X}, \omega)$ of rank $k - 1$ is $H_1^e(\tilde{X})$, which must therefore be finitely generated free abelian of rank $k - 1$. By Proposition 1.2, and Theorem 4.9 and 4.10, the only possible ranks are 0 and 1. □

Proof of Theorem 4.8. Similar, but use Theorem 3.3 in place of Theorem 3.1. □

Proof of Corollary C'. Without loss of generality we may assume that the center of G contains an element w of infinite order, and that $\tilde{X}^1$ is the Cayley graph of G with respect to a finite set of generators $\{g_1, \ldots, g_n\}$. An edge in $\tilde{X}^1$ connects g to each gg_i, where $g \in G$. Pick an edge path τ in $\tilde{X}^1$ joining 1 to w. For any g_i and $g \in G$ there is a loop: g to gw (via $g\tau$) to $gwg_i = gg_iw$ (by an edge) to gg_i (via $gg_i\tau^{-1}$) to g (by an edge). This loop has length $2|w| + 2$ in the word metric, so it bounds a singular 2-disk in $\tilde{X}$ involving a number of 2-cells which is independent of g. Given a compact set C in $\tilde{X}$, there is a compact set D in $\tilde{X}$ such that for any loop α outside D, singular disks of the above kind can be pieced together to build a homotopy between α and $w\alpha$ outside C. Thus w induces an inner automorphism of $\pi_1^e(\tilde{X}, \omega)$.

For the definition of a Cantor set, see Remark 4.1. Hopf's well-known proof that if $\tilde{X}$ has more than two ends then $E(\tilde{X})$ is a Cantor set works on any intermediate covering space which is non-compact and cocompact [21, Theorem 5.4]. Since the action of $G/\langle w \rangle$ on $\tilde{X}/\langle w \rangle$ is cocompact, $E(\tilde{X}/\langle w \rangle)$ has one or two points, or is a Cantor set. Now apply Theorem 4.5. □

5. WRIGHT'S THEOREM, AND APPLICATIONS

Recall the notation $\mathscr{G}(Y, \omega)$ from Section 1, and the definition of pro-isomorphism from Appendix. These occur in the following important theorem of Wright. We only quote the one-ended case:

THEOREM 5.1. [24, Theorem 9.1]. *Let the infinite cyclic group J act as a group of covering transformations on the simply connected one-ended space Y, and let ω be a base ray. If $\mathscr{G}(Y, \omega)$ is essentially monomorphic, then $\mathscr{G}(Y, \omega)$ is pro-isomorphic to an inverse sequence of free groups.*

Essentially monomorphic sequences overlap with semistable sequences precisely in stable sequences. Thus, Theorem 5.1 gives:

THEOREM 5.2. *Let the infinite cyclic group J act as a group covering transformations on Y, where Y is simply connected and strongly connected at ∞. If $\pi_1^e(Y, \omega)$ is discrete (equivalently, if $\mathscr{G}(Y, \omega)$ is stable) then $\pi_1^e(Y, \omega)$ is free and finitely generated.*

Proof. By 5.1, $\mathscr{G}(Y, \omega)$ is a stable sequence of free groups, hence $\mathscr{H}_1(Y)$ is a stable sequence of free abelian groups all of which are finitely generated, by 1.2. So $H_1^e(Y)$ is free abelian of finite rank, say k. Since $\pi_1^e(Y, \omega)$ is free and abelianizes to $H_1^e(Y)$, it is free of rank k. ||

Combining this with Proposition 1.2(iii), Theorems 4.9 and 4.10, we get:

THEOREM 5.3. *Let X be a finite connected complex, and let $G \equiv \pi_1(X, v)$ contain an element of infinite order. Assume G is semistable at ∞. For any base ray ω, if $\pi_1^e(\tilde{X}, \omega)$ is discrete then $\pi_1^e(\tilde{X}, \omega)$ is trivial or infinite cyclic.*

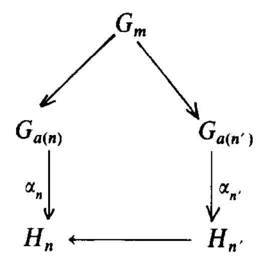
Remark 5.4. In the proofs of Theorem 3.1 [resp. Theorem 3.3] we used the fact that j is properly homotopic to id [resp. induces an inner automorphism] to draw simplifying conclusions about the mapping torus of j. In the absence of that hypothesis on j, the methods of Sections 3 and 4 can still be pursued, and they yield a proof of Theorem 5.3 which is independent of Wright's Theorem.

Remark 5.5. The interested reader can easily prove that if the finitely generated group G has infinitely many ends, if $\mathrm{Aut}(E(\tilde{X}))$ denotes the group of self homeomorphisms of the space of ends, and if $\sigma: G \to \mathrm{Aut}(E(\tilde{X}))$ is the natural representation, then every element of $\ker \sigma$ has finite order in G. In particular, when G is torsion free σ is faithful.

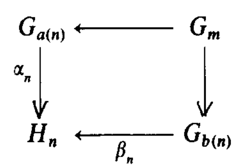
APPENDIX

Here we collect some information about $\pi_1^e(Y, \omega)$ which is well-known in certain circles but is not easy to find in the literature. A full exposition as it pertains to geometric group theory will be included in the first-named author's forthcoming book on the subject [8].

We abbreviate the inverse sequence of groups $G_1 \xleftarrow{\phi_1} G_2 \xleftarrow{\phi_2} \cdots$ to $\{G_n\}$, suppressing the bonds ϕ_n. A *map* $\alpha: \{G_m\} \to \{H_n\}$ consists of an increasing function $a: \mathbb{N} \to \mathbb{N}$ and for each n a homomorphism $\alpha_n: G_{a(n)} \to H_n$ such that whenever $n' \geq n$, there exists $m \geq a(n')$ making the following diagram commute:

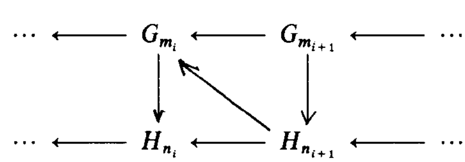


Two maps $\alpha, \beta: \{G_m\} \to \{H_n\}$ are *equivalent* if for each n there exists $m \geq \max\{a(n), b(n)\}$ such that the following diagram commutes:



Here the unmarked arrows are bonds. There is a category called pro-Groups whose objects are inverse systems of groups indexed by directed sets. We will only be interested in the full subcategory generated by inverse sequences (or *towers*). A morphism between the towers $\{G_m\}$ and $\{H_n\}$ is defined to be an equivalence class of maps. (We omit the general definition of a morphism of pro-Groups.). There is an obvious definition of composition of such morphisms between towers of groups. We will denote this full subcategory of pro-Groups by towers-Groups.

The towers $\{G_m\}$ and $\{H_n\}$ are *pro-isomorphic* if they are isomorphic objects of towers-Groups; explicitly, this happens if and only if there exist cofinal subtowers and homomorphisms making the following diagram commute:



Indeed for any category $\mathscr{C}$, there is an entirely analogous category towers-$\mathscr{C}$. A good general reference for pro-categories is [16]. If $\mathscr{C}$ admits countable inverse limits then $\varprojlim$ is a covariant functor towers-$\mathscr{C} \to \mathscr{C}$.

On the full subcategory of semistable towers, $\varprojlim$ does not lose information provided it is taken in the category topological groups, i.e.

$$\mathscr{L} \equiv \varprojlim : \text{Semistable towers} \to \mathscr{M}.$$

where $\mathscr{M}$ was defined in Section 1. To see this consider the functor $\mathscr{P}: \mathscr{M} \to$ Semistable towers which maps the object M of $\mathscr{M}$ to $\{M/I_n\}$, where I_n varies over all open subgroups of M. (See [4] for details: this idea goes back to [1].) The point is that $\mathscr{L} \circ \mathscr{P}$ and $\mathscr{P} \circ \mathscr{L}$ are natural equivalences. This is the context in which one should view $\pi_1^e(Y, \omega)$. One can recover $\mathscr{G}(Y, \omega) \equiv \{\pi_1(Y - L_n, \omega(n))\}$ from $\pi_1^e(Y, \omega)$ up to isomorphism in towers-Groups provided $\mathscr{G}(Y, \omega)$ is semistable.

Let $\{G_m\}$ be a tower of groups. The pointed set $\varprojlim^1 \{G_m\}$ is the orbit space of the action of $\prod_m G_m$ on itself by

$$(x_m)(g_m) = x_m g_m \phi_m(x_{m+1})^{-1}.$$

When every G_m is abelian this has a natural abelian group structure. This $\varprojlim^1$ is a covariant functor towers-Groups $\to$ Pointed Sets (see [7] or [5]), though one rarely needs this. Relevant to the present paper is the fact that $\varprojlim^1 \{G_m\}$ is trivial if $\{G_m\}$ is semistable, and that the less obvious converse is true when every G_m is countable (see [12] for the abelian case and [7] for the general case; also [16, p. 173]). If the base ray of Y is ω, there is a natural bijection (see below) between $\varprojlim^1 \mathscr{G}(Y, \omega)$ and the set of proper homotopy classes of proper rays determining the same end as ω. Hence the open problem discussed in Remark 4.2 is a problem about the vanishing of $\lim^1$.

One can form *strong homotopy groups* ${}^S\pi_i(Y, \omega)$ using base ray preserving proper maps $(S^i, *) \times [0, \infty) \to (Y, \omega)$ in the obvious fashion, and one has a short exact seqnence (of groups when $i > 0$, of pointed sets when $i = 0$):

$$0 \to \varprojlim^1 \{\pi_{i+1}(Y - K_n, \omega(n))\} \to {}^S\pi_i(Y, \omega) \to \varprojlim \{\pi_i(Y - K_n, \omega(n))\} \to 0.$$

See, for example, [2, §2]. When $i = 1$, this can be useful in computing $\pi_1^e(Y, \omega)$.

We end with the promised proof of Proposition 3.6; for more information of this kind, see [9, Proposition 8.3]. Let the compact subsets $L_n \subset Y_1$ be as before, and let a similar collection of compact subsets $M_n \subset Y_2$ be chosen. Choose a proper homotopy inverse g for f. Let $H: g \circ f \simeq \mathrm{id}_{Y_1}$ and $\bar{H}$: $g \circ f \simeq \mathrm{id}_{Y_2}$ be proper homotopies. It may be assumed that for all t and n, $H_t \circ \omega([n, \infty)) \subset Y_1 - L_n$ and $\bar{H}_t \circ f \circ \omega([n, \infty)) \subset Y_2 - M_n$. Let α_n be the path (in Y_1) $\alpha_n(t) = H_t \omega(n)$ and let β_n be the path (in Y_2) $\beta_n(t) = \bar{H}_t f\omega(n)$. The isomorphisms $\pi_1(Y_1 - L_n, gf\omega(n)) \xrightarrow{\alpha_{n\#}} \pi_1(Y_1 - L_n, \omega(n))$ given by $[\sigma_n] \mapsto [\alpha_n^{-1} \cdot \sigma_n \cdot \alpha_n]$ fit together to give an isomorphism of towers-Groups

$$\alpha_\#: \{\pi_1(Y_1 - L_n, gf\omega(n))\} \to \{\pi_1(Y_1 - L_n, \omega(n))\}.$$

There is a similar definition for $\beta_\#$. Note that the proper homotopies H and $\bar{H}$ are needed to establish that these are indeed isomorphisms.

The maps f and g induce morphisms

$$f_\#: \{\pi_1(Y_1 - L_n, \omega(n))\} \to \{\pi_1(Y_2 - M_n, gf\omega(n))\}.$$

and

$$g_\#: \{\pi_1(Y_2 - M_n, f\omega(n))\} \to \{\pi_1(Y_1 - L_n, gf\omega(n))\}$$

(some details are omitted here). Thus,

$$\alpha_\# g_\# f_\# = \mathrm{id} \text{ and } \quad \beta_\# f_\# g_\# = \mathrm{id}. \qquad \square$$

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