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A new analytic solution for fractional chaotic dynamical systems using the differential transform method

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ABSTRACT

Nonlinear differential equations with fractional derivatives give general representations of real life phenomena. In this paper, a modification of the differential transform method (DTM) for solving the nonlinear fractional differential equation is introduced for the first time. The new algorithm is simple and gives an accurate solution. Moreover the new solution is continuous and analytic on each subinterval. A fractional Chen system is considered, to demonstrate the efficiency of the algorithm. The results obtained show good agreement with the generalized Adams–Bashforth–Moulton method.

1. Introduction

Nature is intrinsically nonlinear. So, it is not surprising that most of the systems that we encounter in the real world are nonlinear. And what is interesting is that some of these nonlinear systems can be described using fractional order differential equations which can display a variety of behaviors including chaos and hyperchaos. Moreover, many authors have pointed out that derivatives and integrals of non-integer order are very suitable for the description of properties of various real materials [1]. The advantage of fractional derivatives becomes apparent in modeling mechanical and electrical properties of real materials as well as the fractional Chen system [2]:

$$D_t^{\alpha} x = a(y - x), \tag{1}$$

$$D_t^{\alpha} y = (c - a)x - xz + cy, \tag{2}$$

$$D_t^{\alpha} z = xy - bz, \tag{3}$$

$$x(0) = c_1 \qquad y(0) = c_2, \qquad z(0) = c_3, \tag{4}$$

where D^{α} is the Caputo fractional derivative, a, b, c are from $\Re$ and $0 < \alpha \leq 1$. It is interesting to note that the (positive) Lyapunov exponent for the standard Chen system ($\alpha = 1$) is about 2.0272, whereas the corresponding exponent for the standard Lorenz system is about 0.9056 [3]. In other words, the Chen system is more sensitive to the initial conditions compared to the Lorenz system.

Finding accurate and efficient methods for solving fractional differential equations (FDEs) has been an active research undertaking. Exact solutions of most of the FDEs cannot be found easily; thus analytical and numerical methods must be used [4]. For example, the generalized Adams–Bashforth–Moulton method (GABMM) is one of the methods most used to solve fractional differential equations [5–9]. Some of the analytic methods for solving nonlinear problems include He's homotopy perturbation method (HPM) [10–17], He's variational iteration method (VIM) [18–21], the Adomian

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decomposition method (ADM) [22,23] and the homotopy analysis method (HAM) [24–26]. All of the above analytical methods have been used in a sequence of subintervals; see [27–29].

After Zhou [30] introduced a powerful analytical technique, namely the differential transform method (DTM), Arikoglu and Ozkol [30] and Odibat et al. [31] extended the algorithm to solve fractional differential equations. Keskin and Oturanc [32,33] presented the reduced differential transform method for standard and fractional partial differential equations. Odibat et al. [34] treated the DTM as an algorithm in a sequence of intervals. A fractional chaotic system is one of the systems that cannot be solved by that extension. Thus, a new algorithm should be presented to overcome this limitation.

The aim of this paper is to obtain the solution of the fractional Chen system using the DTM in a sequence of intervals (i.e. time steps) for finding accurate approximate solutions to the nonlinear FDEs. This new algorithm is called the modified differential transform method (MDTM). To the best of our knowledge, this is also the first time that an analytical solution has been obtained for a fractional chaotic system by using the DTM. The Odibat et al. [34] algorithm is a special case of the new algorithm. Numerical results are presented graphically and are found to be in excellent agreement with the GABMM solution.

2. Preliminaries and notation

In this section, we give some definitions and properties of the fractional calculus and differential transformation method. For more details see [1,30].

2.1. Fractional calculus

Definition 1. A real function $f(t)$, $t > 0$, is said to be in the space C_μ, $\mu \in \Re$, if there exists a real number $p > \mu$ such that $f(t) = t^p f_1(t)$, where $f_1(t) \in C(0, \infty)$, and it is said to be in the space C_μ^n if and only if $h^{(n)} \in C_\mu$, $n \in N$.

Definition 2. The Riemann–Liouville fractional integral operator (J^α) of order $\alpha \geq 0$ for a function $f \in C_\mu$, $\mu \geq -1$, is defined as

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(t) \mathrm{d}s \quad (\alpha > 0), \tag{5}$$

$$J^0 f(t) = f(t), \tag{6}$$

where $\Gamma(\alpha)$ is the well-known gamma function.

2.2. The differential transform method

Firstly, expand the analytical function $f(t)$ in terms of a fractional power series as follows:

$$f(t) = \sum_{k=0}^{\infty} F(k)(t-t_0)^{k\alpha} \tag{7}$$

where $0 < \alpha \leq 1$ is the order of the fractional derivative and $F(k)$ is the fractional differential transform of $f(t)$ given as

$$F(k) = \frac{1}{\Gamma(\alpha k + 1)} [(D_{t_0}^\alpha)^k (f(t_0))] \tag{8}$$

where $(D_{t_0}^\alpha)^k = D_{t_0}^\alpha \cdot D_{t_0}^\alpha \cdots D_{t_0}^\alpha$, the k-times-differentiable Caputo fractional derivative.

In our applications, we will approximate the function $f(t)$ by the finite series

$$f(t) = \sum_{k=0}^{N} F(k)(t-t_0)^{(k\alpha)}.$$

The following are the basic properties of the Caputo fractional derivative and the differential transformation [31]:

(1) Let $f \in C_{-1}^n$, $n \in N$; then $D^\alpha f$, $0 \leq \alpha \leq n$, is well defined and $D^{\alpha t_0} f \in C_{-1}$.
(2) If $f(t) = g(t) \pm h(x)$, then $F(k) = G(k) \pm H(k)$.
(3) If $f(t) = g(t)h(t)$, then $F(k) = \sum_{l=0}^{k} G(l)H(k-l)$.
(4) If $f(t) = D^{\alpha t_0}[g(t)]$, then $F(k) = \frac{\Gamma(\alpha(k+1)+1)}{\Gamma(\alpha k+1)} G(\alpha + 1)$.

3. Solution approaches

The differential transform method for a system of FDEs and the new modification of the DTM will be discussed in this section.

3.1. The DTM for a system of FDEs

To illustrate the basic idea of the DTM for a system of fractional differential equations (FDEs), we consider the system of the form

$$N_i[\mathbf{f}(t)] := D^{\alpha} f_i(t) \quad i = 1, 2, \ldots, n, \tag{9}$$

where $N_i[\mathbf{f}(t)]$ are the linear or nonlinear terms of the FDEs, and $D^{\alpha} f_i(t)$ is the α Caputo fractional derivative of unknown function $f_i(t)$. Suppose $f_i(t)$ to be analytic in a domain D and let $t = t_j$ represent any point in D. Then, the function $f_i(t)$ can be represented by power series whose center is located at t_j. The Taylor series expansion function of $f_i(t)$ is of the form

$$f_i(t) = \sum_{k=0}^{\infty} F_i(k)(t - t_0)^{k\alpha}. \tag{10}$$

Taking the transform of Eq. (9), we have

$$\frac{\Gamma(\alpha(k+1)+1)}{\Gamma(\alpha k+1)} F_i(k+1) = R_{i,m}[\overrightarrow{F}_i(k)], \tag{11}$$

where

$$R_{i,m}[\overrightarrow{F}_i(k)] = \frac{1}{\Gamma(\alpha k+1)} [[(D_{t_0}^{\alpha})^k (N_i[\mathbf{f}(t)])]_{t=t_0}], \tag{12}$$

and the vector $\overrightarrow{F}_i(k) = \{F_i(0), F_i(1), F_i(2), \ldots, F_i(k)\}$ for $i = 1, 2, \ldots, n$ and $k = 1, 2, 3, \ldots$. Eq. (11) is a recursion formula starting from $F_i(0)$ which will be considered as the same value as for the initial condition.

3.2. The modified differential transform method

In this subsection, we introduce a new technique as a hybrid of the numerical discretization with the DTM. If we need the solution for $[0, T]$, then the simple idea is to divide the interval $[0, T]$ into subintervals with time step Δt and we get the solution in each subinterval. So in this case we have to satisfy the initial condition for each of the subintervals. Accordingly, the initial values $F_i(0)$ will be changed for each subinterval, i.e. $f_i(t^*) = c_i^* = F_i(0)$. To carry out the solution on every subinterval of equal length Δt, we need to know the value of the initial condition $c_i = f_i(t^*)$. In general, we do not have this information at our disposal except at the initial point $t^* = t_0 = 0$, but we can obtain these values by assuming that the new initial condition is the solution in the previous interval. i.e. if we need the solution in interval $[t_j, t_{j+1}]$, then the initial conditions of this interval will be

$$c_i = f_i(t_j) = \sum_{m=0}^{N} F_i(m)(t_j - t_{j-1})^{(m\alpha)}, \tag{13}$$

where c_i is the initial condition in the interval $[t_j, t_{j+1}]$. In this way we modified the DTM. It is worth mentioning that the Odibat et al. [34] algorithm is a special case of this modification.

4. Applications

In this section, we apply the DTM and its modification for the fractional Chen system.

4.1. Solution by the DTM

Consider the fractional Chen system (1)–(4). The differential transformation for this system is

$$\frac{\Gamma(\alpha(k+1)+1)}{\Gamma(\alpha k+1)} X(k+1) = a(Y(k) - X(k)), \tag{14}$$

$$\frac{\Gamma(\alpha(k+1)+1)}{\Gamma(\alpha k+1)} Y(k+1) = (c-a)X(k) - \sum_{m=0}^{k} X(k)Z(k-m) + cY(k), \tag{15}$$

$$\frac{\Gamma(\alpha(k+1)+1)}{\Gamma(\alpha k+1)} Z(k+1) = \sum_{m=0}^{k} X(m)Y(k-m) - bZ(k), \tag{16}$$

where

$$X(0) = c_1, \qquad Y(0) = c_2, \qquad Z(0) = c_3. \tag{17}$$

Eqs. (14)–(16) give the recursion formula for the differential transformation of the fractional Chen system, starting from $X(0)$, $Y(0)$ and $Z(0)$ of Eq. (17). The first few terms are

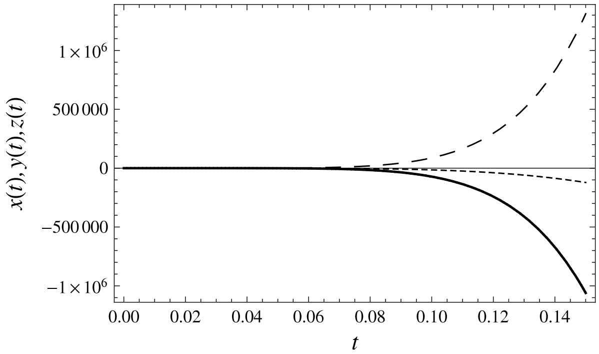

Fig. 1. DTM solution for the Chen system when $\alpha = 0.99$.

$$X(1) = \frac{ac_2 - ac_1}{\Gamma(\alpha+1)},$$
$$Y(1) = \frac{c_1(c-a) - c_1c_3 + c_2c}{\Gamma(\alpha+1)},$$
$$Z(1) = \frac{c_1c_2 - bc_3}{\Gamma(\alpha+1)},$$
$$X(2) = \frac{a(-ac_2 + c_1(c-c_3) + c_2c)}{\Gamma(2\alpha+1)},$$
$$Y(2) = \frac{1}{\Gamma(2\alpha+1)}(a^2(c_1-c_2) + a(c_1(c_3-2c) + c_2(c-c_3)) + c_1(bc_3 - c_3c + c^2) - {c_1}^2c_2 + c_2c^2),$$
$$Z(2) = \frac{1}{\Gamma(2\alpha+1)}(-a({c_1}^2 + c_1c_2 - {c_2}^2) + b^2c_3 - bc_1c_2 + c_1(-c_1c_3 + c_1c + c_2c)),$$

etc. Thus, the N-order solutions with the inverse transform are

$$x(t) = \sum_{m=0}^{N} X(m)(t-t_0)^{(m\alpha)} \tag{18}$$

$$y(t) = \sum_{m=0}^{N} Y(m)(t-t_0)^{(m\alpha)} \tag{19}$$

$$z(t) = \sum_{m=0}^{N} Z(m)(t-t_0)^{(m\alpha)}. \tag{20}$$

Fig. 1 presents the DTM solution for the Chen system; it is clear that the solution is becoming unbounded, which does not agree with the dynamical property of the system. Thus the DTM solution is not effective for a longer time span.

4.2. Solution by the MDTM

The solution for Eqs. (1)–(3) is not effective for larger t. If we need the solution for $[0, 7]$, then the simple idea is to divide the interval $[0, 7]$ into subintervals with time step Δt and we get the solution at each subinterval. So in this case we have to satisfy the initial condition at each of the subintervals. Accordingly, the initial values $x(0), y(0), z(0)$ will be changed for each subinterval, i.e. $x(t_0) = c_1^* = X(0)$, $y(t_0) = c_2^* = Y(0)$ and $z(t_0) = c_3^* = Z(0)$.

Therefore, the solution will be as follows:

$$x(t) = c_1^* + \sum_{m=1}^{N} X(m)(t-t_0)^{(m\alpha)}, \tag{21}$$

$$y(t) = c_2^* + \sum_{m=1}^{N} Y(m)(t-t_0)^{(m\alpha)}, \tag{22}$$

$$z(t) = c_3^* + \sum_{m=1}^{N} Z(m)(t-t_0)^{(m\alpha)}, \tag{23}$$

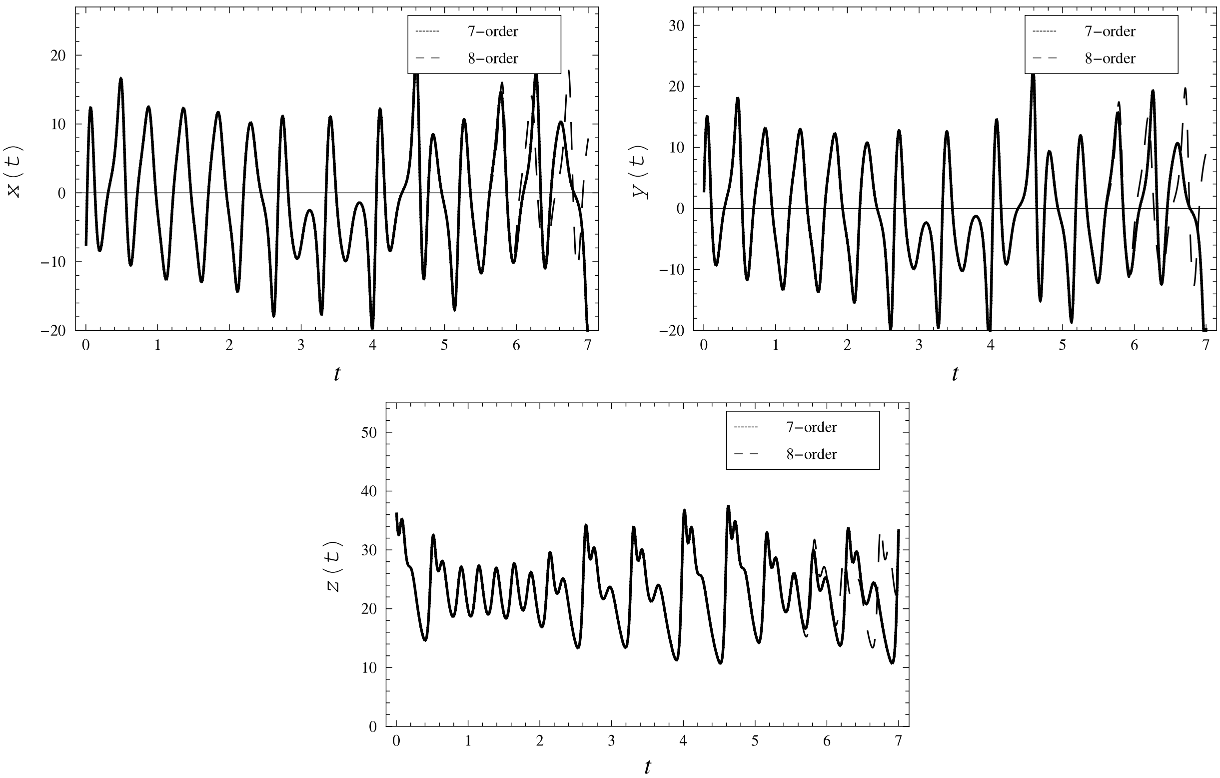

Fig. 2. Time series of the order 7 and order 8 approximations for the Chen system with $\alpha = 0.99$.

where t_0 goes from $t_0 = 0$ to $t_n = T = 7$. To carry out the solution on every subinterval of equal length Δt, we need to know the values of the following initial conditions:

$$c_1 = x(t_0), \quad c_2 = y(t_0), \quad c_3 = z(t_0).$$

In general, we do not have this information at our disposal except at the initial point $t^* = t_0 = 0$, but we can obtain these values by assuming that the new initial condition is the solution in the previous interval, i.e. if we need the solution in interval $[t_j, t_{j+1}]$, then the initial conditions of this interval will be

$$c_1 = x(t_j) = \sum_{m=0}^{N} X(m)(t_j - t_{j-1})^{m\alpha}, \tag{24}$$

$$c_2 = y(t_j) = \sum_{m=0}^{N} Y(m)(t_j - t_{j-1})^{m\alpha}, \tag{25}$$

$$c_3 = z(t_j) = \sum_{m=0}^{N} Z(m)(t_j - t_{j-1})^{m\alpha}, \tag{26}$$

where c_1, c_2 and c_3 are the initial conditions in the interval $[t_j, t_{j+1}]$. The solution obtained by Odibat et al. [34] is a special case of the above solution when $\alpha = 1$.

5. Results and discussion

In this part, we set $a = 35$, $b = 3$, $c = 28$ and we take the initial conditions $x(0) = -10$, $y(0) = 0$ and $z(0) = 37$ as in [24] for the standard case $\alpha = 1$. To observe the convergence of the solution, we present order 7 and order 8 MDTM solutions with $\Delta t = 0.0025$ in Fig. 2. It is clear that the solution of order 7 is like the solution of order 8; therefore we can consider the order 8 one as a good approximate solution. The phase portraits of the MDTM solution and the GABMM solution are given in Figs. 3 and 4, respectively, at different fractional derivatives. The figures show that the MDTM solution is in good agreement with the GABMM solution. The algorithm obtained by Odibat et al. [34] is a special case of the above algorithm when $\alpha = 1$. Thus, the general algorithm for different values of $0 < \alpha \leq 1$ is given.

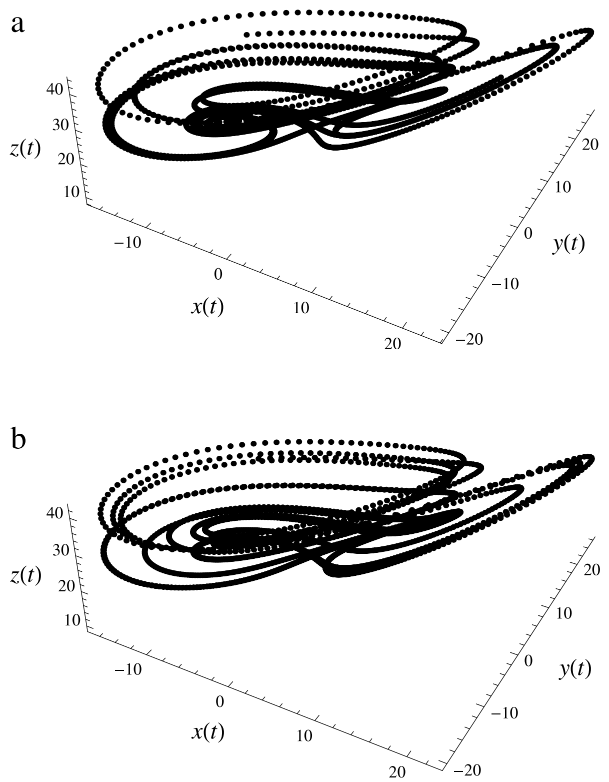

Fig. 3. Comparison of the x–y–z phase portraits when $\alpha = 1$ using (a) the GABMM, (b) the order 8 MDTM.

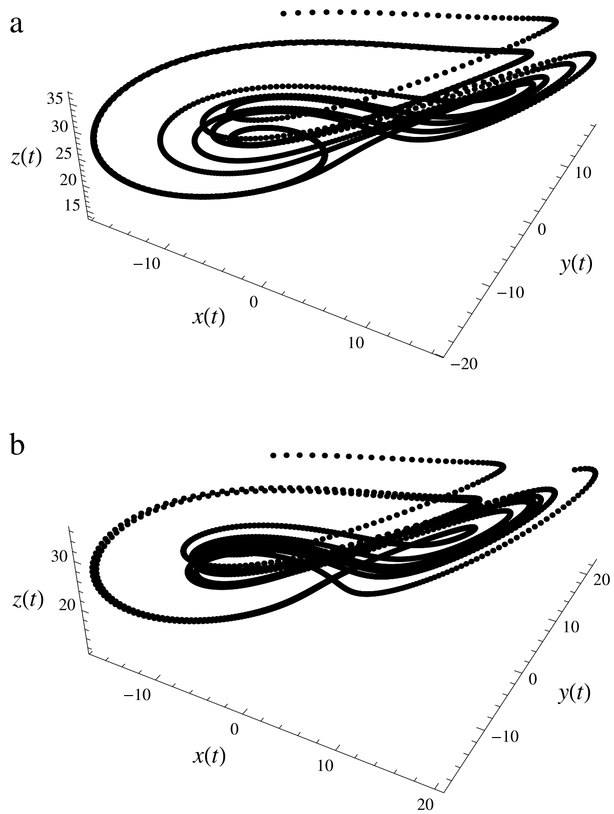

Fig. 4. Comparison of the x–y–z phase portraits when $\alpha = 0.99$ using (a) the GABMM, (b) the order 8 MDTM.

6. Conclusions

In this present work, the modified DTM was introduced to obtain the solutions of the fractional Chen system by time discretization. The modified method has the advantage of giving an analytical form of the solution within each time interval which is not possible using purely numerical techniques like the fourth-order Runge–Kutta method (RK4) or the ABMM. We also note that the MDTM solutions were computed via a simple algorithm without any need for perturbation techniques, special transformations or linearization. The MDTM solutions are in excellent agreement with the GABMM solutions. Solving a system of partial differential equations is one of the open problems for which it would be interesting to see whether the above algorithm can be applied.

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