On the existence of periodic solutions for a class of $p$-Laplacian system

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Abstract
By using generalized Borsuk theorem in coincidence degree theory, some criteria to guarantee the existence of $\omega$-periodic solutions for a class of $p$-Laplacian system are derived.

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1. Introduction

Throughout this paper, $1 < p < \infty$ is a fixed real number. The conjugate exponent of $p$ is denoted by $q$, i.e., $\frac{1}{p} + \frac{1}{q} = 1$. Let $\phi_p : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be the mapping defined by

$$\phi_p(u) = \phi_p(u_1, \ldots, u_n) := (|u_1|^{p-2}u_1, \ldots, |u_n|^{p-2}u_n)^T.$$ 

Then $\phi_p$ is a homeomorphism of $\mathbb{R}^n$ with the inverse $\phi_q$.

In this paper, we will consider the existence of periodic solutions of the following system:

$$\left(\phi_p(u'(t))\right)' + \frac{d}{dt} \text{grad } F(u(t)) + \text{grad } G(u(t)) = e(t), \quad (1.1)$$
where $F \in C^2(\mathbb{R}^n, \mathbb{R})$, $G \in C^1(\mathbb{R}^n, \mathbb{R})$, $e \in C(\mathbb{R}, \mathbb{R}^n)$, $e(t + \omega) \equiv e(t)$ for all $t \in \mathbb{R}$, $\omega > 0$ is fixed.

In recent years, the existence of periodic solutions of (1.1) for $p = 2$ has been extensively studied (see [1–7]). In [5], by using Krasnoselskii’s fixed point theorem, Ding proved the following result.

**Theorem A.** (See [5].) Suppose that there exist constants $c_0 > 0$, $a_0 > 0$, $a_1 > 0$, $b_0 \geq 0$, $b_1 \geq 0$, and $\alpha > 1$, such that

(a) $y^T \frac{\partial^2 F(x)}{\partial x^2} y \geq c_0 |y|^2_2$, $\forall x, y \in \mathbb{R}^n$,
(b) $G(x) \geq 0$ and $G(x) \geq a_0 |x|^\alpha - b_0$, $\forall x \in \mathbb{R}^n$,
(c) $x^T \text{grad} G(x) \geq a_1 G(x) - b_1$, $\forall x \in \mathbb{R}^n$.

Then (1.1) has at least one $\omega$-periodic solution for $p = 2$.

Many results were also given by using topological degree theory; see, for example, [1–4,6,7] and the references therein. Some researchers discussed the existence of periodic solutions to scalar $p$-Laplacian differential equations in [8–11,13,14]. But the existence of periodic solutions of (1.1) for $p \neq 2$ and $n > 1$, as far as we know, has rarely been studied. For general differential systems of $p$-Laplacian type, M.R. Zhang [12] has considered the Dirichlet boundary value problems

$$-(\phi_p(u'(t)))' = f(t, u(t), u'(t)), \quad t \in [0, \omega], \quad u(0) = u(\omega) = 0. \quad (1.2)$$

R. Manásevich and J. Mawhin [13] have discussed the periodic boundary value problems

$$\left(\phi(u'(t))\right)' = f(t, u(t), u'(t)), \quad t \in [0, \omega], \quad u(0) = u(\omega), \quad u'(0) = u'(\omega), \quad (1.3)$$

where the function $\phi : \mathbb{R}^n \to \mathbb{R}^n$ satisfies some monotonicity conditions which ensure that $\phi$ is an homeomorphism onto $\mathbb{R}^n$. They have also given some applications for $\phi = \phi_p$ in [13]. On basis of application of Schauder’s fixed point theorem, Mawhin [14] generalized the Hartman–Knobloch results on the periodic boundary value problem in [15,16] to perturbations of the vector $p$-Laplacian ordinary operator of the form

$$\left(\psi_p(u')\right)' = f(t, u), \quad (1.4)$$

where $\psi_p : \mathbb{R}^n \to \mathbb{R}^n$ is defined by $\psi_p(u) = |u|^{p-2}u$.

The purpose of this paper is to establish some criteria to guarantee the existence of $\omega$-periodic solutions for (1.1) by using coincidence degree theory. The methods used to estimated a priori bound of periodic solutions are different from the corresponding ones in [1–7]. Furthermore, the significance of this paper is that Theorems 3.2 and 3.3 do not impose any other condition on the function $F(x)$ besides $F$ is twice continuously differentiable. When $p = 2$, the results in this paper are also different from those in [1–7].

In what follows, we will use $(\cdot, \cdot)$ to denote the Euclidean inner product in $\mathbb{R}^n$, $|\cdot|_p$ denotes the $l^p$-norm in $\mathbb{R}^n$, i.e.,

$$|x|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}.$$
The norm in $\mathbb{R}^{n\times n}$ is defined by
$$\|A\|_p = \sup_{|x|_p = 1, \; x \in \mathbb{R}^n} |Ax|_p.$$ 

The corresponding $L^p$-norm in $L^p([0, \omega], \mathbb{R}^n)$ is defined by
$$\|x\|_p = \left( \sum_{i=1}^n \int_0^\omega |x_i(t)|^p \, dt \right)^{1/p}.$$ 

The $L^\infty$-norm in $L^\infty([0, \omega], \mathbb{R}^n)$ is
$$\|x\|_{\infty} = \max_{1 \leq i \leq n} \|x_i\|_{\infty},$$ 
where $\|x_i\|_{\infty} = \sup_{t \in [0, \omega]} |x_i(t)| \; (i = 1, \ldots, n)$.

2. Preliminaries

Let $X$ and $Z$ be real normed vector spaces, $L : \text{Dom} \; L \subset X \rightarrow Z$ be a linear mapping, and $N : X \rightarrow Z$ be a continuous mapping. The mapping $L$ will be called a Fredholm mapping of index zero if $\dim \ker L = \text{codim} \; \text{Im} \; L < +\infty$ and $\text{Im} \; L$ is closed in $Z$. If $L$ is a Fredholm mapping of index zero and there exist continuous projections $P : X \rightarrow X$ and $Q : Z \rightarrow Z$ such that $\text{Im} \; P = \ker L$, $\text{Im} \; L = \ker Q = \text{Im}(I - Q)$, it follows that $L|_{\text{Dom} \; L \cap \ker P} : (I - P)X \rightarrow \text{Im} \; L$ is invertible. We denote the inverse of that mapping by $K_P$. If $\Omega$ is an open bounded subset of $X$, the mapping $N$ will be called $L$-compact on $\bar{\Omega}$ if $QN(\bar{\Omega})$ is bounded and $K_P (I - Q)N : \bar{\Omega} \rightarrow X$ is compact.

In the proof of our results on existence of periodic solutions below, we will use the following generalized Borsuk theorem in coincidence degree of Gaines and Mawhin [17, p. 31].

**Lemma 2.1.** Let $L$ be a Fredholm mapping of index zero. $\Omega$ is an open bounded subset of $X$ and $\Omega$ is symmetric with respect to the origin and contains it. Let $\tilde{N} : \tilde{\Omega} \times [0, 1] \rightarrow Z$ be $L$-compact and such that

(a) $\tilde{N}(-x, 0) = -\tilde{N}(x, 0), \; \forall x \in \tilde{\Omega}$,

(b) $Lx \neq \tilde{N}(x, \lambda), \; \forall x \in \text{Dom} \; L \cap \partial \Omega$.

Then for every $\lambda \in [0, 1]$, equation
$$Lx = \tilde{N}(x, \lambda)$$
has at least one solution in $\Omega$.

Let $W = W^{1, p}([0, \omega], \mathbb{R}^n)$ be the Sobolev space.

**Lemma 2.2.** (See [12].) Suppose $u \in W$ and $u(0) = u(\omega) = 0$, then
$$\|u\|_p \leq \frac{\omega}{\pi_p} \|u'|_p,$$
where
$$\pi_p = 2 \int_0^\omega \frac{ds}{(1 - s^p)^{1/p}} = \frac{2\pi (p - 1)^{1/p}}{p \sin \left( \frac{\pi}{p} \right)}.$$
In order to use coincidence degree theory to study the existence of $\omega$-periodic solutions for (1.1), we rewrite (1.1) in the following form:

$$\begin{align*}
    x'(t) &= \phi_q(y(t)), \\
    y'(t) &= -\frac{d}{dt} \text{grad } F(x(t)) - \text{grad } G(x(t)) + e(t).
\end{align*}$$

(2.1)

If $z(t) = (x^T(t), y^T(t))^T$ is an $\omega$-periodic solution of (2.1), then $x(t)$ must be an $\omega$-periodic solution of (1.1). Thus, the problem of finding an $\omega$-periodic solution for (1.1) reduces to finding one for (2.1).

Let $C_\omega = \{x \in C(\mathbb{R}, \mathbb{R}^n): x(t + \omega) \equiv x(t)\}$ with norm $\|x\|_\infty = \max_{1 \leq i \leq n} \|x_i\|_\infty$, $X = Z = \{z = (x^T(\cdot), y^T(\cdot))^T \in C(\mathbb{R}, \mathbb{R}^{2n}): z(t + \omega) \equiv z(t)\}$ with norm $\|z\| = \max\{\|x\|_\infty, \|y\|_\infty\}$. Clearly, $X$ and $Z$ are Banach spaces. Meanwhile, let $L: \text{Dom } L \subset X \to Z, (Lz)(t) = z'(t) = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix}$, $N: X \to Z, (Nz)(t) = \begin{pmatrix} \phi_q(y(t)) \\ -\frac{d}{dt} \text{grad } F(x(t)) - \text{grad } G(x(t)) + e(t) \end{pmatrix} := H(z,t)$.

It is easy to see that $\text{ker } L = \mathbb{R}^{2n}$, $\text{Im } L = \{z \in Z: \int_0^\omega z(s) ds = 0\}$. So $L$ is a Fredholm operator with index zero. Let $P: X \to \ker L$ and $Q: Z \to \text{Im } Q$ be defined by

$$Pu = \frac{1}{\omega} \int_0^\omega u(s) ds, \quad u \in X; \quad Qv = \frac{1}{\omega} \int_0^\omega v(s) ds, \quad v \in Z,$$

and let $K_P$ denote the inverse of $L|_{\ker P \cap \text{Dom } L}$. Obviously, $\ker L = \text{Im } Q = \mathbb{R}^{2n}$ and

$$(K_P z)(t) = \int_0^t z(s) ds - \frac{1}{\omega} \int_0^\omega \int_0^t z(s) ds dt.$$  

(2.2)

From (2.2), one can easily see that $N$ is $L$-compact on $\tilde{\Omega}$, where $\Omega$ is an open bounded subset of $X$.

3. Existence of periodic solutions

**Theorem 3.1.** Suppose that there exist constants $a > 0$, $b > 0$, $c \geq 0$ and $\alpha > 1$, such that

(i) $y^T \frac{\partial^2 F(x)}{\partial x^2} y \geq a \|y\|_2^2$ or $y^T \frac{\partial^2 F(x)}{\partial x^2} y \leq -a \|y\|_2^2$, \forall x, y $\in \mathbb{R}^n$,

(ii) $\langle x, \text{grad } G(x) \rangle \geq b \|x\|_\alpha^\alpha - c$, \forall x $\in \mathbb{R}^n$.

Then (1.1) has at least one $\omega$-periodic solution for $1 < p \leq 2$.

**Proof.** For any $\lambda \in [0, 1]$, let

$$\tilde{N}(z, \lambda)(t) = \frac{1 + \lambda}{2} H(z, t) - \frac{1 - \lambda}{2} H(-z, t).$$

Consider the following parameter equation:

$$(Lz)(t) = \tilde{N}(z, \lambda)(t).$$

(3.1)
Let \( z(t) = (x(t), y(t)) \) be a possible \( \omega \)-periodic solution of (3.1) for some \( \lambda \in [0, 1] \). One can see \( x = x(t) \) is an \( \omega \)-periodic solution of the following system:

\[
(\phi_p(x'(t)))' + \frac{1 + \lambda}{2} \frac{d}{dt} \text{grad } F(x(t)) - \frac{1 - \lambda}{2} \frac{d}{dt} \text{grad } F(-x(t)) \\
+ \frac{1 + \lambda}{2} \text{grad } G(x(t)) - \frac{1 - \lambda}{2} \text{grad } G(-x(t)) = \lambda e(t).
\]

(3.2)

Noticing that \( x(t) \) is an \( \omega \)-periodic solution, we have

\[
-\|x'\|_p^p = \int_0^\omega \langle x, (\phi_p(x'))' \rangle \, dt,
\]

(3.3)

and

\[
\int_0^\omega \left( x(t), \frac{d}{dt} \text{grad } F(x(t)) \right) \, dt = [x(t), \text{grad } F(x(t))]_0^\omega - \int_0^\omega \left( \text{grad } F(x(t)), x'(t) \right) \, dt \\
= -\|F(x(t))\|_0^\omega = 0.
\]

(3.4)

From (ii), by (3.3) and (3.4), we can use (3.2) to obtain

\[
-\|x'\|_p^p + b\|x\|_\alpha^\alpha - c\omega \leq \lambda \int_0^\omega \langle x(t), e(t) \rangle \, dt \leq \|e\|_{\beta} \|x\|_\alpha,
\]

(3.5)

where \( \frac{1}{\alpha} + \frac{1}{\beta} = 1 \).

On the other hand,

\[
\int_0^\omega \langle x'(t), (\phi_p(x'(t)))' \rangle \, dt = \int_0^\omega \langle \phi_q(y(t)), y'(t) \rangle \, dt = 0.
\]

From (3.2), we have

\[
\int_0^\omega \left( x'(t), \frac{1 + \lambda}{2} \frac{d}{dt} \text{grad } F(x(t)) - \frac{1 - \lambda}{2} \frac{d}{dt} \text{grad } F(-x(t)) \right) \, dt = \lambda \int_0^\omega \langle x', e(t) \rangle \, dt.
\]

By (i), one can get

\[
a\|x'\|_2^2 \leq \|e\|_2 \|x'\|_2.
\]

So, we have

\[
\|x'\|_2 \leq \frac{\|e\|_2}{a} := R_1.
\]

(3.6)

It is obvious that there exist \( c_1 > 0 \) and \( c_2 > 0 \) such that

\[
c_1 |x|_2 \leq |x|_p \leq c_2 |x|_2, \quad x \in \mathbb{R}^n.
\]

Thus,
\[ \|x\|_p^p = \int_0^\omega |x'(t)|_p^p \, dt \leq c_2^p \int_0^\omega |x'(t)|^2_2 \, dt \leq c_2^p \left( \int_0^\omega |x'(t)|^2_2 \, dt \right)^{p/2} \omega^{(2-p)/2} \]

\[ \leq (c_2 R_1)^p \omega^{(2-p)/2} := R_2, \quad (3.7) \]

where \(1 < p \leq 2\).

From (3.5), we can see

\[ b\|x\|^2_d - \|e\|_f x - c \omega \leq R_2. \]

from which it follows that there exists a positive number \(R_3\) such that

\[ \|x\|_\alpha \leq R_3. \quad (3.8) \]

From (3.8), there exists \(t_0 \in [0, \omega)\), such that \(|x(t_0)|_\alpha \leq R_3 \alpha^{-1/\alpha}\), so

\[ |x_i(t)| = |x_i(t_0) + \int_{t_0}^t x'_i(s) \, ds| \leq R_3 \alpha^{-1/\alpha} + \sqrt{\omega} \left( \int_0^\omega (x'_i(s))^2 \, ds \right)^{1/2} \]

\[ \leq R_3 \alpha^{-1/\alpha} + \sqrt{\alpha} R_1 := R_4. \]

Therefore \(\|x\|_\infty \leq R_4\) and \(|x(t)|_p \leq n^{1/p} R_4\).

Since \(F \in C^2(\mathbb{R}^n, \mathbb{R})\), \(G \in C^1(\mathbb{R}^n, \mathbb{R})\), there exist \(R_5\) and \(R_6\) such that \(\|F(x)/\alpha x^2\|_p \leq R_5\), \(|\text{grad} \, G(x)|_p \leq R_6\) for \(|x|_p \leq n^{1/p} R_4\). From (3.2), we have

\[ \int_0^\omega |(\phi_p(x'))'|_p \, dt \leq R_5 \int_0^\omega |x'|_p \, dt + R_6 \omega + \int_0^\omega |e(t)|_p \, dt \]

\[ \leq R_5 \alpha^{1/\alpha} \|x\|_p + R_6 \omega + \int_0^\omega |e(t)|_p \, dt \]

\[ \leq R_5 \alpha^{1/\alpha} R_2^{1/p} + R_6 \omega + \int_0^\omega |e(t)|_p \, dt := R_7. \quad (3.9) \]

Clearly, for each \(i = 1, \ldots, n\), there exists \(t_i \in (0, \omega)\), such that \(x'_i(t_i) = 0\). Thus, for any \(t \in [0, \omega]\), we have

\[ |y_i(t)| = |\phi_p(x'_i(t))| = |\phi_p(x'_i(t)) - \phi_p(x'_i(t_i))| = \left| \int_{t_i}^t (\phi_p(x'_i(s)))' \, ds \right| \leq R_7. \]

Therefore \(\|y\|_\infty \leq R_7\).

Choose a number \(R_8 > \max(R_4, R_7)\), let \(\Omega = \{z \in X: \|z\| < R_8\}\), then \(Lz \neq \tilde{N}(z, \lambda)\) for any \(z \in \text{Dom} \, L \cap \partial \Omega\), \(\lambda \in [0, 1]\). It is easy to see \(\tilde{N}\) is \(L\)-compact on \(\hat{\Omega} \times [0, 1]\), \(Lz = \tilde{N}(z, 1)\) is (2.1) and \(\tilde{N}(-z, 0) = -\tilde{N}(z, 0)\). From Lemma 2.1, (2.1) has at least one \(\omega\)-periodic solution \(\tilde{z}(t) = (\tilde{x}(t), \tilde{y}(t))\), \(\tilde{x}(t)\) is an \(\omega\)-periodic solution of (1.1). \(\square\)

**Remark 3.1.** We see that the conditions in Theorem A are also valid to (1.1) for \(1 < p < 2\). When \(p = 2\), the conditions in Theorem 3.1 are weaker than those in Theorem A.
Theorem 3.2. Suppose that there exist constants \( b \geq 0, \ c \geq 0 \) and \( d > 0 \), such that

(I) \( \langle x, \text{grad} \ G(x) \rangle \leq b|x|^p + c, \ \forall x \in \mathbb{R}^n \),

(II) \( \forall i \in \{1, \ldots, n\} \), either \( x_i[\frac{\partial G(x)}{\partial x_i} - \bar{e}_i] > 0 \) or \( x_i[\frac{\partial G(x)}{\partial x_i} - \bar{e}_i] < 0 \) for \( |x_i| > d \), where \( \bar{e}_i = \frac{1}{\omega} \int_0^\omega e_i(t) \, dt \).

Then (1.1) has at least one \( \omega \)-periodic solution for \( b < \left( \frac{\pi_p}{\omega} \right)^p \).

Proof. We also consider Eqs. (3.1). Let \( z(t) = \left( \begin{array}{c} x(t) \\ y(t) \end{array} \right) \) be a possible \( \omega \)-periodic solution of (3.1), from (I) and (3.2), we have

\[
-\|x'|^p + b\|x|^p + c\omega \geq \lambda \int_0^\omega \langle x(t), e(t) \rangle \, dt \geq -\|e\|_q \|x\|_p,
\]

i.e.,

\[
\|x'|^p \leq b\|x|^p + \|e\|_q \|x\|_p + c\omega. \tag{3.10}
\]

Integrating both sides of (3.2) over \([0, \omega]\), we get

\[
\frac{1 + \lambda}{2} \int_0^\omega \left[ \frac{\partial G(x(t))}{\partial x_i} - \bar{e}_i \right] \, dt - \frac{1 - \lambda}{2} \int_0^\omega \left[ \frac{\partial G(-x(t))}{\partial x_i} - \bar{e}_i \right] \, dt = 0, \quad i = 1, \ldots, n.
\]

So there exist \( \tilde{t}_i \in [0, \omega] \) such that

\[
\frac{1 + \lambda}{2} \left[ \frac{\partial G(x(\tilde{t}_i))}{\partial x_i} - \bar{e}_i \right] - \frac{1 - \lambda}{2} \left[ \frac{\partial G(-x(\tilde{t}_i))}{\partial x_i} - \bar{e}_i \right] \, dt = 0, \quad i = 1, \ldots, n.
\]

From (II), one can see \( |x_i(\tilde{t}_i)| \leq d \). Let \( \chi_i(t) = x_i(t + \tilde{t}_i) - x_i(\tilde{t}_i), \chi(t) = (\chi_1(t), \ldots, \chi_n(t))^T \), then \( \chi(0) = \chi(\omega) = 0 \), by Lemma 2.2, one can obtain

\[
\|\chi\|_p \leq \frac{\omega}{\pi_p} \|x'|_p. \tag{3.11}
\]

Noticing the periodicity of \( x(t) \), we have

\[
\|x_i\|^p = \int_0^\omega |x_i(t)|^p \, dt = \int_0^\omega |x_i(t + \tilde{t}_i)|^p \, dt \leq \int_0^\omega (|x_i(t)| + d)^p \, dt \leq (\|x_i\|_p + \omega^{1/p} d)^p.
\]

So from Minkovski’s inequality, we have

\[
\|x\|_p = \left( \sum_{i=1}^n \|x_i\|^p \right)^{1/p} \leq \left( \sum_{i=1}^n (\|x_i\|_p + \omega^{1/p} d)^p \right)^{1/p} \\
\leq \|\chi\|_p + (n\omega)^{1/p} d \leq \frac{\omega}{\pi_p} \|x'|_p + (n\omega)^{1/p} d = \frac{\omega}{\pi_p} \|x'|_p + (n\omega)^{1/p} d. \tag{3.12}
\]

In view of (3.10), we get

\[
\|x'|_p \leq b\left( \frac{\omega}{\pi_p} \|x'|_p + (n\omega)^{1/p} d \right)^p + \|e\|_q \left( \frac{\omega}{\pi_p} \|x'|_p + (n\omega)^{1/p} d \right) + c\omega. \tag{3.13}
\]

Since \( b\left( \frac{\omega}{\pi_p} \right)^p < 1 \), from (3.13), there exists a constant \( R_9 > 0 \), such that

\[
\|x'|_p \leq R_9. \tag{3.14}
\]
Therefore,
\[ \|x\|_p \leq \frac{\omega}{\pi p} R_9 + (n\omega)^{1/p} d := R_{10}. \] (3.15)

From (3.14) and (3.15), we know that the rest of the proof of the theorem is similar to that of Theorem 3.1. □

**Theorem 3.3.** Suppose that there exist constants \( b > 0, c \geq 0 \) and \( \alpha > 1 \), such that
\[ \langle x, \text{grad } G(x) \rangle \leq -b|x|_\alpha^\alpha + c, \quad \forall x \in \mathbb{R}^n. \]
Then (1.1) has at least one \( \omega \)-periodic solution.

**Proof.** Consider the parameter equation (3.1), suppose that \( z(t) = (x(t), y(t)) \) is a possible \( \omega \)-periodic solution of (3.1). From the condition of this theorem and (3.2), we have
\[ -\|x'\|_p^p - b\|x\|_\alpha^\alpha + c\omega \geq \lambda \int_0^\omega \langle x(t), e(t) \rangle \, dt \geq -\|e\|_\beta \|x\|_\alpha, \]
i.e.,
\[ 0 \leq \|x'\|_p^p \leq -b\|x\|_\alpha^\alpha + \|e\|_\beta \|x\|_\alpha + c\omega, \] (3.16)
where \( \frac{1}{\alpha} + \frac{1}{\beta} = 1. \)
It follows that there exist two constants \( R_{11} > 0 \) and \( R_{12} > 0 \) such that
\[ \|x\|_\alpha \leq R_{11}, \quad \|x'\|_p \leq R_{12}. \] (3.17)
From the proof of Theorem 3.1, we know that (1.1) has at least one \( \omega \)-periodic solution. □

As applications, we list the following examples.

**Example 3.1.** Consider the following system:
\[ \left( \phi_p (u'(t)) \right)' + \frac{d}{dt} \text{grad } F(u(t)) + \text{grad } G(u(t)) = \left( \frac{1 + \sin(2t)}{2 - \cos(2t)} \right), \] (3.18)
where \( F \in C^2(\mathbb{R}^2, \mathbb{R}), G \in C^1(\mathbb{R}^2, \mathbb{R}). \)
Let
\[ x = (x_1, x_2)^T, \]
\[ F(x_1, x_2) = x_1^2 + x_2^2 - \frac{x_1 x_2}{2} - \sqrt{1 + x_1^2}, \]
\[ G(x_1, x_2) = x_1^4 + x_2^3 - \frac{1}{4} x_1^2 x_2^2 + x_2^4, \]
then
\[ y^T \frac{\partial^2 F(x)}{\partial x^2} y \geq \frac{3 - \sqrt{2}}{2} |y|^2, \quad y = (y_1, y_2)^T, \quad \langle x, \text{grad } G(x) \rangle \geq \frac{1}{2} |x|^4 - 3, \quad x \in \mathbb{R}^2. \]
By Theorem 3.1, (3.18) has at least one \( \pi \)-periodic solution when \( 1 < p \leq 2. \)
Example 3.2. We also consider (3.18). Let $p = 4$, $\omega = \pi$, $G(x_1, x_2) = \frac{1}{8} x_1^4 - \frac{1}{2} x_2^2$, then
\[
\langle x, \text{grad } G(x) \rangle \leq \frac{1}{2} |x|^4 + \frac{1}{2}, \quad \left( \frac{\pi_4}{\omega} \right)^4 = \frac{3}{4},
\]
\[
\lim_{|x_1| \to \infty} x_1 \left( \frac{\partial G(x)}{\partial x_1} - \bar{e}_1 \right) = \lim_{|x_1| \to \infty} x_1 \left( \frac{1}{2} x_1^3 - 1 \right) = +\infty,
\]
\[
\lim_{|x_2| \to \infty} x_2 \left( \frac{\partial G(x)}{\partial x_2} - \bar{e}_2 \right) = \lim_{|x_2| \to \infty} (-x_2^2 - 2x_2) = -\infty,
\]
so there exists $d > 0$, such that $x_1 \left( \frac{\partial G(x)}{\partial x_1} - \bar{e}_1 \right) > 0$ for $|x_1| > d$, $x_2 \left( \frac{\partial G(x)}{\partial x_2} - \bar{e}_2 \right) < 0$ for $|x_2| > d$.

By Theorem 3.2, (3.18) has at least one $\pi$-periodic solution for any $F \in C^2(\mathbb{R}^2, \mathbb{R})$.

If we set $G(x_1, x_2) = -x_1^4 - x_3^3 + \frac{1}{4} x_1^2 x_2^2 - x_2^4$, then $\langle x, \text{grad } G(x) \rangle \leq -\frac{1}{2} |x|^4 + 3$, from Theorem 3.3, one can see that (3.18) has at least one $\pi$-periodic solution for any $F \in C^2(\mathbb{R}^2, \mathbb{R})$ and $p > 1$.

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