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Berezin–Toeplitz quantization on Lie groups

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This article is dedicated to Professor Paul Malliavin, for showing the way

Abstract

Let K be a connected compact semisimple Lie group and $K_{\mathbb{C}}$ its complexification. The generalized Segal–Bargmann space for $K_{\mathbb{C}}$ is a space of square-integrable holomorphic functions on $K_{\mathbb{C}}$, with respect to a K-invariant heat kernel measure. This space is connected to the "Schrödinger" Hilbert space $L^2(K)$ by a unitary map, the generalized Segal–Bargmann transform. This paper considers certain natural operators on $L^2(K)$, namely multiplication operators and differential operators, conjugated by the generalized Segal–Bargmann transform. The main results show that the resulting operators on the generalized Segal–Bargmann space can be represented as Toeplitz operators. The symbols of these Toeplitz operators are expressed in terms of a certain subelliptic heat kernel on $K_{\mathbb{C}}$. I also examine some of the results from an infinite-dimensional point of view based on the work of L. Gross and P. Malliavin.

Keywords: Berezin–Toeplitz quantization; Segal–Bargmann transform; Heat kernel

1. Introduction

The Berezin–Toeplitz quantization is a standard method of quantizing a symplectic manifold $\mathcal{M}$ that admits a Kähler structure. In such cases, the quantum Hilbert space is a space of square-integrable holomorphic sections of an appropriate complex line bundle. Let P denote the orthogonal projection operator from the space of all square-integrable sections to the holomorphic subspace. Then for any bounded measurable function ϕ, we can construct the

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Toeplitz operator with symbol ϕ, acting on the space of holomorphic sections, as $T_\phi s = P(\phi s)$. That is, T_ϕ consists of multiplication by ϕ followed by projection back into the holomorphic subspace.

The map sending ϕ to T_ϕ is called the *Berezin–Toeplitz quantization*, and it may be thought of as a generalization of anti-Wick-ordered quantization. (See [19] for discussion.) There is a large literature devoted to this quantization scheme, including early works such as [2,31,32], continuing with specific examples in [4,6,29], and then developing into a general theory in [3,5], to mention just a few examples.

In this paper, we will examine the case in which $\mathcal{M}$ is the cotangent bundle $T^*(K)$ of a connected compact Lie group K, which we assume for simplicity to be semisimple. (In the torus case, the results are essentially the same as in the $\mathbb{R}^n$ case, but the semisimple case displays some interesting new phenomena.) There is a natural way to identify $T^*(K)$ with the *complexification* $K_{\mathbb{C}}$ of K and in this case, the relevant line bundle is holomorphically trivial. Thus, in this case, the quantum Hilbert space is identified with $\mathcal{H}L^2(K_{\mathbb{C}}, \nu_t)$, the space of square-integrable holomorphic functions on $K_{\mathbb{C}}$ with respect to a certain measure $\nu_t(g)\,dg$, where dg is a Haar measure on $K_{\mathbb{C}}$ and ν_t is a K-invariant heat kernel. Here t is a positive parameter that plays the role of Planck's constant.

Of course, since $T^*(K)$ is a cotangent bundle, there is another commonly used method of quantizing it, namely, Schrödinger-style quantization in which the Hilbert space is $L^2(K)$ with respect to a Haar measure. The goal of the present paper is to compare the two approaches to quantization. There is a natural unitary map between $L^2(K)$ and $\mathcal{H}L^2(K_{\mathbb{C}}, \nu_t)$, called the (generalized) Segal–Bargmann transform and introduced in [16]. The goal of the present paper is to show that certain natural operators on $L^2(K)$, when conjugated by the Segal–Bargmann transform, become Toeplitz operators on $\mathcal{H}L^2(K_{\mathbb{C}}, \nu_t)$.

In Section 3, we consider M_V, multiplication by V, acting as an operator on $L^2(K)$. The operator M_V should be thought of as the Schrödinger quantization of the function $V \circ \pi$, where π is the projection from $T^*(K)$ to K. That is to say, multiplication operators are the Schrödinger quantization of functions that are constant along the fibers of the cotangent bundle.

If V is very regular, then $C_t M_V C_t^{-1}$ can be expressed as a Toeplitz operator with symbol ϕ_V, which is computed as follows. We first apply the *backward* heat operator $e^{-t\Delta/4}$ to V, obtaining $\tilde{V} := e^{-t\Delta/4}V$. (For this to make sense, V must be very regular.) In the case where K is commutative, ϕ_V is given simply by $\phi_V(xe^{iy}) = \tilde{V}(x)$. This is essentially the same as what one has in the $\mathbb{R}^n$ case. On the other hand, if K is semisimple, then

$$\phi_V(g) = \frac{\int_K \mu_{t/2,t}(gx^{-1})\tilde{V}(x)\,dx}{\nu_t(g)}, \quad g \in K_{\mathbb{C}},$$

where $\mu_{t/2,t}$ is the heat kernel associated to a certain *subelliptic* Laplace-type operator on $K_{\mathbb{C}}$.

Even in the $\mathbb{R}^n$ case, the formula for ϕ_V involves applying the backward heat operator $e^{-t\Delta/4}$ to V, so the assumption that V is very regular (in the domain of the backward heat operator) seems unavoidable. This assumption reflects a sort of smoothing property of the Berezin–Toeplitz quantization, namely that even very rough symbols give rise to nice operators. For example, it is easy to have a highly singular distributional symbol ϕ for which the associated Toeplitz operator T_ϕ is bounded. It follows that the inverse operation, trying to represent an operator on $\mathcal{H}L^2(K_{\mathbb{C}}, \nu_t)$ as a Toeplitz operator, will be quite singular.

In Section 4, we consider differential operators on K, which we do not assume are invariant under either the left- or right-action of K. Differential operators of degree at most N should be thought of as the quantization of functions on $T^*(K)$ that are polynomials of degree at most N on each fiber. (Specifically, one may consider a generalization to K of the Weyl quantization, which indeed maps the space of functions that are polynomials of degree at most N on each fiber into the space of differential operators of degree at most N.) A differential operator on K can be represented as a linear combination of left-invariant differential operators multiplied by functions on K, that is, as linear combinations of left-invariant differential operators composed with multiplication operators. If α is a left-invariant differential operator and V is a sufficiently regular function, then $C_t M_V \alpha C_t^{-1}$ can be represented as a Toeplitz operator with a certain symbol. The results of Section 4 are a substantial generalization of the results of [27], which treats only the invariant case.

Finally, in Section 5, we look at the results of Section 3 from an infinite-dimensional point of view. The work of L. Gross and P. Malliavin [13], as refined in [7], allows the Segal–Bargmann transform for the compact group K to be viewed as a special case of the Segal–Bargmann transform for an infinite-dimensional Euclidean space. In Section 5, I explain how the results of Section 3 can be derived, at least formally, from the infinite-dimensional perspective.

2. Berezin–Toeplitz and Schrödinger quantization for $T^*(K)$

The Hilbert space we will consider will ultimately be identified with the space of square-integrable holomorphic functions on $K_{\mathbb{C}}$ with respect to a K-invariant heat kernel measure $\nu_t(g)\,dg$. This space is denoted $\mathcal{H}L^2(K_{\mathbb{C}}, \nu_t)$. As I will now explain, however, this space can also be obtained by applying the method of geometric quantization with half-forms, thus connecting with much of the literature on Berezin–Toeplitz quantization. (See, for example, the work [31] of J. Rawnsley, who interprets the work of Berezin [2] in terms of geometric quantization.)

Let K be a connected compact Lie group, assumed to be semisimple. We choose once and for all a bi-invariant Riemannian metric on K. This is equivalent to choosing an Ad-invariant inner product on the Lie algebra $\mathfrak{k}$ of K. There is then a natural "adapted complex structure" on $T^*(K)$, described independently and in slightly different language by [14,15] and [30,34]. This complex structure fits together with the canonical symplectic structure on $T^*(K)$ to form a Kähler structure. Furthermore, $T^*(K)$ with its adapted complex structure is biholomorphic in a natural way to the complexification $K_{\mathbb{C}}$ of K [17, Section 3]. Here $K_{\mathbb{C}}$ is the unique connected complex Lie group that has Lie algebra $\mathfrak{k}_{\mathbb{C}} := \mathfrak{k} + i\mathfrak{k}$ and that contains K as a maximal compact subgroup; for example, if $K = SU(n)$ then $K_{\mathbb{C}} = SL(n, \mathbb{C})$.

We now apply the method of geometric quantization (with half-forms) with respect to the adapted complex structure. This amounts to constructing a certain holomorphic line bundle over $T^*(K)$ and giving a certain recipe for computing the norm of such a section. The quantum Hilbert space is then the space of holomorphic sections of finite norm. In the case at hand, this bundle is holomorphically trivial. Upon choosing a natural trivialization, the quantum Hilbert space becomes the space of holomorphic functions that are square integrable with respect to a certain measure. If we identify $T^*(K)$ with $K_{\mathbb{C}}$ then this measure turns out to coincide, up to an irrelevant constant, with the K-invariant heat kernel measure $\nu_t(g)\,dg$ considered in [16]. (Details on the definition of ν_t will be given in Section 3.) Here dg is a Haar measure on $K_{\mathbb{C}}$ and t is a positive parameter that plays the role of Planck's constant. See [22] for the details of this calculation. Our quantum Hilbert space is thus identified with the space of holomorphic functions on $K_{\mathbb{C}}$ that are square integrable with respect to ν_t. We denote this space $\mathcal{H}L^2(K_{\mathbb{C}}, \nu_t)$ and refer to it as the Segal–Bargmann space.

Now that we have the Segal–Bargmann space $\mathcal{H}L^2(K_{\mathbb{C}}, \nu_t)$, we consider Toeplitz operators on it. Let P_t denote the orthogonal projection from $L^2(K_{\mathbb{C}}, \nu_t)$ onto the holomorphic subspace. If ϕ is a bounded measurable function, then we define the Toeplitz operator T_ϕ with symbol ϕ, as an operator from $\mathcal{H}L^2(K_{\mathbb{C}}, \nu_t)$ to itself, by

$$T_\phi(F) = P_t(\phi F).$$

That is, T_ϕ consists of multiplication by ϕ followed by projection back into the holomorphic subspace. If ϕ is an unbounded measurable function, T_ϕ may still be defined by the same formula, but restricted to the domain of those F's for which ϕF is in $L^2(K_{\mathbb{C}}, \nu_t)$. The operator T_ϕ will typically be unbounded. In the cases we will consider in this paper, T_ϕ will always be a densely defined operator on $\mathcal{H}L^2(K_{\mathbb{C}}, \nu_t)$.

As an alternative to the Segal–Bargmann space, one has the Schrödinger-type Hilbert space, $L^2(K)$. (If no other measure is specified, $L^2(K)$ is understood to be with respect to the Riemannian volume measure dx, which is a Haar measure.) Use of $L^2(K)$ as the Hilbert space leads to a natural way of quantizing certain functions. For example, suppose ϕ is a function on $K_{\mathbb{C}} \cong T^*(K)$ that is constant along each cotangent space. Then ϕ is of the form $\phi = V \circ \pi$, where π is the projection from $T^*(K)$ to K. It is natural to quantize such a function as multiplication by V acting on $L^2(K)$. Similarly, functions on $T^*(K)$ that are polynomials of degree at most N on each fiber get quantized as differential operators of degree at most N acting as (unbounded) operators on $L^2(K)$.

In [16], I introduced a generalized Segal–Bargmann transform for K, which is a unitary map C_t of $L^2(K)$ onto $\mathcal{H}L^2(K_{\mathbb{C}}, \nu_t)$. This operator is defined by applying the heat operator $e^{t\Delta/2}$ to a function f in $L^2(K)$ and then analytically continuing the resulting function to $K_{\mathbb{C}}$. In [22], I show that C_t coincides (up to a constant) with the "pairing map" of geometric quantization. See also [8,9].

Since we have a natural unitary map between $L^2(K)$ and $\mathcal{H}L^2(K_{\mathbb{C}}, \nu_t)$, it is natural to compare the quantization procedures associated to these two Hilbert spaces. The main goal of the present paper is to demonstrate how multiplication operators and differential operators on $L^2(K)$, when conjugated by the Segal–Bargmann transform, can be expressed as Toeplitz operators on $\mathcal{H}L^2(K_{\mathbb{C}}, \nu_t)$.

3. Multiplication operators

In this section, we will consider multiplication by a function $V : K \to \mathbb{C}$ as an operator on $L^2(K)$ and then conjugate this operator by the Segal–Bargmann transform. If V is sufficiently regular, we will show that $C_t M_V C_t^{-1}$ can be expressed as a Toeplitz operator with symbol ϕ_V and give a formula for ϕ_V in terms of V. This formula involves a certain subelliptic heat kernel on $K_{\mathbb{C}}$.

It is helpful to consider first the $\mathbb{R}^n$ case, with a Segal–Bargmann transform C_t mapping from $L^2(\mathbb{R}^n)$ onto $\mathcal{H}L^2(\mathbb{C}^n, \nu_t)$, where

$$\nu_t(x + iy) = (\pi t)^{-d/2} e^{-|y|^2/t}.$$

Here $C_t f$ is the analytic continuation (from $\mathbb{R}^n$ to $\mathbb{C}^n$) of $e^{t\Delta/2} f$. See [19] or [18, Section 3] for a comparison of the normalization conventions I am using here to those used in [1,33].

It is well known (e.g., [10, Propositions 2.96, 2.97]) that for reasonable symbols ϕ, the operator $C_t^{-1} T_\phi C_t$ coincides with the Weyl quantization of the function $\hat{\phi}$, where $\hat{\phi} = e^{t\Delta/4}\phi$. If $\phi(x+iy)$ depends only on x, then the same is true of $\hat{\phi}(x+iy)$. In that case, the Weyl quantization of $\hat{\phi}$ is simply the operation of multiplication by $\hat{\phi}(x)$, acting on $L^2(\mathbb{R}^n)$. Thus, given a multiplication operator M_V on $L^2(\mathbb{R}^n)$, if there exists a function $\tilde{V}$ such that $e^{t\Delta/4}\tilde{V} = V$, then we have

$$C_t M_V C_t^{-1} = T_{\tilde{V}}. \tag{1}$$

On the right-hand side of (1), we abuse notation slightly and allow $\tilde{V}$ to stand for the function on $\mathbb{C}^n$ given by $x + iy \mapsto \tilde{V}(x)$. Of course, in order for such a $\tilde{V}$ to exist, V itself must be extremely regular. (See, for example, [28] or [24] for some discussion of how regular V must be.)

We now proceed to the case of a connected compact Lie group K. In the interests of notational simplicity, we assume K is semisimple. The results in the torus case are essentially the same as in the $\mathbb{R}^n$ case, whereas the semisimple case involves a subelliptic heat kernel that does not show up in the $\mathbb{R}^n$ case.

We fix an Ad-invariant inner product on $\mathfrak{k}$, which determines a bi-invariant Riemannian metric on K. We let dx denote the Riemannian volume measure on K, which is a Haar measure. We let Δ_K denote the Laplacian with respect to this metric, take to be a *negative* operator. This operator can be computed as $\Delta_K = \sum X_k^2$, where the X_k's form an orthonormal basis for $\mathfrak{k}$ and are viewed as left-invariant differential operators. We let ρ_t be the fundamental solution at the identity of the heat equation $\partial\rho_t/\partial t = \frac{1}{2}\Delta_K\rho_t$. For each fixed $t > 0$, the function ρ_t admits an analytic continuation (also denoted ρ_t) to $K_{\mathbb{C}}$. Let $\mathcal{H}(K_{\mathbb{C}})$ denote the space of holomorphic functions on $K_{\mathbb{C}}$ and let C_t be the map from $L^2(K)$ (with respect dx) into $\mathcal{H}(K_{\mathbb{C}})$ given by

$$C_t f(g) = \int_K \rho_t\bigl(gx^{-1}\bigr) f(x)\, dx, \quad g \in K_{\mathbb{C}}. \tag{2}$$

If $\{X_k\}_{k=1}^{\dim K}$ is an orthonormal basis for $\mathfrak{k}$, then $\{X_k, JX_k\}_{k=1}^{\dim K}$ forms a basis of $\mathfrak{k}_{\mathbb{C}}$. We now regard each X_k and each JX_k as a left-invariant differential operator on $K_{\mathbb{C}}$. The function ν_t is the solution (in $L^2(K_{\mathbb{C}}, dg)$) to the heat equation

$$\frac{\partial \nu}{\partial t} = \frac{1}{4}\sum_{k=1}^{\dim K}\bigl(X_k^2 + (JX_k)^2\bigr)\nu_t$$

subject to the initial condition

$$\lim_{t\to 0^+}\int_{K_{\mathbb{C}}} f(g)\nu_t(g)\, dg = \int_K f(x)\, dx.$$

Here dg is a fixed Haar measure on $K_{\mathbb{C}}$.

The function ν_t is the heat kernel at the identity coset for the symmetric space $K_{\mathbb{C}}/K$, viewed as a right-K-invariant function on $K_{\mathbb{C}}$. We let $L^2(K_{\mathbb{C}}, \nu_t)$ denote the L^2 space with respect to the measure $\nu_t(g)\, dg$, and we let $\mathcal{H}L^2(K_{\mathbb{C}}, \nu_t)$ denote the holomorphic subspace thereof. According to Theorem 2 of [16], C_t is a unitary map of $L^2(K)$ onto $\mathcal{H}L^2(K_{\mathbb{C}}, \nu_t)$. We let P_t

denote the orthogonal projection of $L^2(K_{\mathbb{C}}, \nu_t)$ onto the holomorphic subspace. For any bounded measurable function ϕ on $K_{\mathbb{C}}$, we let T_ϕ denote the Toeplitz operator on $\mathcal{H}L^2(K_{\mathbb{C}}, \nu_t)$ given by $T_\phi(F) = P_t(\phi F)$. If F_1 and F_2 belong to $\mathcal{H}L^2(K_{\mathbb{C}}, \nu_t)$ then

$$\langle F_1, T_\phi F_2\rangle_{\mathcal{H}L^2(K_{\mathbb{C}},\nu_t)} = \int_{K_{\mathbb{C}}} \overline{F_1(g)}\phi(g)F_2(g)\nu_t(g)\,dg \tag{3}$$

because P_t is self-adjoint and $P_t F_1 = F_1$.

In [7] (see also [18]), B. Driver and I consider a family $A_{s,t}$ of operators on $K_{\mathbb{C}}$ parameterized by two positive numbers s and t with $s \geqslant t/2$,

$$A_{s,t} = \left(s - \frac{t}{2}\right)\sum_{k=1}^{\dim K} X_k^2 + \frac{t}{2}\sum_{k=1}^{\dim K}(JX_k)^2,$$

where J is the "multiplication by i" map on $\mathfrak{k}_{\mathbb{C}}$. If $s > t/2$, then $A_{s,t}$ is an elliptic operator. We now consider the heat equation $\partial u/\partial r = \frac{1}{2}A_{s,t}u$ on $K_{\mathbb{C}}$ and we let $\mu_{s,t}$ denote the fundamental solution of this equation at the identity, evaluated at $r = 1$. This is equivalent to saying that

$$\mu_{s,t} = e^{A_{s,t}/2}(\delta_e),$$

where δ_e is a Dirac delta-function at the identity.

Since we are assuming that K is semisimple, the "borderline" operator $A_{t/2,t}$ satisfies Hörmander's condition and is therefore hypoelliptic. In the semisimple case, the heat kernel $\mu_{s,t}$ is still a well-defined, smooth, strictly positive function on $K_{\mathbb{C}}$ when $s = t/2$. The subelliptic heat kernel

$$\mu_{t/2,t} = \exp\left\{\frac{t}{4}\sum_{k=1}^{\dim K}(JX_k)^2\right\}(\delta_e) \tag{4}$$

plays an essential role in all the main results of this paper.

Meanwhile, the Casimir operator $\sum((JX_k)^2 - X_k^2)$ commutes with each X_j and each JX_j. It follows that $\sum X_k^2$ commutes with $\sum(JX_k)^2$. We then have (at least formally)

$$e^{A_{s,t}/2} = e^{A_{s-r,t}/2}e^{r\Delta_K/2}.$$

It is then not hard to show (cf. [16, Section 8]) that for $s - r \geqslant t/2$ we have

$$\mu_{s,t}(g) = \int_K \mu_{s-r,t}\big(gx^{-1}\big)\rho_r(x)\,dx, \quad g \in K_{\mathbb{C}}. \tag{5}$$

For any $s \geqslant t/2$ we have

$$\nu_t(g) = \int_K \mu_{s,t}\big(gx^{-1}\big)\,dx. \tag{6}$$

Furthermore,

$$\lim_{s\to\infty}\mu_{s,t}(g)=\mathrm{Vol}(K)\nu_t(g), \tag{7}$$

where $\mathrm{Vol}(K)$ is the Riemannian volume of K.

We are now ready to state the main result of this section.

Theorem 1. *Suppose that V is a function on K and that there exists a bounded measurable function $\tilde{V}$ on K such that $V=e^{t\Delta/4}\tilde{V}$. Let ϕ_V be the function on $K_{\mathbb{C}}$ defined by*

$$\phi_V(g)=\frac{\int_K \mu_{t/2,t}(gx^{-1})\tilde{V}(x)\,dx}{\nu_t(g)}.$$

Then ϕ_V is a bounded function and

$$C_t M_V C_t^{-1}=T_{\phi_V}.$$

In the commutative case, the formula would be $\phi_V(xe^{iY})=\tilde{V}(x)$, essentially the same as what we have in the $\mathbb{R}^n$ case.

Proof. Applying (6) with $s=t/2$, we see that

$$\left|\phi_V(g)\right|\leqslant\sup|\tilde{V}|,$$

establishing the boundedness of ϕ_V.

Since C_t (as defined in (2)) is an integral operator, its adjoint is easily computed as

$$\left(C_t^*\Phi\right)(x)=\lim_{n\to\infty}\int\limits_{E_n}\overline{\rho_t\left(gx^{-1}\right)}\Phi(g)\,dg, \tag{8}$$

for any $\Phi\in L^2(K_{\mathbb{C}},\nu_t)$. Here E_n is any increasing sequence of compact, K-invariant subsets of $K_{\mathbb{C}}$ whose union is $K_{\mathbb{C}}$ and the limit is in the norm topology of $L^2(K)$. (See [16, Section 8].)

For any fixed s, we consider the heat kernel measure $\rho_s(x)\,dx$ and the resulting L^2 space, which we denote $L^2(K,\rho_s)$. We assume for the moment that $s>t/2$ (we will eventually let s tend to $t/2$) and we consider the transform denoted $B_{s,t}$ in [7,18]. This is the map from $L^2(K,\rho_s)$ into $\mathcal{H}(K_{\mathbb{C}})$ given by

$$\begin{aligned}(B_{s,t}f)(g)&=\int\limits_K\rho_t\left(gx^{-1}\right)f(x)\,dx\\&=\int\limits_K\frac{\rho_t(gx^{-1})}{\rho_s(x)}f(x)\,\rho_s(x)\,dx.\end{aligned} \tag{9}$$

Note that the formula for $B_{s,t}$ is the same as the formula for C_t and is independent of s; only the inner product on the domain space depends on s.

According to [7, Theorem 5.3] or [18, Theorem 1.2], $B_{s,t}$ is an isometric map of $L^2(K,\rho_s)$ into $L^2(K_{\mathbb{C}})$ with respect to the measure $\mu_{s,t}(g)\,dg$, whose image is precisely the holomorphic subspace $\mathcal{H}L^2(K_{\mathbb{C}},\mu_{s,t})$. Since $B_{s,t}$ is isometric, its adjoint is a one-sided inverse, where the adjoint is readily computed from (9). Given $f\in L^2(K,\rho_s)$, if we let $F=B_{s,t}f$ then we have an inversion formula given by

$$f(x)=\lim_{n\to\infty}\int_{E_n}F(g)\frac{\overline{\rho_t(gx^{-1})}}{\rho_s(x)}\mu_{s/2,t}(g)\,dg. \tag{10}$$

Since $\rho_s(x)$ is independent of g, we may pull this factor outside the integral in (10) and then multiply both sides by ρ_s to obtain

$$\begin{aligned}\rho_s(x)f(x)&=\lim_{n\to\infty}\int_{E_n}F(g)\overline{\rho_t\bigl(gx^{-1}\bigr)}\mu_{s,t}(g)\,dg\\&=\lim_{n\to\infty}\int_{E_n}\frac{\mu_{s,t}(g)}{\nu_t(g)}F(g)\overline{\rho_t\bigl(gx^{-1}\bigr)}\nu_t(g)\,dg.\end{aligned}\tag{11}$$

Now, the proof of the "averaging lemma" in [16] applies to $\mu_{s,t}$ for $s>t/2$ and yields that for each fixed s,t with $s>t/2$ we have positive constants c_1 and c_2 such that

$$c_1\nu_t(g)\leqslant\mu_{s,t}(g)\leqslant c_2\nu_t(g)$$

for all $g\in K_{\mathbb{C}}$. (This result follows fairly easily from (5) and (6).) It follows that the function Φ on $K_{\mathbb{C}}$ given by

$$\Phi(g)=\frac{\mu_{s,t}(g)}{\nu_t(g)}F(g)$$

belongs to $L^2(K_{\mathbb{C}},\nu_t)$. Comparing (11) to (8) we obtain

$$\rho_s f=C_t^*\Phi.$$

Now, since C_t is isometric and its image is precisely the holomorphic subspace $\mathcal{H}L^2(K_{\mathbb{C}},\nu_t)$, the adjoint map may be computed as

$$C_t^*=C_t^{-1}P_t,$$

where P_t is the orthogonal projection of $L^2(K_{\mathbb{C}},\nu_t)$ onto $\mathcal{H}L^2(K_{\mathbb{C}},\nu_t)$. Here C_t^{-1} denotes the inverse of C_t as a map of $L^2(K)$ onto $\mathcal{H}L^2(K_{\mathbb{C}},\nu_t)$. We conclude, then, that

$$\rho_s f=C_t^{-1}P_t\left(\frac{\mu_{s,t}}{\nu_t}F\right). \tag{12}$$

Now, the density ρ_s is bounded and bounded away from zero, so if f belongs to $L^2(K,\rho_s)$ then it also belongs to $L^2(K)$. Furthermore, the formula for $B_{s,t}f$ is the same as for C_tf. Thus,

F, which was defined to be $B_{s,t}f$, coincides with C_tf and (12) becomes $\rho_s f = C_t^{-1}T_\phi C_t f$, where $\phi = \mu_{s,t}/\nu_t$. Thus,

$$C_t M_{\rho_s} C_t^{-1} = T_\phi, \quad \phi = \frac{\mu_{s,t}}{\nu_t}. \tag{13}$$

Since the transform C_t commutes with left- and right-translations by elements of K (since Δ_K is bi-invariant), it is easy to see that

$$C_t M_{R_x\rho_s} C_t^{-1} = T_{\phi_x}, \quad \phi_x = \frac{R_x\mu_{s,t}}{\nu_t} \tag{14}$$

where for any function f on K or $K_{\mathbb{C}}$, we set $(R_y f)(g) = f(gy^{-1})$. (Recall that ν_t is invariant under the right action of K.)

Recall now that V is a function on K of the form $V = e^{t\Delta/4}\tilde{V}$, where $\tilde{V}$ is a bounded measurable function. For $s > t/2$, let us integrate (14) against $\tilde{V}$. (On the right-hand side, write things in terms of the matrix entries as in (3) and use Fubini.) Note that $e^{s\Delta/2}\tilde{V}$ may be computed as $\int_K (R_x\rho_s)f(x)\,dx$. We obtain, then,

$$C_t M_{e^{s\Delta_K/2}\tilde{V}} C_t^{-1} = T_{\phi_{s,V}} \tag{15}$$

where

$$\phi_{s,V}(g) = \frac{\int_K \mu_{s,t}(gx^{-1})\tilde{V}(x)\,dx}{\nu_t(g)}. \tag{16}$$

Now, as s decreases to $t/2$, $e^{s\Delta_K/2}\tilde{V}$ converges uniformly to $e^{t\Delta_K/4}\tilde{V} = V$. Meanwhile, from (5) with $s = t/2$ we see that $\mu_{s,t}$ tends pointwise to $\mu_{t/2,t}$ as s decreases to $t/2$. Furthermore, (6) tells us that $|\phi_{s,V}(g)|$ is bounded by $\sup|\tilde{V}|$, independently of s. It then follows from Dominated Convergence (and (3)) that $T_{\phi_{s,V}}$ tends weakly to T_{ϕ_V} as $s \to t/2$. Thus, letting s decrease to $t/2$ in (15) gives the desired result. □

4. Differential operators

In this section, we will consider differential operators acting as unbounded operators on $L^2(K)$. We *do* not assume that the operators are right- or left-invariant, but we do assume that the coefficients (say, when expanded in terms of left-invariant vector fields) are very regular. By differentiating the results of Section 3 and then integrating by parts, we will see that a differential operator, when conjugated by the Segal–Bargmann transform, can be expressed as a Toeplitz operator.

Now, in the previous section, under sufficiently stringent assumptions on V, we were able to express $C_t M_V C_t^{-1}$ as a Toeplitz operator with a bounded symbol. Differential operators, however, are unbounded and the corresponding Toeplitz symbols are necessarily unbounded as well. In general, we will *not* obtain equality of domains between a differential operator α (conjugated by C_t) and the associated Toeplitz operator. (Nevertheless, see [27, Remark 19] for examples where equality of domains does occur.) Rather, we will content ourselves by establishing equality of the operators on the space of finite linear combinations of matrix entries.

By the results of Section 3, we have

$$Vf = C_t^{-1} P_t M_{\phi_V} C_t f$$

for sufficiently nice V and $f \in L^2(K)$. Letting $F = C_t f$ and recalling that $C_t^* = C_t^{-1} P_t$, we have

$$V(x)f(x) = \lim_{n\to\infty} \int_{E_n} F(g)\overline{\rho_t\left(gx^{-1}\right)} \int_K \mu_{t/2,t}\left(gy^{-1}\right)\tilde{V}(y)\,dy\,dg. \tag{17}$$

(Compare (8).)

Now let A be a left-invariant differential operator on K, a linear combination of products of left-invariant vector fields. Those left-invariant vector fields can be extended to left-invariant vector fields on $K_{\mathbb{C}}$, and so A may be regarded as a left-invariant operator on $K_{\mathbb{C}}$. Since C_t commutes with left- and right-translations by elements of K, $C_t Af = AC_t f$. We may therefore replace f by Af and F by AF in (17), assuming f is in the domain of A.

For a left-invariant vector field X, let $X_{\mathbb{C}}$ be the holomorphic vector field given by

$$X_{\mathbb{C}} = \frac{1}{2}(X - iJX).$$

Then $X_{\mathbb{C}}F = XF$ if F is holomorphic and $X_{\mathbb{C}}F = 0$ if F is antiholomorphic. If A is written as a sum of products of left-invariant vector fields, we may produce a holomorphic differential operator $A_{\mathbb{C}}$ by replacing each vector field X by $X_{\mathbb{C}}$. Then $A_{\mathbb{C}}F = F$ for all holomorphic functions F.

Let us now replace f by Af and F by AF in (17), and then change AF to $A_{\mathbb{C}}F$. We obtain, then,

$$V(x)Af(x) = \lim_{n\to\infty} \int_{E_n} \left(A_{\mathbb{C}}F(g)\right)\overline{\rho_t\left(gx^{-1}\right)} \int_K \mu_{t/2,t}\left(gy^{-1}\right)\tilde{V}(y)\,dy\,dg. \tag{18}$$

We now want to integrate by parts on the right-hand side of (18). Let $B \mapsto B^{\mathrm{tr}}$ be the linear (*not* conjugate-linear) map on left-invariant operators satisfying

$$(X_1 X_2 \cdots X_N)^{\mathrm{tr}} = (-1)^N X_N \cdots X_2 X_1.$$

If F is nice enough, the limit in (18) will become simply integration over $K_{\mathbb{C}}$. If we then integrate by parts repeatedly, assuming boundary terms may be neglected, we get

$$\begin{aligned} V(x)Af(x) &= \int_{K_{\mathbb{C}}} F(g)\overline{\rho_t\left(gx^{-1}\right)}\left[A_{\mathbb{C}}^{\mathrm{tr}} \int_K \mu_{t/2,t}\left(gy^{-1}\right)\tilde{V}(y)\,dy\right]dg \\ &= \int_{K_{\mathbb{C}}} F(g)\overline{\rho_t\left(gx^{-1}\right)}\phi_{V,A}(g)\nu_t(g)\,dg, \end{aligned} \tag{19}$$

where

$$\phi_{V,A} = \frac{A_{\mathbb{C}}^{\mathrm{tr}} \int_K \mu_{t/2,t}(gx^{-1})\tilde{V}(x)\,dx}{\nu_t(g)}. \tag{20}$$

Note that since $\overline{\rho_t(gx^{-1})}$ is antiholomorphic as a function of g, the holomorphic operator $A_{\mathbb{C}}^{\mathrm{tr}}$ does not "see" this factor.

Assume that the boundary terms in the integration by parts may indeed be neglected. Assume also that $F\phi_{V,A}$ is in $L^2(K_{\mathbb{C}}, \nu_t)$. Then (19) tells us that

$$\begin{aligned} M_V A f &= C_t^* M_{\phi_{V,A}} C_t f \\ &= C_t^{-1} T_{\phi_{V,A}} C_t f. \end{aligned}$$

Thus, on some as yet undetermined domain in $\mathcal{H}L^2(K_{\mathbb{C}}, \nu_t)$, we have

$$C_t M_V A C_t^{-1} = T_{\phi_{V,A}}.$$

If $A = I$, then this simply reproduces the results of Section 3. On the other hand, if V is the constant function $\mathbf{1}$, then by (6)

$$\phi_{\mathbf{1},A} = \frac{A_{\mathbb{C}}^{\mathrm{tr}} \nu_t}{\nu_t},$$

reproducing a result of [27]. Because there is a fairly simple explicit expression for ν_t, it is possible to compute $\phi_{\mathbf{1},A}$ quite explicitly in some cases. For example, it easy to see that if A is a power of the Laplacian, then $\phi_{\mathbf{1},A}(xe^{iY})$ is a polynomial in $|Y|^2$. See [27, Section 3] for more information.

Theorem 2. *Suppose f is a finite linear combination of matrix entries and let $F = C_t f$. Suppose V is of the form $V = e^{t\Delta_K/4}\tilde{V}$ for some bounded measurable function $\tilde{V}$ on K. Then $\phi_{V,A}F$ belongs to $L^2(K_{\mathbb{C}}, \nu_t)$ and*

$$C_t^{-1} P_t(\phi_{V,A} F) = M_V A f,$$

where $\phi_{V,A}$ is given in (20).

Proof. The main issue in the proof of Theorem 2 is to obtain reasonable estimates on the subelliptic heat kernel $\mu_{t/2,t}$ and its derivatives. Such estimates can be obtained by using an appropriate parabolic Harnack inequality, such as Theorem V.3.1 in [36]. This inequality bounds derivatives of the heat kernel (or any positive solution of the heat equation) at fixed point and some time t by a constant times the heat kernel itself at the same point and some slightly later time. In the group case, the homogeneity of the problem means that the constant can be taken to be independent of the point. Thus, for any $\tau > t > 0$ and any left-invariant differential operator α, we have a constant c (depending on t, τ, and α but not on g) such that

$$\alpha\mu_{t/2,t}(g) \leqslant c\mu_{\tau/2,\tau}(g)$$

for all $g \in K_{\mathbb{C}}$.

We now make use of the pointwise bounds for $\mu_{\tau/2,\tau}$, which are expressed in terms of a sub-Riemannian distance function on $K_{\mathbb{C}}$. The distance is the infimum of lengths of paths joining two points, where we allow only paths whose tangent vectors at each point lie in the span of the left-invariant vector fields $\{JX_k\}_{k=1}^{\dim K}$. (Any two points can be joined by such a path.) The length of an allowed path is computed by identifying $J\mathfrak{k}$ with $\mathfrak{k}$ and using the fixed Ad-invariant inner product on $\mathfrak{k}$.

Let us think about this distance function in terms of the polar decomposition for $K_{\mathbb{C}}$, which expresses each $g \in K_{\mathbb{C}}$ uniquely as $g = xe^{iY}$, with $x \in K$ and $Y \in \mathfrak{k}$. The length of any allowed path in $K_{\mathbb{C}}$ is the equal to the length of its projection into $K_{\mathbb{C}}/K$, where we use on $K_{\mathbb{C}}/K$ an obvious left-$K_{\mathbb{C}}$-invariant Riemannian metric. What this means is that

$$d\left(e, xe^{iY}\right) \geqslant d\left(eK, xe^{iY}K\right) = |Y|.$$

Meanwhile, the distance from e to $x \in K$ is bounded, from which it follows that $d(e, xe^{iY}) \leqslant |Y| + C$.

Meanwhile, according to [36, Theorem VIII.4.3], for all $\varepsilon > 0$ there exists C_ε

$$\mu_{t/2,t}(g) \leqslant C_\varepsilon t^{-n/2} e^{-d(e,g)^2/(t+\varepsilon)},$$

where n is the "local dimension." (The absence of a 4 in the denominator in the exponent is due to our normalization of the subelliptic Laplacian; see (4).) On the other hand, there is an explicit formula for $\nu_t(xe^{iY})$ (due to R. Gangolli [11]) and it is a Gaussian in Y, multiplied by an exponentially decaying factor. So $1/\nu_t$ is bounded by $e^{|Y|^2/t}$ times a factor that grows no more than exponentially.

So, $\mu_{t/2,t}(xe^{iY})$ is bounded by a constant times a Gaussian in Y, where the constant in the exponent of the Gaussian is arbitrarily close to what one has in the Euclidean case. By the parabolic Harnack inequality, the same is true of any left-invariant derivatives of $\mu_{t/2,t}$. From this and the bounds on ν_t, we see that $\phi_{V,A}(xe^{iY})$ is bounded by a constant times a Gaussian in $|Y|$, where the constant in the exponent in the Gaussian can be made as close to zero as we like.

Now, $\rho_t(xe^{iY})$ can be bounded using the calculations in [17] (compare also [35]), by $e^{|Y|^2/2t}$ times an exponentially decaying factor. All of these estimates together show that the integral in (19) is absolutely convergent. (The function F has at most exponential growth, ρ_t grows like $e^{|Y|^2/2t}$, $\phi_{V,A}$ grows at most like $e^{\varepsilon|Y|^2}$ and ν_t decays like $e^{-|Y|^2/t}$.) Similar remarks apply to the integral on the right-hand side of (18), which means that the limit can be replaced by an integral over all of $K_{\mathbb{C}}$. Using the arguments in Section 4 of [27], there is no problem in justifying the integration by parts to establish the correctness of (19). Since also $\phi_{V,A}F \in L^2(K_{\mathbb{C}}, \nu_t)$, (19) amounts to saying that $C_t^{-1}P_t(\phi_{V,A}F) = VAf$. □

5. The infinite-dimensional perspective

In this section, we will look at the results of Section 3 from an infinite-dimensional point of view. (Presumably the point of view could be extended to the results of Section 4, but I will not consider that problem here.) We will derive (nonrigorously) a formula for the Toeplitz symbol ϕ_V in terms of Gaussian measures and the Itô map, and then verify (rigorously) that this infinite-dimensional prediction indeed reproduces the results of Theorem 1. See (26) and (31). In particular, the measure $\mu_{t/2,t}(g)\,dg$ can be seen to arise from a certain application of the Itô map.

5.1. The work of Gross and Malliavin

The motivation for the introduction (in [16]) of the generalized Segal–Bargmann transform for compact Lie groups was the "J-perp" theorem of Leonard Gross [12]. That theorem of Gross is an analog for a compact Lie group K of the Fock space (symmetric tensor) decomposition on Euclidean space. Gross obtained this theorem by looking at the pathgroup $W(K)$ along with a Wiener measure ρ on $W(K)$. He then considered the space of functions in $L^2(W(K), \rho)$ that are invariant under the left action of the finite-energy loop group—what we will call loop-invariant functions.

As expected, the space of loop-invariant functions turns out to consist entirely of functions of the endpoint, but proving this is no small task. Gross first linearized the problem by mapping paths in the group K to paths in the Lie algebra $\mathfrak{k}$ by means of the Itô map. The loop invariant functions in $W(K)$ correspond to functions on $W(\mathfrak{k})$ that are invariant under a certain action of the loop group, which we will also call loop-invariant functions. Gross then expands a loop-invariant function on $W(\mathfrak{k})$ in a "chaos expansion," the infinite-dimensional linear version of the Fock space decomposition. The Fock space decomposition for the infinite-dimensional linear space $W(\mathfrak{k})$, when restricted to loop-invariant functions gives rise to the J-perp expansion for the compact group K. This analysis also leads to a proof that the only loop-invariant functions are endpoint functions, that is, functions of the endpoint of the Itô map. See [7,26] for further results in this direction, and [23], [25], and [21] for additional exposition.

The existence of an analog of the Fock space decomposition for K led Gross to suggest that I look for an analog for K of the Segal–Bargmann transform. My work on that subject became my PhD thesis and led to the paper [16]. Although the motivation for this work was in stochastic analysis, the paper [16] was purely finite-dimensional. Later on, Gross and Paul Malliavin showed [13] that the Segal–Bargmann transform for K could be understood in much the same way as the J-perp expansion. Roughly speaking, the main result of [13] asserts that the Segal–Bargmann transform for K coincides with the Segal–Bargmann transform for the infinite-dimensional linear space $W(\mathfrak{k})$, when restricted to functions of the endpoint of the Itô map. This result is in the vein of much of the work of Malliavin: using infinite-dimensional analysis to obtain results in finite-dimensional analysis.

5.2. A two-parameter version of the Gross–Malliavin result

In the above discussion of the work of Gross and Malliavin, I have glossed over the distinction between different forms of the Segal–Bargmann transform for K. The paper [13] actually deals with the B_t form of the transform (Theorem 1′ in [16]), which is nothing but the $s = t$ case of the transform $B_{s,t}$ discussed in the proof of Theorem 1. Driver and I generalize the work of Gross and Malliavin to work for the transform $B_{s,t}$ and then obtain the transform C_t as the $s \to \infty$ limit of $B_{s,t}$. This work was motivated in part by the work of Gross and Malliavin and in part by the work of K. Wren [37] on the quantization of $(1 + 1)$-dimensional Yang–Mills theory. See also [20].

We now examine the details of the construction in [7], with some small notational changes. Let $H(\mathfrak{k})$ denote the space of absolutely continuous paths $B : [0, 1] \to \mathfrak{k}$ having one (distributional) derivative in L^2 and satisfying $B_0 = 0$. Let $W(\mathfrak{k})$ denote the space of continuous paths $B : [0, 1] \to \mathfrak{k}$ satisfying $B_0 = 0$, so that $H(\mathfrak{k})$ is a dense subspace of $W(\mathfrak{k})$. Let P_s denote the Wiener measure of variance s on $W(\mathfrak{k})$, which is characterized by the property that

$$\int_{W(\mathfrak{k})} e^{i\phi(B)} dP_s(B) = \exp\left(-\frac{s}{2}\|\phi\|^2_{H(\mathfrak{k})}\right)$$

for all continuous linear functionals ϕ on $W(\mathfrak{k})$, where $\|\phi\|_{H(\mathfrak{k})}$ denotes the norm of ϕ as a linear functional on $H(\mathfrak{k})$. We let $W(K)$ denote the set of continuous maps $x : [0,1] \to K$ satisfying $x_0 = e$ and we let $\theta : W(\mathfrak{k}) \to W(K)$ denote the Itô map. The Itô map is the almost-everywhere-defined map sending $B \in W(\mathfrak{k})$ to $x \in W(K)$ given by solving the Stratonovich stochastic differential equation

$$dx_t = x_t \circ dB_t.$$

We now let $H(\mathfrak{k}_{\mathbb{C}})$, $W(\mathfrak{k}_{\mathbb{C}})$, and $W(K_{\mathbb{C}})$ denote the analogously defined spaces with values in the complexified group or Lie algebra. We let $M_{s,t}$ denote the Wiener measure on $W(\mathfrak{k}_{\mathbb{C}}) = W(\mathfrak{k}) \oplus W(\mathfrak{k})$ with variance $(s - t/2)$ in the real directions and variance $t/2$ in the imaginary directions, which is nothing but the product measure

$$dM_{s,t}(A, B) = dP_{s-t/2}(A)\, dP_{t/2}(B).$$

We also have the complex version of the Itô map, $\theta_{\mathbb{C}} : W(\mathfrak{k}_{\mathbb{C}}) \to W(K_{\mathbb{C}})$ given by solving the Stratonovich differential equation

$$dg_t = g_t \circ dZ_t.$$

For positive real numbers s and t with $s > t/2$, we now have a Segal–Bargmann transform $S_{s,t} : L^2(W(\mathfrak{k}), P_s) \to \mathcal{H}L^2(W(\mathfrak{k}_{\mathbb{C}}), M_{s,t})$ given formally by

$$S_{s,t}(f) = \text{analytic continuation of } e^{t\Delta/2} f,$$

where Δ is supposed to represent the sum of squares of derivatives with respect to an orthonormal basis for $H(\mathfrak{k})$ and the analytic continuation is from $W(\mathfrak{k})$ to $W(\mathfrak{k}_{\mathbb{C}})$ with t fixed. The above description of $S_{s,t}$ may be taken more or less literally on polynomial cylinder functions and the map $S_{s,t}$ then extends to a unitary map of $L^2(W(\mathfrak{k}), P_s)$ onto $\mathcal{H}L^2(W(\mathfrak{k}_{\mathbb{C}}), M_{s,t})$. Here $\mathcal{H}L^2$ is defined as the L^2 closure of the space of holomorphic polynomial cylinder functions. I refer to [7, Section 4] for details.

Given a function f on K, we can form the "endpoint function" on $W(\mathfrak{k})$ given by $B \mapsto f(\theta(B)_1)$. The endpoint of the Itô map, $\theta(B)_1$, is distributed as the heat kernel measure $\rho_s(x)\, dx$ on K, which means that the norm of $f(\theta(B)_1)$ in $L^2(W(\mathfrak{k}), P_s)$ is equal to the norm of f in $L^2(K, \rho_s)$. Similarly, the endpoint of the complex Itô map is distributed as $\mu_{s,t}(g)\, dg$, so that the norm of $F(\theta_{\mathbb{C}}(Z)_1)$ in $L^2(W(\mathfrak{k}_{\mathbb{C}}), M_{s,t})$ is equal to the norm of F in $L^2(K_{\mathbb{C}}, \mu_{s,t})$. Furthermore, $F \in L^2(K_{\mathbb{C}}, \mu_{s,t})$ belongs to the holomorphic subspace if and only if $F(\theta_{\mathbb{C}}(\cdot)_1) \in L^2(W(\mathfrak{k}_{\mathbb{C}}), M_{s,t})$ belongs to the holomorphic subspace.

Theorem 3. *Given $f \in L^2(K, \rho_s)$, let $F = B_{s,t} f$, which means that F is the analytic continuation to $K_{\mathbb{C}}$ of $e^{t\Delta/2} f$. Consider the endpoint function $f(\theta(B)_1) \in L^2(W(\mathfrak{k}), P_s)$. Then*

$$S_{s,t}\big(f\big(\theta(\cdot)_1\big)\big) = F\big(\theta_{\mathbb{C}}(\cdot)_1\big).$$

That is to say, the Segal–Bargmann transform $S_{s,t}$ for the infinite-dimensional linear space $W(\mathfrak{k})$, when restricted to endpoint functions, becomes the Segal–Bargmann transform $B_{s,t}$ for the compact group K. This result is Theorem 5.2 of [7]. The case $s=t$ is a variant of one of the main results of [13].

Note that the C_t version of the Segal–Bargmann transform does not make sense in the infinite-dimensional linear case. This is because in the C_t version, the measure on the domain space should be Riemannian volume measure, which would be a Lebesgue measure in the linear case, and there is no Lebesgue measure when the dimension of the space is infinite. Driver and I introduced the two-parameter transform $S_{s,t}$ with the idea that the large s limit of this transform would be an approximation to the nonexistent transform C_t.

Note that the formula for $B_{s,t}$ is independent of s; only the norms on the domain and range depend on s. Furthermore, as on any compact manifold, the heat kernel measure $\rho_s(x)\,dx$ tends to a constant multiple of the Riemannian volume measure dx as s tends to infinity. Thus, roughly speaking, the C_t form of the Segal–Bargmann transform can be obtained from the infinite-dimensional linear case by applying $S_{s,t}$ to endpoint functions and then letting s tend to infinity.

5.3. *Toeplitz operators on $K_{\mathbb{C}}$ from the infinite-dimensional perspective*

We now (finally) arrive at the matter of Toeplitz operators. If ϕ is a bounded measurable function on $W(\mathfrak{k}_{\mathbb{C}})$ then we can define the Toeplitz operator T_ϕ on $\mathcal{H}L^2(W(\mathfrak{k}_{\mathbb{C}}), M_{s,t})$ precisely as in the finite-dimensional case as $T_\phi(\Psi) = P_{s,t}(\phi\Psi)$, where $P_{s,t}$ is the orthogonal projection from $L^2(W(\mathfrak{k}_{\mathbb{C}}), M_{s,t})$ to the holomorphic subspace. (Recall that the holomorphic subspace is defined to be the L^2 closure of the space of holomorphic polynomial cylinder functions.) For $\Psi_1, \Psi_2 \in \mathcal{H}L^2(W(\mathfrak{k}_{\mathbb{C}}), M_{s,t})$, we write (as in (3))

$$\langle \Psi_1, T_\phi \Psi_2\rangle = \int_{W(\mathfrak{k}_{\mathbb{C}})} \overline{\Psi_1(Z)}\phi(Z)\Psi_2(Z)\,dM_{s,t}(Z). \tag{21}$$

Up to now, things have been rigorous, but the time has come to shift to a heuristic viewpoint. It is certainly possible that all of what is to come could be done rigorously, but for now we content ourselves with using a heuristic infinite-dimensional argument to give additional insight into the rigorous finite-dimensional proofs of the preceding section.

Let us now consider how Toeplitz operators relate to multiplication operators. Suppose U is a function on $W(\mathfrak{k})$ and U is of the form $U = e^{t\Delta/4}\tilde{U}$ for some other function $\tilde{U}$ (assuming we can make sense of $e^{t\Delta/4}$). Now, given functions ψ_1 and ψ_2 on $W(\mathfrak{k})$, we let Ψ_1 and Ψ_2 denote the analytic continuations of $e^{t\Delta/2}\psi_1$ and $e^{t\Delta/2}\psi_2$, respectively. We use the $s\to\infty$ limits of the Hilbert spaces $L^2(W(\mathfrak{k}), P_s)$ and $\mathcal{H}L^2(W(\mathfrak{k}_{\mathbb{C}}), M_{s,t})$ and the map $S_{s,t}$ as approximations to the nonexistent spaces $L^2(W(\mathfrak{k}), \mathcal{D}A)$, $\mathcal{H}L^2(W(\mathfrak{k}_{\mathbb{C}}), \nu_t)$ and the nonexistent transform C_t connecting them. In light of (21) and the finite-dimensional result (1), it is reasonable to expect that we will have

$$\lim_{s\to\infty}\langle \psi_1, U\psi_2\rangle_{L^2(W(\mathfrak{k}),P_s)} = \lim_{s\to\infty}\int_{W(\mathfrak{k}_{\mathbb{C}})} \overline{\Psi_1(A+iB)}\tilde{U}(A)\Psi_2(A+iB)\,dM_{s,t}(A,B). \tag{22}$$

Now, there are at least two reasons why (22) does not make sense in general. First, the heat operator $e^{t\Delta/4}$ is very ill behaved in the infinite-dimensional case. Second, we have to regard the *same* functions ψ_1 and ψ_2 as functions in various *different* L^2 spaces of the form $L^2(W(\mathfrak{k}), P_s)$. However, for $s \neq s'$, the measures P_s and $P_{s'}$ are mutually singular, so it does not make sense to think of an element of $L^2(W(\mathfrak{k}), P_s)$ as also being an element of $L^2(W(\mathfrak{k}), P_{s'})$. A similar issue applies to the functions Ψ_1 and Ψ_2. As a result, if we hope to apply (22), we need to restrict to a case where we can make sense of $e^{t\Delta/4}\tilde{U}$ and where it makes sense to think of the same function as belonging to various different L^2 spaces, with respect to pairwise singular measures.

The case we are really interested in is the one in which ψ_1, ψ_2, and U are all functions of the endpoint of the Itô map. So we assume $\psi_j(A) = f_j(\theta(A)_1)$, $j = 1, 2$, where f_1 and f_2 are functions on K and we assume that $U(A) = V(\theta(A)_1)$. If A is distributed as the measure P_s then $\theta(A)_1$ is distributed as the heat kernel measure $\rho_s(x)\,dx$ on K. These heat kernel measures are equivalent for different values of s. Thus, for f an almost-everywhere defined function on K, it makes sense to think of $f(\theta(A)_1)$ as a function in various different spaces $L^2(W(\mathfrak{k}), P_s)$.

We compute Ψ_1 and Ψ_2 using the transform $S_{s,t}$. By Theorem 3, we have $\Psi_j(Z) = F_j(\theta_{\mathbb{C}}(Z))$, where F_j is the analytic continuation to $K_{\mathbb{C}}$ of $e^{t\Delta/2} f_j$. Now, Theorem 3 is one rigorous way of interpreting the heuristic formula

$$\Delta\big[f\big(\theta(\cdot)_1\big)\big] = (\Delta_K f)\big(\theta(\cdot)_1\big). \tag{23}$$

Here on the left-hand side Δ refers to the infinite-dimensional Laplacian for $W(\mathfrak{k})$ (formally, sum of squares of derivatives with respect to an orthonormal basis of $H(\mathfrak{k})$) and on the right-hand side, Δ_K refers to the finite-dimensional Laplacian for K. (See also the appendix of [7] for another way of interpreting this formula.) Suppose that there exists a function $\tilde{V}$ on K such that $V = e^{t\Delta/4}\tilde{V}$. Then (23) implies, at least formally, that

$$e^{t\Delta/4}\big[\tilde{V}\big(\theta(\cdot)_1\big)\big] = V\big(\theta(\cdot)_1\big).$$

We see, then, that in the case of endpoint functions, we can make sense of both sides of (22). Of course, this does not prove that the two sides are equal, but it seems reasonable to expect this to be the case. In the case of endpoint functions, (22) becomes

$$\begin{aligned} &\lim_{s\to\infty} \int_{W(\mathfrak{k})} \overline{f_1\big(\theta(A)_1\big)}\, V\big(\theta(A)_1\big) f_2\big(\theta(A)_1\big)\, dP_s(A) \\ &\quad = \lim_{s\to\infty} \int_{W(\mathfrak{k}_{\mathbb{C}})} \overline{F_1\big(\theta_{\mathbb{C}}(A+iB)_1\big)}\, F_2\big(\theta_{\mathbb{C}}(A+iB)_1\big)\tilde{V}\big(\theta(A)_1\big)\, dM_{s,t}(A,B). \end{aligned}$$

We can write this as

$$\lim_{s\to\infty} \int_K \overline{f_1(x)}\, V(x) f_2(x)\rho_s(x)\, dx = \lim_{s\to\infty} \int_{K_{\mathbb{C}}} \overline{F_1(g)}\, F_2(g)\, d\mu_{s,t}^{\tilde{V}}(g), \tag{24}$$

where $d\mu_{s,t}^{\tilde{V}}$ is (in general complex) measure defined by

$$d\mu_{s,t}^{\tilde{V}} = E_*\big(\tilde{V}\big(\theta(A)\big)\, dM_{s,t}(A,B)\big),$$

where E_* is the push-forward under the "endpoint map" $E : W(\mathfrak{k}_{\mathbb{C}}) \to K_{\mathbb{C}}$ given by

$$E(Z) = \theta_{\mathbb{C}}(Z)_1.$$

Let us now assume that $\mu_{s,t}^{\tilde{V}}$ has a density $\mu_{s,t}^{\tilde{V}}(g)$ with respect to the Haar measure dg. Since ρ_s tends to $1/\mathrm{Vol}(K)$ as s tends to infinity, letting s tend to infinity in (24) gives

$$\frac{1}{\mathrm{Vol}(K)} \int_K \overline{f_1(x)} V(x) f_2(x)\, dx = \int_{K_{\mathbb{C}}} \overline{F_1(g)} F_2(g) \left[\lim_{s\to\infty} \frac{\mu_{s,t}^{\tilde{V}}(g)}{\nu_t(g)} \right] \nu_t(g)\, dg. \tag{25}$$

If all of this heuristic arguing actually leads in the end to the right answer, (25) tells us that $C_t M_V C_t^{-1}$ can be represented as a Toeplitz operator with symbol ϕ_V given by

$$\phi_V(g) = \mathrm{Vol}(K) \lim_{s\to\infty} \frac{\mu_{s,t}^{\tilde{V}}(g)}{\nu_t(g)}. \tag{26}$$

The infinite-dimensional approach thus at least gives us a prediction of what the Toeplitz symbol of $C_t M_V C_t^{-1}$ should be. We will now verify (rigorously) that the right-hand side of (26) agrees with the expression for ϕ_V given in Section 3. We restrict ourselves to the semisimple case; the commutative case is similar. I am grateful to Bruce Driver for pointing out to me the relation (27) and its proof.

We begin by observing that

$$\theta_{\mathbb{C}}(A + iB) = \theta_{\mathbb{C}}\big(iB^{\theta(A)}\big)\theta(A), \tag{27}$$

almost surely, where

$$B_t^{\theta(A)} = \int_0^t \mathrm{Ad}_{\theta(A)_s}\, dB_s.$$

This result follows from the stochastic differential equation for $\theta_{\mathbb{C}}$. (If A and B were smooth paths, then a simple computation shows that the right-hand side of (27) would satisfy the same differential equation as the left-hand side. Stratonovich stochastic differential equations are such that the same result holds in the stochastic case, with no correction terms.)

Since the increments of B are distributed in an Ad-K-invariant fashion, the distribution of $(A, B^{\theta(A)})$ is the same as the distribution of (A, B). Using this fact and (27), we have

$$\begin{aligned}
&\int_{W(\mathfrak{k}_{\mathbb{C}})} f\big(\theta_{\mathbb{C}}(A+iB)\big)\tilde{V}\big(\theta(A)\big)\, dM_{s,t}(A,B) \\
&\quad= \int_{W(\mathfrak{k}_{\mathbb{C}})} f\big(\theta_{\mathbb{C}}\big(iB^{\theta(A)}\big)\theta(A)\big)\tilde{V}\big(\theta(A)\big)\, dM_{s,t}(A,B) \\
&\quad= \int_{W(\mathfrak{k}_{\mathbb{C}})} f\big(\theta_{\mathbb{C}}(iB)\theta(A)\big)\tilde{V}\big(\theta(A)\big)\, dM_{s,t}(A,B) \qquad (28)
\end{aligned}$$

for any continuous function f of compact support on $K_{\mathbb{C}}$. Recall that $M_{s,t}$ decomposes as the product measure $dP_{s-t/2}(A) \times dP_{t/2}(B)$. Furthermore, $\theta_{\mathbb{C}}(iB)$ is distributed as the heat kernel measure $\mu_{s,t}(g)\,dg$ on $K_{\mathbb{C}}$ and $\theta(A)$ is distributed as the heat kernel measure $\rho_{s-t/2}(x)\,dx$ on K. Thus, (28) becomes

$$\int_{W(\mathfrak{k}_{\mathbb{C}})} f\big(\theta_{\mathbb{C}}(A+iB)\big)\tilde{V}\big(\theta(A)\big)\,dM_{s,t}(A,B) = \int_K \int_{K_{\mathbb{C}}} f(gx)\tilde{V}(x)\;\mu_{s,t}(g)\,dg\;\rho_{s-t/2}(x)\,dx. \tag{29}$$

After making the change of variable $g \to gx^{-1}$ in the inner integral of (29) and reversing the order of integration, we see that the pushed-forward measure (which we are denoting $d\mu_{s,t}^{\tilde{V}}(g)$) is given by

$$d\mu_{s,t}^{\tilde{V}}(g) = \left[\int_K \mu_{s,t}\big(gx^{-1}\big)\tilde{V}(x)\rho_{s-t/2}(x)\,dx\right]dg. \tag{30}$$

If we now let s tend to infinity in (30), $\rho_{s-t/2}$ becomes $1/\mathrm{Vol}(K)$ and $\mu_{s,t}$ becomes the subelliptic heat kernel $\mu_{t/2,t}$. Thus, we have

$$\mathrm{Vol}(K)\lim_{s\to\infty}\frac{\mu_{s,t}^{\tilde{V}}(g)}{\nu_t(g)} = \frac{\int_K \mu_{t/2,t}(gx^{-1})\tilde{V}(x)\,dx}{\nu_t(g)}, \tag{31}$$

which means that (26) is in agreement with the results of Section 3.

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