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G_δ-Embeddings in Hilbert Space*

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It is shown that a separable Banach space X has the *point of weak to norm continuity property* (resp. *the Radon–Nikodym property*) if and only if there exists a compact G_δ-embedding (resp. an H_δ-embedding) from X into l_2. This solves several questions of J. Bourgain and H. P. Rosenthal (*J. Funct. Anal.* **52** (1983)). It is also shown that every non-relatively compact sequence in a Banach space with property (PC) has a difference subsequence which is a boundedly complete basic sequence. This solves a question of Pelczynski and extends some results of W. B. Johnson and H. P. Rosenthal (*Studia Math.* **43** (1972), 77–92). Various related questions asked in the above Bourgain–Rosenthal reference and by G. A. Edgar and R. F. Wheeler (*Pac. J. Math.* **115** (1984)) and N. Ghoussoub and H. P. Rosenthal (*Math. Ann.* **264** (1983), 321–332) are also settled.

INTRODUCTION

Let X and Y be two Banach spaces and let $S: X \to Y$ be a one-to-one bounded linear operator. S is said to be:

(i) A *semi-embedding* if the image of the unit ball of X by S is norm closed in Y.

(ii) An F_σ-*embedding* (resp. a *nice* F_σ-*embedding*) if the image of every norm open set in X by S is a norm F_σ (resp. a weak F_σ) in Y.

(iii) A G_δ-*embedding* (resp. a *nice* G_δ-*embedding*) if the image of

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every norm closed bounded and separable subset of X by S is a norm G_δ (resp. a weak G_δ).

(iv) An *H_δ-embedding* if for every norm closed convex bounded and separable subset C of X, we have that $\overline{S(C)}\backslash S(C)$ is a countable union of closed convex bounded sets.

It is easy to see that if X is separable, then a semi-embedding is a nice F_σ-embedding which in turn is a nice G_δ-embedding. Moreover, if S is an F_σ-embedding then there exists an equivalent norm on X which makes S a semi-embedding (Proposition 1.6 of [5]). This result, which is due to Saint-Raymond, immediately implies the following:

PROPOSITION. *If X is separable then any F_σ-embedding is a nice F_σ-embedding and an H_δ-embedding.*

However, we shall see in this paper that the two notions of G_δ-embeddings are essentially different since we construct a Banach space X and a G_δ-embedding from X into l_2 such that no operator from X into l_2 is a nice G_δ-embedding (Example IV.1).

The first section is devoted to the proof of a topological characterization for G_δ-embeddings. It may be omitted on a first reading as it is independent of the rest of the paper. In it we show that a one-to-one operator $S: X \to Y$ is a G_δ-embedding if and only if the image of any δ-separated sequence in X has an isolated point in Y. This shows that S is a G_δ-embedding whenever for every separable closed bounded non-empty subset K of X, $S^{-1}_{|S(K)}$ has a point of continuity. This answers a question in [5] where the statement is proved under the additional assumption that the image of the unit ball of X is a G_δ.

Recall that a Banach space X is said to have:

(i) *The point of continuity property (PC)* if every weakly closed bounded subset of X contains a point of weak to norm continuity.

(ii) *The Radon–Nikodym property (R.N.P.)* if every weakly closed bounded subset of X contains a denting point.

In Section II we show that the Banach spaces which nicely G_δ-embed in l_2 are exactly those separable Banach spaces with property (PC). This settles at once questions (1), (2) and (3) of Bourgain and Rosenthal [5]. Indeed.

(1) Every G_δ-embedding into l_2 of a space with (PC) but failing (R.N.P.) cannot be the composition of a finite number of semi-embeddings since these operators preserve the (R.N.P.).

(2) Every Banach space with property (PC) but failing the (R.N.P.) is a counterexample to question (2) since they G_δ-embed in spaces with the (R.N.P.) while failing to have such a property.

(3) Since spaces with (R.N.P.) have property (PC), they nicely G_δ-embed in l_1.

A counterexample to questions (1) and (2) is for example, the predual B of the James-tree space and was given in [11]. The $\mathscr{L}_\infty$-space with (R.N.P.) constructed by Bourgain and Delbaen [2] is also shown to be a counterexample to questions in [5] and [10].

Following Bourgain and Rosenthal [4] we shall say that a Banach space X has a *boundedly complete skipped blocking finite-dimensional decomposition* provided there exists a sequence (G_i) of finite-dimensional subspaces of X such that the following conditions are satisfied:

(a) $X = [G_i]_{i=1}^{\infty}$.

(b) $G_i \cap [G_j]_{j \neq i} = \{0\}$ for every i.

(c) If (m_k) and (n_k) are sequences of positive integers so that $m_k < n_k + 1 < m_{k+1}$ and (x_k) is a sequence such that x_k belong to the finite-dimensional subspace $G[m_k, n_k]$ generated by $\{G_i; m_k \leqslant i \leqslant n_k\}$ then the series $\sum_k x_k$ converges whenever its partial sums are bounded.

We also show in Section II that separable Banach spaces with (PC) are exactly the ones with a boundedly complete skipped blocking finite-dimensional decomposition, from which follows that every non-relatively compact sequence in such a space has a difference subsequence which is a boundedly complete basis. This answers positively a question of Pelczynski [7] (see also Bourgain and Rosenthal [4]).

In Section III, we give a more precise characterization for spaces with the (R.N.P.) in terms of G_δ-embeddings in l_2. We show that a separable Banach space X has the (R.N.P.) if and only if there exists an H_δ-embedding of X into l_2. Note that from the results of Bourgain and Rosenthal [5], a space X is isomorphic to a separable dual if and only if there is an F_σ-embedding from X into l_2.

Finally, in Section IV we show that every Banach space which G_δ-embeds in l_2 has an infinite-dimensional subspace which nicely G_δ-embeds in l_2 which in turn has an infinite-dimensional subspace which F_σ-embeds in l_2. In particular every space that G_δ-embeds in l_2 is also somewhat separable dual.

Several of the concepts of this paper were motivated by those of a recent joint work of G. A. Edgar and R. F. Wheeler. We refer the interested reader to their fundamental paper [8].

For unexplained notions and notations we refer to the books of Diestel and Uhl [7] and Lindenstrauss and Tzafriri [18].

I. A Topological Characterization of G_δ-Embeddings

In [5], Bourgain and Rosenthal prove that a one-to-one operator $T: X \to Y$ is a G_δ-embedding if $T(B_X)$ is a G_δ and if for every closed bounded non-empty subset K of X, $T^{-1}_{|TK}$ has a point of continuity. They ask whether the assumption that $T(B_X)$ is a G_δ can be omitted. This section is devoted to give a positive answer to this question.

We shall say that a sequence (x_n) in a metric space (X, d) is δ-separated if $d(x_m, x_n) \geqslant \delta > 0$ whenever $m \neq n$.

THEOREM I.1. *Let X and Z be two Polish spaces and let f be a continuous function which maps every δ-separated sequence in X into a sequence which is not dense in itself in Z, for every $\delta > 0$. Then $f(X)$ is a G_δ in Z.*

First some notations. We set $Y = f(X) \subseteq Z$ and say that a subset $A \subseteq Z$ is *Y-dense* if A is non-empty and $Y \cap A$ is dense in A. We say that $S \subseteq X$ is *ε-small* if S is contained in a finite union of closed balls of radius ε in X. Note that if S is ε-small, so is $\bar{S}$.

If $A \subset Z$ is Y-dense, we define a function k_A on A by $k_A(a) = \inf\{\varepsilon > 0; \exists V$ neighborhood of a with $f^{-1}(V \cap A)$ ε-small$\}$.

It is clear that $B = \{k_A < \varepsilon\}$ is an open subset of A for every $\varepsilon > 0$. In other words k_A is upper-semi-continuous on A. Note also that if B is non-empty, B is then Y-dense and $k_B = k_A$ on B.

For the proof of Theorem I.1, we shall need the following four lemmas.

LEMMA I.1. *Under the above assumptions. If A is a Y-dense subset of Z, then the set $\{k_A < \varepsilon\}$ is non-empty for every $\varepsilon > 0$.*

Proof. Suppose not. We have then $k_A \geqslant \varepsilon$ on A. We shall construct an $\varepsilon/2$-separated sequence (x_n) in S, such that $\{f(x_n)\} \subseteq A$ and is dense in itself, which is obviously a contradiction to the assumption.

Suppose $x_1, x_2, \ldots, x_n$ constructed with $a_i = f(x_i) \in A$, $i = 1, \ldots, n$, and $d(x_i, x_j) > \varepsilon/2$ whenever $1 \leqslant i < j \leqslant n$. Let $V = B(a_1, 2^{-n})$. Since $k_A(a_1) \geqslant \varepsilon$, $f^{-1}(V \cap A)$ is not $\varepsilon/2$-small, thus not contained in $\bigcup_{i=1}^n B(x_i, \varepsilon/2)$. Let $x_{n+1} \in f^{-1}(V \cap A) \setminus \bigcup_{i=1}^n B(x_i, \varepsilon/2)$. We can proceed the same way for $x_2, \ldots, x_n$, thus finding $x_{n+1}, \ldots x_{2n}$ with $a_i = f(x_i) \in A$, $i = 1, 2, \ldots, 2n$, $d(x_i, x_j) > \varepsilon/2$ if $1 \leqslant i < j \leqslant 2n$ and $d(a_i, a_{n+i}) < 2^{-n}$ for $i = 1, \ldots, n$.

By repeating this procedure we clearly obtain an $\varepsilon/2$-separated sequence whose image is dense in itself.

Remark I.1. If Y is dense in Z, then Y contains a G_δ-dense in Z. Indeed suppose $Z = \bar{Y}$, then $k = k_Z$ is upper-semi-continuous. Moreover the set

$\{k < 1/n\}$ is dense for every n, for otherwise we have an open set $\omega \subset Z$ with $k \geqslant 1/n$ on ω, thus $k_\omega = k \geqslant 1/n$ on ω, contradicting Lemma I.1. It follows that $\{k=0\}$ is a dense G_δ in Z. If $k(z)=0$, one can find a decreasing sequence (V_n) of neighborhood of z such that $\operatorname{diam}(\bar{V}_n) \leqslant 2^{-n}$ and $f^{-1}(\bar{V}_n)\, 2^{-n}$-small. Then $K = \bigcap_n f^{-1}(\bar{V}_n) = f^{-1}(\bigcap_n \bar{V}_n) = f^{-1}(\{z\})$ shows that $z \in Y$. Furthermore, for every sequence (x_m) in X with $f(x_m) \to z$, we have $d(x_m, K) \to 0$. Note that $\{x_m\}$ is precompact: given n, all x_m's but finitely many belong to V_n, thus $\{x_m\} \subseteq f^{-1}(\bar{V}_n)$ except for a finite set, and $f^{-1}\ (\bar{V}_n)$ is 2^{-n}-small. If $x_{m_k} \to x$, $f(x_{m_k}) \to z$ so $f(x) = z$ and $x \in K$. It follows that the set $\{k=0\}$ is a dense G_δ in Z consisting of the points of continuity of $f^{-1} \,|\, f(X)$.

We will call a Y-dense subset A of Z *ε-moderate* ($\varepsilon > 0$) if $k_A < \varepsilon$ on A. If B is Y-dense, $A = \{k_B < \varepsilon\}$ is non-empty by Lemma I.1, open in B, therefore A is Y-dense and $k_A = k_B$ on A which shows that A is ε-moderate.

LEMMA I.2. *Let A be a Y-dense subset of Z and $\varepsilon > 0$ given. There exists then a transfinite family (A_α) of subsets of A such that for every $\alpha < \Omega$ the following property (H_α) holds.*

(H_α)
- (a) For every $\beta \leqslant \alpha$, A_β is open in $F_\beta = \overline{Y \cap A \backslash \bigcup_{\gamma<\beta} A_\gamma}$ (where the closure is taken relative to A), and A_β is ε-moderate.
- (b) The sets $(A_\beta)_{\beta \leqslant \alpha}$ are disjoint and for every $\beta \leqslant \alpha$, $\bigcup_{\gamma<\beta} A_\gamma$ is G_δ in A.
- (c) If $F_\beta \neq \phi$, then A_β is non-empty, for every $\beta \leqslant \alpha$.

Proof. Suppose the family has been constructed for every $\beta < \alpha$ with the property H_β.

If $Y \cap A \subseteq \bigcup_{\beta<\alpha} A_\beta$, then $F_\alpha = \overline{Y \cap A \backslash \bigcup_{\beta<\alpha} A_\beta}$ is empty and we set $A_\alpha = \phi$.

Otherwise F_α is Y-dense and $A_\alpha = \{k_{F_\alpha} < \varepsilon\}$ is open in F_α and is ε-moderate.

We must show that $A_\beta \cap A_\alpha = \phi$ when $\beta < \alpha$. Actually $A_\beta \cap F_\alpha = \phi$; because A_β is open in $F_\beta, A_\beta \cap F_\alpha$ is open in $F_\alpha \subset F_\beta$, so $A_\beta \cap F_\alpha \neq \phi \Rightarrow A_\beta \cap F_\alpha$ meets $Y \cap A \backslash \bigcup_{\gamma<\alpha} A_\gamma$, which is absurd.

It remains to show that $\bigcup_{\beta<\alpha} A_\beta$ is G_δ. To this end consider

$$B = \bigcap_{\beta<\alpha} \left\{ \left(\bigcup_{\gamma<\beta} A_\gamma \right) \cup F_\beta \right\}.$$

Since $A_\gamma \subset F_\gamma \subset F_\beta$ for $\beta < \gamma < \alpha$,

$$B_\beta = \left(\bigcup_{\gamma<\beta} A_\gamma \right) \cup F_\beta \supset A' = \bigcup_{\gamma<\alpha} A_\gamma.$$

From $H_\beta(b)$, we have that B_β and hence B is G_δ in A. We have $B \supset A'$, and if $z \in B \backslash A'$, $z \in F = \bigcap_{\beta<\alpha} F_\beta$, which is disjoint from A', thus

$$A' = B \backslash F \text{ is } G_\delta \text{ in } A.$$

Also note that each A_α is relatively open in A, thus $F_\sigma - G_\delta$ in A, and $\bigcup_{\beta<\alpha} A_\beta$ is $F_\sigma - G_\delta$ in A for every α.

LEMMA I.3. *Let A be an $F_\sigma - G_\delta$ Y-dense subset of Z and $\varepsilon > 0$. There exists then a sequence (A_n) of subsets of A such that*

(a) *the A_n's are disjoint $F_\sigma - G_\delta$ ε-moderate subsets of Z;*

(b) *$A' = \bigcup_n A_n$ is $F_\sigma - G_\delta$ in Z and contains $Y \cap A$.*

Proof. Since A is metrizable and separable the decreasing family F_α of closed subsets of A must be stationary at some countable ordinal but clearly $F_{\alpha+1} = F_\alpha \Rightarrow F_\alpha = \phi$ from our construction.

If $F_\alpha = \phi$ we have $Y \cap A \subseteq \bigcup_{\beta<\alpha} A_\beta$, which proves Lemma I.3.

LEMMA I.4. *There exists a double sequence $(A_{k,n})$ of subsets of Z such that:*

(1) *For every fixed k, the $A_{k,n}$'s are disjoint $F_\sigma - G_\delta$ 2^{-k}-moderate subsets of Z, and $A_k = \bigcup_n A_{k,n}$ is an $F_\sigma - G_\delta$ containing Y.*

(2) *For every k and n, $A_{k+1,n}$ is a subset of some $A_{k,l}$.*

Proof. We start with $A = \bar{Y}$ and apply Lemma I.3 with $\varepsilon = 1/2$ to get $(A_{1,n})$ satisfying (1) for $k = 1$.

Apply again Lemma I.3 to each $A_{1,n}$ with $\varepsilon = 1/4$ to produce a family $(A_{2,n,m})_m$ with the properties (a) and (b) of Lemma I.3.

Set $A'_{2,n} = \bigcup_m A_{2,n,m} \supseteq Y \cap A_{1,n}$.

Consider the family $(A_{2,n,m})_{n,m}$ and $A_2 = \bigcup_{n,m} A_{2,n,m} = \bigcup_n A'_{2,n}$. We have $A_2 \supseteq \bigcup_n (Y \cap A_{1,n}) \supseteq Y \cap A_1 = Y$.

Everything is clear in (1) and (2) for the family $(A_{2,n,m})$ except that A_2 is a G_δ.

But $\bigcup_{m>n} A_{1,m}$ is G_δ since A_1 is G_δ and each $A_{1,j}$ is F_σ, thus

$$C_n = \left(\bigcup_{j=1}^{n} A'_{2,j}\right) \cup \left(\bigcup_{m>n} A_{1,m}\right) \text{ is } G_\delta,$$

and

$$\bigcap_n C_n = \bigcup_n A'_{2,n} = A_2 \text{ is } G_\delta.$$

The family $A_{2,n}$ in Lemma I.4 is just the family $(A_{2,n,m})$ after some re-indexing. The step from k to $k+1$ is identical to what we just made.

Proof of Theorem I.1. Let (A_k) be as in Lemma I.4. We claim that $Y = \bigcap_k A_k$ thus proving that $Y = f(X)$ is a G_δ subset of Z. Indeed, we have $Y \subseteq \bigcap_k A_k$ by Lemma I.4. If now $z \in \bigcap_k A_k$, there exists by (2) in Lemma I.4 a sequence $A_{1,n_1} \supseteq A_{2,n_2} \supseteq \cdots \supseteq A_{k,n_k} \supseteq \cdots$ of sets containing z, with $B_k = A_{k,n_k}$ being 2^{-k}-moderate.

Using the definition of moderation, it is possible to find a decreasing sequence (V_k) of neighborhoods of z, with $\operatorname{diam}(\bar{V}_k) \leqslant 2^{-k}$ and $f^{-1}(V_k \cap B_k)$ 2^{-k}-small and non-empty for every k.

It follows that

$$\bigcap_k \overline{f^{-1}(V_k \cap B_k)} = K \text{ is non-empty (and compact).}$$

If now $x \in K, f(x) \in \bigcap_k \bar{V}_k = \{z\}$, showing that $z \in Y$.

Remark. If one assumes that $f(X)$ is co-analytic (in particular Borel) in Y, then Theorem I.1 would follow easily from a celebrated result of Hurewicz [13].

THEOREM I.2. *Let X and Z be two Banach spaces and let $T: X \to Z$ be a one-to-one operator. Then the following assertions are equivalent*:

(1) *T is a G_δ-embedding.*

(2) *For every closed bounded separable subset K of X, $T^{-1}_{|T(K)}$ has a point of continuity.*

(3) *For every $\delta > 0$ and every δ-separated sequence $\{x_n\}$ in B_X, the sequence $\{Tx_n\}$ is not dense in itself.*

Proof. Note first that we can suppose X and Z separable since we can restrict ourselves to the separable subspaces generated by the separable subsets involved in the discussion. (1) ⇒ (2) is clear. For (2) ⇒ (3) take any point of continuity of T^{-1} on $T(D)$ whenever D is a δ-separated countable subset of B_X. It is clearly an isolated point of $T(D)$. For (3) ⇒ (1) it is enough to apply Theorem I.1 to each closed bounded separable subset of X.

THEOREM I.3. *Let X and Z be two Banach spaces such that the ball of Z equipped with the weak topology is a Polish space. Let $T: X \to Z$ be a one-to-one operator. The following assertions are then equivalent.*

(1) *T is a nice G_δ-embedding.*

(2) *For every closed bounded separable subset K of X, $T^{-1}_{|T(K)}$ has a point of weak to norm continuity.*

(3) *For every $\delta > 0$ and every δ-separated sequence (x_n) in B_X the sequence $\{Tx_n\}$ is not weakly dense in itself.*

Remark. Note that (1) implies (2) under the weaker assumption that every weakly closed bounded subset of Z is a Baire space for the weak topology. In view of a result of Edgar and Wheeler [8] this is verified whenever Z has the (PC) property.

We now give an application of Theorem I.2 to a three-space type problem.

THEOREM I.4. *Given a separable Banach space X and a subspace Y of X, let $Q: X \to X/Y$ be the quotient map. Let T be an operator from X into a Banach space Z such that the restriction of T to Y is a G_δ-embedding, then the operator $(T, Q): X \to Z \oplus X/Y$ is a G_δ-embedding.*

Proof. Suppose (x_n) is a bounded sequence in X that is δ-separated for some $\delta > 0$ and such that $\{(Tx_n, Qx_n)\}_n$ is dense in itself in $Z \oplus X/Y$. We may assume without loss of generality that $x_0 = 0$. Consider $M = \{m \in M; \|Qx_m\| < \delta/10\}$. The sequence $\{(Tx_m, Qx_m); m \in M\}$ is again dense in itself. Let now R be a continuous lifting from X/Y into X such that $\|RQx_m\| \leqslant \delta/4$ for each $m \in M$. Note that the sequence $y_m = x_m - RQx_m$ is $\delta/2$-separated and is contained in Y. Given now m_0 in M, there exists a sequence (m_k) such that $(Tx_{m_k}, Qx_{m_k}) \to (Tx_{m_0}, Q_{m_0})$ when $k \to \infty$. Hence $RQx_{m_k} \to RQx_{m_0}$ and $Ty_{m_k} = T(x_{m_k} - RQx_{m_k}) \to T(x_{m_0} - RQx_{m_0}) = Ty_{m_0}$ which shows that $(Ty_m)_m$ is dense in itself hence contradicting the assumption on $T_{|Y}$.

II. SPACES WITH THE POINT OF CONTINUITY PROPERTY

Recall first that a Banach space X has property (PC) iff for every $\varepsilon > 0$ and every bounded non-empty subset F of X, there exists a weakly open set $V \subset X$ such that $V \cap F \neq \phi$ and $\operatorname{diam}(V \cap F) \leqslant \varepsilon$.

The following lemma reduces most of the problems considered here to the study of the unit ball.

LEMMA II.1. *Let X be a separable subspace of a dual space Y^*, then for every norm closed subset F of B_X, we have*

$$B_{Y^*} \backslash F = (B_{Y^*} \backslash B_X) \cup \bigcup_n K_n$$

where the K_n's are weak-compact convex subsets of Y^*.*

Proof. For each x in X, denote by $r(x)$ the distance $d(x, F)$ of x to F, and let $D_X(x, r(x)) = \{y \in X;\ \|y\| \leqslant 1 \text{ and } \|y - x\| < r(x)/2\}$. These are open balls in the polish space $(B_{X'}\| \ \|)$. Since, $B_X \backslash F = \bigcup_{x \in B_X \backslash F} D_X(x, r(x))$, there exists a countable subfamily (x_n) such that $B_X \backslash F = \bigcup_n D(x_n, r(x_n))$.

Let now $\bar{D}_{Y^*}(x, r(x)) = \{y \in Y^*;\ \|y\| \leqslant 1 \text{ and } \|y - x\| \leqslant r(x)/2\}$. Note that each $\bar{D}_{Y^*}(x, r(x))$ is weak*-compact and convex in Y^*. We claim that $F \cap \bigcup_n \bar{D}_{Y^*}(x_n, r(x_n)) = \phi$. Indeed, if not there will be y in F and an x_n in $B_X \backslash F$ such that $\|y - x_n\| \leqslant r(x_n)/2 = \frac{1}{2} d(x_n, F)$, which is absurd. Finally, we get $B_{Y^*} \backslash F = (B_{Y^*} \backslash B_X) \cup \bigcup_n \bar{D}_{Y^*}(x_n, r(x_n))$.

The following is a version of a result of Edgar and Wheeler [8] that is crucial to what follows. We include a proof for completeness. Recall first that an elementary w^*-open neighborhood of x is a set of the form $V(x, G, \alpha) = \{x^{**} \in X^{**};\ \sup_{x^* \in G} |(x^{**} - x)(x^*)| < \alpha\}$ where $\alpha > 0$ and G is a finite subset of X^*.

LEMMA II.2. *If X is a separable Banach space with property (PC), then for every bounded closed subset F of X, there exists a sequence (γ_n) of countable ordinals, sequences $\{K_{\alpha,n};\ \alpha \leqslant \gamma_n, n \in \mathbb{N}\}$ of weak*-compact sets in X^{**} and sequences $\{V_{\alpha,n};\ \alpha \leqslant \gamma_n, n \in \mathbb{N}\}$ of elementary weak*-open sets such that*

$$F = \bigcap_n \bigcap_{\alpha \leqslant \gamma_n} \Big(K_{\alpha,n} \cup \bigcup_{\beta < \alpha} V_{\beta,n} \Big).$$

Proof. Given $\varepsilon > 0$, we construct by transfinite induction a decreasing family of norm closed subsets (F_α) of F in the following way:

(i) $F_0 = F$.

(ii) If $\alpha = \beta + 1$ and $F_\beta \neq \phi$, use property (PC) to find an elementary w^*-open set V_β such that $V_\beta \cap F_\beta \neq \phi$ and $\operatorname{diam}(F_\beta \cap V_\beta) < \varepsilon$. Set $F_\alpha = F_\beta \backslash V_\beta$.

(iii) If α is a limit ordinal, let $F_\alpha = \bigcap_{\beta < \alpha} F_\beta$.

Since X is separable, there is $\gamma < \Omega$ (the first uncountable ordinal) such that $F_\gamma = \phi$. Let now K_α equal the weak*-closure of F_α in X^{**}.

Note that $F \subseteq K_\alpha \cup (\bigcup_{\beta < \alpha} V_\beta)$ for each $\alpha \leqslant \gamma$. On the other hand suppose $x^{**} \in \bigcap_{\alpha \leqslant \gamma} (K_\alpha \cup (\bigcup_{\beta < \alpha} V_\beta))$ and let α_0 be the first ordinal $\alpha_0 \leqslant \gamma$ such that $x^{**} \notin K_{\alpha_0}$. We have then $x^{**} \in \bigcup_{\beta < \alpha_0} V_\beta$. That is there is $\beta < \alpha_0$ with $x^{**} \in K_\beta \cap V_\beta$. Let now a net (x_j) in F_β with $x_j \to x^{**}$ weak*. For large enough j, the x_j's are also in V_β. Take now such an x_j in V_β and we have

$$\|x_j - x^{**}\| \leqslant \varliminf_i \|x_j - x_i\| \leqslant \varepsilon \qquad \text{hence } d(x^{**}, F) \leqslant \varepsilon.$$

By taking $\varepsilon = 1/n$ in the above construction, we get a sequence of countable ordinals (γ_n), weak*-compact sets $\{K_{\alpha,n};\ \alpha \leqslant \gamma_n,\ n \in \mathbb{N}\}$ and elementary weak*-open sets $\{V_{\alpha,n}; \alpha \leqslant \gamma_n, n \in \mathbb{N}\}$ such that

$$F = \bigcap_n \bigcap_{\alpha \leqslant \gamma_n} \left(K_{\alpha,n} \cup \bigcup_{\beta < \alpha} V_{\beta,n}\right).$$

LEMMA II.3. *If X is a separable Banach space with property (PC), then there exists a separable Banach space Y and an isometry $T: X \to Y^*$ such that $T(B_X)$ is a weak* $- G_\delta$ in B_{Y^*}.*

Proof. By Lemma II.2 applied to B_X, we write

$$B_X = \bigcap_n \bigcap_{\alpha \leqslant \gamma_n} \left(K_{\alpha,n} \cup \left(\bigcup_{\beta < \alpha} V_{\beta,n}\right)\right)$$

where $K_{\alpha,n}$ is weak*-compact and $V_{\beta,n}$ is an elementary weak*-open set of the form $V(x_{\beta,n}, G_{\beta,n}, \varepsilon_{\beta,n})$ where $\varepsilon_{\beta,n} > 0$, $x_{\beta,n} \in X$ and $G_{\beta,n}$ is a finite subset in X^*. Let now D be a countable norming subset of X^* containing $G_{\beta,n}$ for each $n \in \mathbb{N}$ and $\beta < \gamma_n$ and let Y be the separable closed subspace of X^* generated by D.

Consider the inclusion map $S: Y \to X^*$ and $T = S^*: X^{**} \to Y^*$. Then $T_{|X}$ is an isometry. If $x \in X$, $\alpha > 0$ and G is a finite subset of Y, consider $W(Tx, G, \alpha) = \{y^* \in Y^*; \sup_{y \in G} |(y^* - Tx)(y)| < \alpha\}$. It is a weak*-open subset of Y^*. Note that

$$T^{-1}(W(Tx, G, \alpha)) = \left\{x^{**} \in X^{**};\ \sup_{y \in G} |T(x^{**} - x)(y)| < \alpha\right\}$$

$$= \left\{x^{**} \in X^{**};\ \sup_{y \in G} |(x^{**} - x)(y)| < \alpha\right\}.$$

This shows that each $V_{\alpha,n}$ is equal to $T^{-1}(W_{\alpha,n})$ where $W_{\alpha,n}$ is a weak*-open subset of Y^*, and $T(V_{\alpha,n}) = W_{\alpha,n}$ since T is onto. The same also holds for unions of elementary weak*-open sets. That is, since we can write

$B_X = \bigcap_m (K_m \cup O_m)$ where the K_m's are weak*-compact in X^{**} and the O_m's are countable unions of elementary weak*-open sets determined by functionals in Y we can now write

$$B_X = \bigcap_m (K_m \cup T^{-1}(W_m))$$

where W_m is weak*-open in Y^*. Note now that

$$T(B_X) = T\left(\bigcap_m (K_m \cup T^{-1}(W_m))\right) \subseteq \bigcap_m (T(K_m) \cup W_m).$$

Conversely if $y^* \in \bigcap_m (T(K_m) \cup W_m)$, there exists a subset M of $\mathbb{N}$ such that

$$y^* \in \left(\bigcap_{m \in M} T(K_m) \right) \cup \left(\bigcap_{m \notin M} W_m \right).$$

By the compactness of the K_m's we have $\bigcap_{m \in M} T(K_m) = T(\bigcap_{m \in M} K_m)$. On the other hand

$$\bigcap_{m \notin M} W_m = T\left(T^{-1} \left(\bigcap_{m \notin M} W_m \right) \right) = T\left(\bigcap_{m \notin M} T^{-1}(W_m) \right).$$

Hence $y^* \in T(\bigcap_{m \in M} K_m) \cup T(\bigcap_{m \notin M} T^{-1}(W_m)) \subseteq T(\bigcap_m (K_m \cup T^{-1}(W_m))) = T(B_X)$.

It follows that $T(B_X) = \bigcap_m (T(K_m) \cup W_m)$.

Since Y is separable, B_{Y^*} is a metrizable weak*-compact set, hence each K_m is a weak*-G_δ in Y^* from which follows that $T(B_X)$ is a weak*-G_δ in Y^*.

Now, we can prove the following:

THEOREM II.1. *For a Banach space X, the following properties are equivalent*:

(a) *X is separable and has the (PC) property.*

(b) *X has a boundedly complete skipped blocking finite-dimensional decomposition.*

(c) *There exists a nice G_δ-embedding of X into l_2.*

Proof. (a) $\Rightarrow$ (b) Use Lemma II.3 to find a separable Banach space Y such that X is a closed subspace of Y^* and $Y^* \backslash X = \bigcup_n K_n$ with (K_n) being an increasing sequence of weak*-compact subsets of Y^*. Let (x_n) be a dense sequence in X and let Y_0 be a finite subset of Y such that $X = [x_1] \oplus Y_0^\perp$. (All the annihilators will be taken in X.) We shall say that a subset M of Y, C-norms a subspace Z of X for some constant C if for every x in Z we have

$$\|x\| \leqslant (1 + C) \sup\{\langle x, y \rangle; y \in M \text{ and } \|y\| \leqslant 1\}.$$

Set now $X_1 = [x_1]$. Let $x_2 = u_2 + v_2$ where $u_2 = \lambda x_1$ and $v_2 \in Y_0^\perp$. We shall choose a finite subset Y_1 of Y such that:

(a) $Y_0 \subset Y_1$.

(b) Y_1 $\frac{1}{2}$-norms X_1.

(c) If $v_2 \neq 0$ then $v_2 \notin Y_1^\perp$.

(d) The topology $\sigma(Y^*, Y_1)$ separates $\text{Ball}(X_1)$ from K_1: that is, every

x in Ball(X_1) has a w^*-open neighborhood W determined by functionals in Y_1 such that $W \cap K_1 = \phi$.

Indeed since Ball(X_1) is compact and $K_1 \cap \text{Ball}(X_1) = \phi$, there exists a finite covering $\{V_j; j \in F_1\}$ for Ball(X_1) such that V_j is a $\sigma(X, Y)$ elementary open set disjoint of K_1 for each j. Let N_1 be the finite set of functionals in Y that determines $\{V_j; j \in F_1\}$. Let M_1 be a $\frac{1}{2}$-norming set for X_1. Let y_2 in Y such that $\langle y_2, v_2 \rangle \neq 0$ and set $Y_1 = Y_0 \cup N_1 \cup M_1 \cup \{y_2\}$. It is clear that Y_1 verifies (a), (b), (c) and (d).

Let now X_2 be the finite-dimensional subspace containing v_2 such that $Y_0^\perp = X_2 \oplus Y_1^\perp$. We get that $x_2 \in X_1 \oplus X_2$. Suppose now $X_1, \ldots, X_n$, $Y_0, \ldots, Y_{n-1}$ are constructed such that for $2 \leqslant j \leqslant n$: $Y_{j-2} \subset Y_{j-1}$ and are finite subsets of Y, $Y_{j-2}^\perp = X_j \oplus Y_{j-1}^\perp$ with $\dim X_j < \infty$, $x_j \in X_1 \oplus X_2 \oplus \cdots \oplus X_j$ and $X = X_1 \oplus X_2 \oplus \cdots \oplus X_j \oplus Y_{j-1}^\perp$. Write now $x_{n+1} = u_{n+1} + v_{n+1}$, with $u_{n+1} \in X_1 \oplus \cdots \oplus X_n$ and $v_{n+1} \in Y_{n-1}^\perp$. Choose as above a finite subset $Y_n \subset Y$ such that:

(a) $Y_{n-1} \subset Y_n$.

(b) Y_n 2^{-n}-norms $X_1 \oplus X_2 \oplus \cdots \oplus X_n$.

(c) If $v_{n+1} \neq 0$, $v_{n+1} \notin Y_n^\perp$.

(d) The topology $\sigma(Y^*, Y_n)$ separates Ball($X_1 \oplus X_2 \oplus \cdots \oplus X_n$) from K_n.

Find $X_{n+1} \subset Y_{n-1}^\perp$ such that $v_{n+1} \in X_{n+1}$, $Y_{n-1}^\perp = X_{n+1} \oplus Y_n^\perp$, hence $x_{n+1} \in X_1 \oplus \cdots \oplus X_{n+1}$.

Finally, we get a sequence (X_n) of finite-dimensional subspaces of X such that $X_n \cap (\sum_{m \neq n} \oplus X_m) = \{0\}$ and $x_n \in X_1 \oplus X_2 \cdots \oplus X_n$ for each n, hence $X = \overline{\bigcup_{n=1}^\infty X_1 \oplus X_2 \oplus \cdots \oplus X_n}$.

To prove that (X_n) is a boundedly complete skipped blocking decomposition for X we shall only consider a sequence $v_i = u_{2i+1} \in X_{2i+1}$ such that $s_n = \sum_{i=0}^n v_i$ is in the ball of X and prove that s_n is convergent. The same reasoning will hold for the even case and the general case.

Let s be a weak*-limit of (s_n) in Y^*. If $s \notin X$, there exists n_0 so that $s \in K_{2n_0+1}$. Let $G_n = X_1 \oplus \cdots \oplus X_n$. For each $n \geqslant n_0$, we have $s_n = s_{n_0} + w$ where $w \in X_{2n_0+3} \oplus X_{2n_0+5} \cdots \oplus X_{2n+1} \subset Y_{2n_0+1}^\perp$. But property ($d$) gives a w^*-open neighborhood of s_{n_0} of the form $W = \{y^*; \sup_\alpha |\langle y^* - s_{n_0}, y_\alpha \rangle| \leqslant \varepsilon\}$ such that $(y_\alpha) \subset Y_{2n_0+1}$ and $W \cap K_{2n_0+1} = \phi$. Hence $s_n \in W$ for all $n \geqslant n_0$ and $s \in W$, which is a contradiction.

It follows that $s \in X$; hence there exists m_0 such that $x \in G_{2m_0+1}$ and $\|s - x\| \leqslant \varepsilon$. By (b) there exists y in Y_{2m_0+1} such that $\|y\| \leqslant 1$ and $\langle y, s_{m_0} - x \rangle \geqslant \|s_{m_0} - x\| (1 + 2^{-2m_0-1})^{-1}$. Note that for $n \geqslant m_0$ we have for such a y:

$$\langle y, s_n - x \rangle = \langle y, s_{m_0} - x \rangle \qquad \text{hence} \qquad \langle y, s_{m_0} - x \rangle = \langle y, s - x \rangle$$

and

$$\|s_{m_0}-x\| \leqslant \|s-x\|(1+2^{-2m_0-1})$$

and

$$\|s_{m_0}-s\| \leqslant 2\varepsilon(1+2^{-2m_0-1}).$$

It follows that (s_n) norm converges to s.

(b) ⇒ (a) by the results of Bourgain and Rosenthal [4].

(a) ⇒ (c) By Lemma II.3, there exists a separable Banach space Y and an isometry $T: X \to Y^*$ such that $T(B_X)$ is a weak*-G_δ in B_{Y^*}. Write $B_{Y^*} \backslash T(B_X) = \bigcup_n K_n$ where the K_n's are weak*-compact. By applying Lemma II.1 to any closed subset F of B_X we have $B_{Y^*} \backslash T(F) = (\bigcup_n K_n) \cup (\bigcup_n D_n)$ where the D_n's are also weak*-compact. Let now S be a dense range operator from l_2 into Y, then $S^*: Y^* \to l_2$ is one-to-one and weak* to weak continuous. Moreover for each closed F in B_X, $S^*(B_{Y^*} \backslash T(F))$ is a weak F_σ in l_2 hence the operator S^*T is a nice G_δ-embedding of X into l_2.

(c) ⇒ (a) Let $R: X \to l_2$ be a nice G_δ-embedding. Suppose first X non-separable. There exists then an uncountable family (x_α) in B_X such that $\|x_\alpha - x_{\alpha'}\| > \delta$ whenever $\alpha \neq \alpha'$. Since the ball of l_2 is a Polish space, there exists a countable subfamily $(x_{\alpha_n})_n$ so that $(x_{\alpha_n})_n$ is dense in itself. But it is also a G_δ since $F = \{x_{\alpha_n}; n \in \mathbb{N}\}$ is closed and R is a G_δ-embedding. This is clearly a contradiction.

To prove that X has (PC) it is enough to take for any closed bounded subset F of X, a point x in F such that Tx is a point of weak to norm continuity for $R^{-1}_{|R(F)}$. It is clearly a point of weak to norm continuity relative to F.

Remark II.1. We may directly construct a nice G_δ-embedding into l_2 using the boundedly complete skipped blocking decomposition (X_n). Indeed, let P_n be a projection from X into X_n, T_n an embedding of X_n into a finite-dimensional Hilbert space H_n and $\varepsilon_n > 0$ such that $\sum_{n=1}^{\infty} \varepsilon_n \|T_n P_n\| < \infty$. Let $T: X \to (\sum_n \oplus H_n)_{l_2}$ defined by $Tx = (\varepsilon_n T_n P_n x)_n$. T is a nice G_δ-embedding. Indeed, suppose (x_α) δ-separated and $(Tx_\alpha)_\alpha$ weakly dense in itself. It follows that for each n, $(P_n x_\alpha)_\alpha$ is norm dense in itself. A reasoning similar to the proof of Theorem (5) of [4] shows that there exists a subsequence $(x_{\alpha_k})_k$ such that $x_{\alpha_{k+1}} - x_{\alpha_k}$ is arbitrarily close to some w_{k+1} in $x_{n_{k+1}+1} \oplus \cdots \oplus X_{n_{k+2}}$, which is obviously a contradiction.

Remark II.2. The proof of Lemma II.3 gives immediately the following "local" result: If C is a closed convex bounded subset of a separable Banach space X, then C has the (PC) property if and only if there exists a separable Banach space Y and an isometry $T: X \to Y^*$ such that $T(C)$ is a w^*-G_δ subset of Y^*.

COROLLARY II.1. *Every Banach space X with property (PC) is somewhat separable dual. Moreover every non-relatively compact sequence in X has a difference subsequence which is a boundedly complete basic sequence.*

Proof. Follows immediately from the above theorem and Proposition (3) of [4].

Remark II.3. Note that Corollary II.1 gives the following result proved by Edgar and Wheeler [8]: A Banach space with property (PC) and a separable dual is somewhat reflexive.

COROLLARY II.2. Every separable Banach space whose dual has a subspace with property (PC) has a separable quotient with a shrinking basic sequence.

Proof. Follows from Theorem II.1 and the results of Johnson and Rosenthal [15].

COROLLARY II.3. *If X is a separable Banach space with the (PC) property, then there exists a compact nice G_δ-embedding $R: X \to l_2$ such that for any Banach space Z containing X as a closed subspace, there exists a compact operator $\tilde{R}: Z \to l_2$ with $\tilde{R}|X = R$.*

Proof. Let Y be the separable Banach space associated to X by Lemma II.3. Let T be the isometry from X into Y^*. Let Q be a quotient map from l_1 onto Y, and let S be a compact, dense range operator from l_2 into l_1. Note that $R = S^*Q^*T$ is a compact nice G_δ-embedding from X into l_2. If now Z is a Banach space containing X, then by a well-known property of l_∞ there exists an operator $N: Z \to l_\infty$ such that $N_{|X} = Q^*T$. It follows that $\tilde{R} = S^*N$ is compact and $\tilde{R}_{|X} = R$.

In general, we cannot expect $\tilde{R}$ to be one-to-one. Indeed we have the following result.

COROLLARY II.4. *For a Banach space X, the following properties are equivalent:*

(a) *X has property (PC) and X^* is separable.*

(b) *There exists a one-to-one operator $R: X^{**} \to l_2$ such that R is weak* to weak continuous and $R_{|X}$ is a nice G_δ-embedding.*

Note that such spaces are the ones whose unit ball is Polish for the weak topology [8].

Proof. (a) $\Rightarrow$ (b) By the results of Edgar and Wheeler [8], the ball of X is a weak*-G_δ in X^{**} hence if S is a dense range operator from l_2 into X^*

(which exists since X^* is separable) then $S^*: X^{**} \to l_2$ is one-to-one and $S^*_{|X}$ is a nice G_δ-embedding by Lemma II.1.

(b) $\Rightarrow$ (a) X has (PC) by Theorem II.1 and since $(R_{|X})^*: l_2 \to X^*$ has a dense range, X^* is weakly compactly generated hence separable since X is [7].

Remark II.4. Note that Corollary II.3 combined with Theorem I.4 gives that a separable Banach space X G_δ-embeds in $l_2 \oplus X/Y$ whenever Y is a subspace of X with property (PC).

COROLLARY II.5. *If X is a Banach space with property (PC), then every operator from L_1 into X is a Dunford–Pettis operator.*

Proof. Let $T: L_1 \to X$. Let Y be the separable subspace of X generated by $T(L_1)$. Let $R: Y \to l_2$ be a G_δ-embedding. By Theorem II.6 of [10], T is a Dunford–Pettis operator since RT obviously is D.P.

The following corollaries solve various questions asked in [5].

COROLLARY II.6. *If X is a separable Banach space with property (PC) but failing the (R.N.P), then there exists a nice G_δ-embedding of X into l_2 which is not the composition of a finite number of semi-embeddings.*

Proof. It follows immediately from Theorem II.1 and the fact proved in [5] that a Banach space belonging to the smallest class $\mathscr{R}$ of spaces stable under semi-embeddings and containing the space l_2 has the (R.N.P.).

Examples of Banach spaces having (PC) but failing (R.N.P.) are:

(1) The space B predual of the James-tree space JT [17].

(2) The subspace BR of the Hagler space JH considered in [4].

The following solves affirmatively question (3) of [5]. We shall give a more precise result in the next section.

COROLLARY II.7. *A separable Banach space with the Radon–Nikodym property G_δ-embeds in l_1.*

Proof. Since spaces with (R.N.P.) have the (P.C.P.), there exists a nice G_δ-embedding S_1 from such a space X into l_2. Consider now any one-to-one operator $S_2: l_2 \to l_1$. It is easy to see that $S_2 S_1$ is a compact nice G_δ-embedding from X into l_1.

Recall from [10] that the class $\mathscr{S}$ (resp. $\mathscr{G}$) is the smallest class of separable Banach spaces stable under semi-embeddings (resp. G_δ-embeddings) and containing the space L^1. The following solves negatively questions in [5] and [10].

COROLLARY II.8. *There exists a* $\mathscr{L}_\infty$*-space BD and a nice* G_δ*-embedding* $S: BD \to l_2$ *such that*:

(i) *S does not fix an infinite-dimensional* $\mathscr{L}_\infty$*-subspace of BD.*

(ii) *BD belongs to the class* $\mathscr{G}$ *but not to the class* $\mathscr{S}$.

Proof. It is enough to take the $\mathscr{L}_\infty$-space with the (R.N.P.) constructed by Bourgain and Delbaen [2]. Note that by Proposition I.5 of [5] such an S is another G_δ-embedding which is not the product of semi-embeddings. For (2) it is enough to notice that l_2 embeds in L_1, hence $BD \in \mathscr{G}$. It is noted in [10] that BD does not belong to $\mathscr{S}$.

LEMMA II.4. *If X nicely* G_δ*-embeds in a Banach space Y having the (PC) property, then X has the (PC) property as well.*

Proof. Let $S: X \to Y$ be such a G_δ-embedding and let F be a closed separable bounded subset of X, then $S(F)$ is a weak G_δ. Since Y has (PC), a result of Edgar and Wheeler [8] shows that $\overline{S(F)}^{\text{weak}}$ is a Baire space for the weak topology, hence $(S(F), \text{weak})$ is a Baire space. It follows that $S^{-1}_{|S(F)}$ has a weak to norm point of continuity Sx.

It is clear that such an x is a point of weak to norm continuity for the set F.

COROLLARY II.9. *The smallest class of Banach spaces stable under nice* G_δ*-embeddings and containing the space* l_2 *is exactly the class of separable Banach spaces with property (PC).*

Remark II.5. Note that the smallest class of spaces stable under F_σ-embeddings and containing the space l_2 is strictly larger than the class of spaces which F_σ-embeds in l_2, since it was noted in [5] that the $\mathscr{L}_1$-spaces with (R.N.P.) constructed by Johnson and Lindenstrauss [14] semi-embed in a separable dual which semi-embed in l_2 while these spaces do not F_σ-embed in l_2 since they do not embed in separable duals.

In [8] it is shown that the dual of the James-tree space JT has property (PC). It follows from the above discussion that every separable subspace of JT^* nicely G_δ-embeds in l_2. However, in this case we can do better and find a nice G_δ-embedding from the whole space JT^* into $l_2 \oplus l_2(\Gamma)$ where Γ is uncountable. This answers negatively a question in [9] since JT^* is a dual space without the (R.N.P.). Note that the range space cannot be taken separable and the G_δ-embedding cannot be weak* to weak continuous since in either case the space $(JT)^*$ would have the (R.N.P.) [9].

To construct the nice G_δ-embedding, recall that there exists a quotient map $Q: JT^* \to l_2(\Gamma)$ (Γ uncountable) such that $(JT)^{**} = JT \oplus Q^*(l_2(\Gamma))$. (For details see [17].) Let now $T: l_2 \to JT$ be a dense range operator which exists since JT is separable. We shall prove the folowing.

PROPOSITION II.1. *The map* $(T^*, Q): JT^* \to l_2 \oplus l_2(\Gamma)$ *is a nice* G_δ-*embedding.*

Proof. Let F be a separable bounded closed subset of $(JT)^*$. Let x be a point of weak to norm continuity in F. We shall show that $(T^*, Q)x$ is a point of weak to norm continuity for $(T^*, Q)^{-1}_{(T^*,Q)(F)}$. Indeed, if $(T^*, Q)x_n$ converges weakly to $(T^*, Q)x$ and $x_n \in F$, (x_n) converges to x $\sigma(JT^*, JT)$ since T^* is weak* to weak continuous and is one-to-one, and $x_n \to x$ $\sigma(JT^*, Q^*(l_2(\Gamma)))$ which implies that $x_n \to x$ weakly hence $x_n \to x$ strongly. Theorem I.3 applies to the separable closed linear space X of F since its range is a separable Hilbert space hence Polish for the weak topology and we get that (T, Q) is a nice G_δ-embedding.

We do not know if an analog of Theorem I.4 holds for nice G_δ-embeddings. On the other hand, J. Bourgain showed that the three-space property holds for Banach spaces having a boundedly complete skipped blocking finite-dimensional decomposition (personal communication). In view of Theorem II.1, this gives the following result which answers positively a question of Edgar and Wheeler [8].

PROPOSITION II.2. *Let X be a Banach space and let Y be a subspace of X such that Y and X/Y are separable and have property (PC) then X has property (PC).*

Remark II.6. If Y is any separable Banach space with property (PC) but without the (R.N.P.), then by a result of Bourgain and Pisier [1], Y embeds in a $\mathscr{L}_\infty$-space X in such a way that X/Y has the (R.N.P.). It follows then by Proposition II.2 that X is a $\mathscr{L}_\infty$-space with property (PC) but failing the (R.N.P.). This pathology does not exist in $\mathscr{L}_1$-spaces since in this case the two properties are equivalent [4].

III. SPACES WITH THE RADON–NIKODYM PROPERTY

Let A be a non-empty bounded subset of a Banach space X. If $x^* \in X^*$, let $M(x^*, A) = \sup x^*(A)$. A slice of A is a set of the form $S(x, \alpha, A) = \{x \in A; x^*(x) \geqslant M(x^*, A) - \alpha\}$ where $x^* \in X^*$ and $\alpha > 0$. We denote by $\mathring{S}(x^*, \alpha, A)$ the set $\{x \in A; x^*(x) > M(x^*, A) - \alpha\}$. We recall that X has (R.N.P.) if every closed bounded convex subset of X has slices of arbitrarily small diameter. We shall call a weak*-open half space of X^{**} a set of the form $H(x^*, \delta) = \{x^{**} \in X^{**}; x^{**}(x^*) > \delta\}$.

LEMMA III.1. *If X is a separable Banach space with the (R.N.P.), then there exists a sequence (γ_n) of countable ordinals and sequences $\{K_{\alpha,n}; \gamma_n, n \in \mathbb{N}\}$, $\{H_{\alpha,n}; \alpha \leqslant \gamma_n, n \in \mathbb{N}\}$ such that:*

(i) *Each* $K_{\alpha,n}$ *is a weak*-compact convex subset of* $B_{X^{**}}$.

(ii) *Each* $H_{\alpha,n}$ *is a weak*-open half space of* X^{**} *such that* $H_{\alpha,n} \cap K_{\alpha,n}$ *is a slice of* $K_{\alpha,n}$.

(iii) $B_X = \bigcap_n \bigcap_{\alpha \leqslant \gamma_n} (K_{\alpha,n} \cup (\bigcup_{\beta<\alpha} H_{\beta,n}))$.

Proof. Given $\varepsilon > 0$, we construct by transfinite induction a decreasing family of norm closed convex subsets (F_α) of B_X in the following way:

(i) $F_0 = B_X$.

(ii) If $\alpha = \beta + 1$ and $F_\beta \neq \phi$, use the (R.N.P.) to find a slice S_β of F_β such that $\operatorname{diam}(S_\beta) < \varepsilon$. Set $F_\alpha = F_\beta \backslash \dot{S}_\beta$. It is norm closed and convex.

(iii) If α is a limit ordinal, let $F_\alpha = \bigcap_{\beta<\alpha} F_\beta$.

Since X is separable, there is $\gamma < \Omega$ (the first uncountable ordinal) such that $F_\gamma = \phi$. Let K_α be the weak*-closure of F_α. It is a weak*-compact convex subset of $B_{X^{**}}$. For each slice $S_\beta = S(x_\beta^*, F_\beta, \delta_\beta) = \{x \in F_\beta ; x_\beta^*(x) \geqslant M(x_\beta^*, F_\beta) - \delta_\beta\}$, let H_β be the weak*-open half space.

$$H(x_\beta^*, M(x_\beta^*, F_\beta) - \delta_\beta) = \{x^{**} \in X^{**}; x^{**}(x_\beta^*) > M(x_\beta^*, F_\beta) - \delta_\beta\}.$$

Note that $H_\beta \cap F_\beta = \dot{S}_\beta$.

Clearly $B_X \subseteq K_\alpha \cup \bigcup_{\beta<\alpha} H_\beta$ for each $\alpha \leqslant \gamma$.

Suppose now $x^{**} \in \bigcap_{\alpha\leqslant\gamma} \{K_\alpha \cup (\bigcup_{\beta<\alpha} H_\beta)\}$. Let α_0 be the first ordinal $\alpha_0 \leqslant \gamma$ such that $x^{**} \notin K_{\alpha_0}$. We have then $x^{**} \in \bigcup_{\beta<\alpha_0} H_\beta$. That is, there is $\beta < \alpha_0$ with $x^{**} \in K_\beta \cap H_\beta$. Let now (x_j) be a net in F_β with $x_j \to x^{**}$ weak*. For large enough j, the x_j's are also in H_β, hence in $F_\beta \cap H_\beta = \dot{S}_\beta$. Take now such an x_j and we have

$$\|x_j - x^{**}\| \leqslant \varliminf_i \|x_j - x_i\| \leqslant \varepsilon \qquad \text{hence } d(x^{**}, B_X) \leqslant \varepsilon.$$

By taking $\varepsilon = 1/n$ in the above construction, we get a sequence of countable ordinals (γ_n), weak*-closed convex sets $\{K_{\alpha,n}; \alpha \leqslant \gamma_n, n \in \mathbb{N}\}$ and weak*-open half spaces $\{H_{\alpha,n};\ \alpha \leqslant \gamma_n, n \in \mathbb{N}\}$ such that $B_X = \bigcap_n \bigcap_{\alpha\leqslant\gamma_n} \{K_{\alpha,n} \cup (\bigcup_{\beta<\alpha_n} H_{\beta,n})\}$.

LEMMA III.2. *If X is a separable Banach space with the (R.N.P.), then there exists a separable Banach space Y and an isometry* $T: X \to Y^*$ *such that* $T(B_X) = \bigcap_n O_n$ *where the complement of each* O_n *is weak*-closed and convex in* Y^*.

Proof. By Lemma III.1 applied to B_X, we write $B_X = \bigcap_n \bigcap_{\alpha\leqslant\gamma_n} K_{\alpha,n} \cup (\bigcup_{\beta<\alpha} H_{\beta,n})$ where $K_{\alpha,n}$ is weak*-compact and convex and $H_{\beta,n}$ is a weak*-open half space of X^{**} of the form $H(x_{\beta,n}^*, \delta_{\beta,n})$.

Let now D be a countable norming subset of X^* containing $\{x^*_{\beta,n}; \beta < \gamma_n$ and $n \in \mathbb{N}\}$. Let Y be the separable closed subspace of X^* generated by D. Consider the inclusion map $S: Y \to X^*$ and $T = S^*: X^{**} \to Y^*$. Then $T_{|X}$ is an isometry.

If $y \in Y$, consider the half space in Y^*, $W(y, \delta) = \{y^* \in Y^*; y^*(y) > \delta\}$. Note that

$$T^{-1}(W(y, \delta)) = \{x^{**} \in X^{**}; Tx^{**}(y) > \delta\} = \{x^{**} \in X^{**}; x^{**}(y) > \delta\}.$$

This shows that each $H_{\beta,n}$ is equal to $T^{-1}W_{\beta,n}$ where $W_{\beta,n}$ is a weak*-open half space in Y. Moreover $T(H_{\beta,n}) = W_{\beta,n}$ since T is onto. The same also holds for unions of open half spaces. That is, since we can write

$$B_X = \bigcap_m (K_m \cup O_m)$$

where K_m is weak*-convex compact and O_m is a countable union of weak*-open half spaces of X^{**}, we have

$$B_X = \bigcap_m K_m \cup T^{-1}(W_m)$$

where W_m is a countable union of weak*-open half spaces in Y^*. The same proof as in Lemma II.3 shows that

$$T(B_X) = \bigcap_m T(K_m) \cup W_m.$$

Since $T(K_m)$ is a weak*-compact convex subset of B_{Y^*}, and B_{Y^*} is weak*-metrizable, $T(K_m) = \bigcap_n V_{m,n}$ where $V_{m,n}$ are weak*-open half spaces of Y^*. Hence

$$T(B_X) = \bigcap_m \bigcap_n (V_{m,n} \cup W_m).$$

In other words

$$T(B_X) = \bigcap_k O_k$$

where O_k is a countable union of weak*-open half spaces of Y^*. Note now that the complements of the O_n's are convex and weak*-closed.

Now we can prove the following:

THEOREM III.1. *For a Banach space X, the following properties are equivalent*:

(a) *X is separable and has the (R.N.P.).*

(b) *There exists an H_δ-embedding of X into l_2.*

Proof. (a) $\Rightarrow$ (b) By Lemma III.2, there exists a separable Banach space Y and an isometry $T: X \to Y^*$ such that $T(B_X) = \bigcap_n O_n$ with the complements of the O_n's being weak*-compact and convex. Let $\widetilde{T(B_X)}$ be the weak*-closure of $T(B_X)$ in Y^*. Note that

$$\widetilde{T(B_X)} \backslash T(B_X) = \bigcup_n (\widetilde{T(B_X)} \cap O_n^c) \qquad \text{and} \qquad \widetilde{T(B_X)} \cap O_n^c$$

is convex and weak*-compact in Y^*.

Let now S be a dense range operator from l_2 into Y then $S^*: Y^* \to l_2$ is one-to-one. The same proof as in Theorem II.1 shows that $R = S^*T$ is a nice G_δ-embedding. On the other hand $\overline{R(B_X)} = S^*(\widetilde{T(B_X)})$, hence

$$\overline{R(B_X)} \backslash R(B_X) = S^*(\widetilde{T(B_X)} \backslash T(B_X)) = \bigcup_n S^*(\widetilde{T(B_X)} \cap O_n^c)$$

where each $K_n = S^*(\widetilde{T(B_X)} \cap O_n^c)$ is convex and weakly-compact.

(b) $\Rightarrow$ (a) Let (Ω, Σ, μ) be a probability space and let $F: \Sigma \to B_X$ be a vector measure with $\|F(E)\| \leqslant \mu(E)$ for all $E \in \Sigma$. The vector measure $R \circ F(E)$ is valued in l_2, hence there exists a Bochner integrable function $\Phi: \Omega \to \overline{R(B_X)}$ such that

$$R \circ F(E) = \int_E \Phi(t)\, d\mu(t) \qquad \text{for each } E \in \Sigma.$$

We shall prove that Φ has almost all its values in $R(B_X)$. Indeed, suppose not and write $\overline{R(B_X)} \backslash R(B_X) = \bigcup_n K_n$ where the K_n's are convex and weakly compact.

For each n, the sets $D_n = \Phi^{-1}(K_n)$ belong to Σ, and if $\mu(D_n) > 0$, $F(D_n)/\mu(D_n)$ is in B_X, hence $R \circ F(D_n)/\mu(D_n)$ is in $R(B_X)$ but not in K_n. On the other hand, $R \circ F(D_n)/\mu(D_n) = (1/\mu(D_n)) \int_{D_n} \Phi(t)\, d\mu(t)$ which belongs to K_n since the latter is closed and convex.

It follows that $\mu(D_n) = 0$ for each n, so $\phi(t) \in R(B_x)$ for almost all t. A theorem of Lusin guarantees then that $R^{-1}\phi$ is measurable and is a Bochner derivative in X for F. Hence X has the (R.N.P.).

The proof of (b) $\Rightarrow$ (a) is essentially the same as the one used by Edgar and Wheeler [8] to show that a Banach space X has the Radon–Nikodym property whenever $X^{**} \backslash X = \bigcup_n K_n$ with each K_n being w^*-compact and convex. Note that if X^* is separable then the space Y considered in Lemma

III.2 can be taken to be the dual of X. We get then the following converse of the result of Edgar and Wheeler [8].

COROLLARY III.1. *Let X be a separable Banach space then the following properties are equivalent*:

(1) *X has the (R.N.P.) and X^* is separable.*

(2) *$X^{**}\backslash X = \bigcup_n K_n$ where each K_n is w^*-compact and convex.*

Remark III.1. The proof of Lemma III.2 gives immediately the following local result: If C is a closed convex bounded subset of a separable Banach space X, then C has the (R.N.P.) if and only if there exists a separable Banach space Y and an isometry $T: X \to Y^*$ such that $T(C)$ is a w^*-H_δ: that is, $Y^*\backslash T(C) = \bigcup_n K_n$ where each K_n is w^*-compact and convex.

Remark III.2. Note that if D is a closed convex bounded w^*-H_δ set in Y^*, then the same proof as in (b) $\Rightarrow$ (a) implies, without any assumption of separability on D, that every D-valued vector measure has a w^*-measurable derivative valued almost everywhere in D. Moreover, if one considers the image $T^*(D)$ in l_2 then it is L_1-convex in the sense of Rosenthal [20] without being necessarily closed. The above proof gives, however, that bounded, L_1-convex, H_δ-subsets of l_2 have the Radon–Nikodym property as defined in [20] for non-necessarily closed sets.

IV. G_δ-EMBEDDINGS IN l_2

In this section we shall investigate the relation between G_δ and nice G_δ-embeddings.

LEMMA IV.1. *Let X be a Banach space such that none of its subspaces is isomorphic to l_2. If T is a G_δ-embedding from X into l_2 then there exists an infinite-dimensional closed subspace Y of X such that $T_{|Y}$ is a nice G_δ-embedding.*

Proof. Since T is not an isomorphism on any subspace of X, it is standard to show the existence of a basic sequence (e_n) in X such that $\lim_n \|Te_n\| = 0$ (Lemma I.a. 6 of [18]). From which follows that T restricted to the closed linear span Y of (e_n) is a compact G_δ-embedding, hence a nice G_δ-embedding.

THEOREM IV.1. *Every Banach space X that G_δ-embeds in l_2 is somewhat separable dual.*

Proof. Let Y be any subspace of X. Either l_2 embeds in Y or there exists a subspace Z of Y which has property (PC) by Lemma IV.1. Hence Theorem II.1 applies to give a separable dual isomorphic to a subspace of Z.

We now show that for a large class of Banach spaces the notions of G_δ embeddings and nice G_δ-embeddings are equivalent.

The key idea is the following result due to H. P. Rosenthal [19]. We sketch a proof for completeness.

PROPOSITION IV.1. *Let X be a Banach space such that every closed convex bounded subset of X with the (PC) property has the (R.N.P.). Let S be a G_δ-embedding of X into a Banach space Y. Then an operator T from L_1 into X is representable if and only if ST is representable.*

Proof. Suppose T is a non-representable operator from L_1 into X such that ST is representable. Then there exists a closed convex subset A of the unit ball of L_1 such that $\overline{T(A)}$ fails the (R.N.P.) while $ST(A)$ is relatively norm compact. This implies that $S_{|\overline{T(A)}}$ is a nice G_δ-embedding and that $\overline{T(A)}$ has the (PC) property hence the (R.N.P.), which is a contradiction.

Recently J. Diestel proved that subspaces of weakly sequentially complete Banach lattices verify the hypothesis of the above proposition. The case of L_1 was observed by Bourgain and Rosenthal [3]. In [12], we give proofs of these results using the methods introduced in this paper.

COROLLARY IV.1 (a) *If X is a subspace of a weakly sequentially complete Banach lattice, and X G_δ-embeds in l_2, then X has the Radon–Nikodym property, hence it H_δ-embeds in l_2.*

(b) *If X is a Banach lattice that G_δ-embeds in l_2, then X is isometric to a dual and separable Banach lattice, hence it F_σ-embeds in l_2.*

Proof. It follows immediately from the above discussion and the recent result of Talagrand stating that separable Banach lattices with the (R.N.P.) are dual Banach lattices [21].

Remark IV.1. The above discussion shows, for instance, that the subspaces of L_1 with the strong-Schur property constructed by Bourgain and Rosenthal [3] do not G_δ-embed in l_2. Moreover this shows that the Banach lattice MT constructed by Talagrand [22] does not G_δ-embed in l_2 even though every operator from L_1 into MT is a Dunford–Pettis operator. Note that in view of the results in [10], this property is a necessary condition for a space that G_δ-embeds in l_2.

The following example shows, however, that the two notions are not equivalent:

EXAMPLE IV.1. There exists a Banach space B_∞ which G_δ-embeds in l_2

but fails property (PC) hence no operator from B_∞ to l_2 is a nice G_δ-embedding.

Proof. We assume the reader is familiar with the construction of the James-tree space JT and its predual B as analyzed in Lindenstrauss and Stegall [17]. In [11] we showed that the space B nicely G_δ-embeds in l_2. This was mostly due to the fact that in such a space one considers a tree with finitely many branching points: that is, a tree T_1 so that for each $t \in T_1$, the set of immediate successors of t in T_1 is finite, its cardinality may depend on t but it is always larger or equal to 2. To construct our counterexample, we shall use a tree T_∞ with infinitely many branching points.

For that consider the tree $T_\infty = \bigcup_{k=0}^{\infty} \mathbb{N}^k$. If $t = (n_1, n_2, ..., n_k) \in T_\infty$, set $|t| = k$ and for $j \leqslant k$ set $t|j = (n_1, n_2, ..., n_j)$. Define the partial order on T_∞ by $s \leqslant t$ if $|s| \leqslant |t|$ and $s = t|\,|s|$. For each element $(n_k) \in \mathbb{N}^{\mathbb{N}}$, we associate the branch $\gamma = \{\phi, (n_1), (n_1, n_2), ..., (n_1, n_2, ..., n_k), ...\} \subset T_\infty$. Set $\gamma|k = (n_1, n_2, ..., n_k) \in T_\infty$.

Define now on the space of real valued, finitely supported functions on T_∞ the norm

$$\|x\| = \sup \left(\sum_{l=1}^{n} \left(\sum_{t \in S_l} x_t \right)^2 \right)^{1/2},$$

the supremum being taken over all families $(S_1, S_2, ..., S_n)$ of disjoint segments in T_∞. Let JT_∞ be the completion of such a space. Let $(e_t)_{t \in T_\infty}$ be the canonical basis; let $(e_t^*)_{t \in T_\infty}$ be the biorthogonal functionals. Denote by B_∞ the closed subspace of JT_∞^* generated by the family $(e_t^*)_{t \in T_\infty}$.

We shall say that $A \subset T_\infty$ is full if $S \cap A$ is a segment of T_∞ for each segment S of T_∞. Note that if Π_A denotes the natural projection on $[e_t]_{t \in A}$, then $\|\Pi_A\| = 1$.

Moreover, for each $t \in T_\infty$, we shall set $A_t = \{s \in T_\infty ; s \geqslant t\}$ and $\Pi_t = \Pi_{A_t}$. Note that A_t is then full and $\|\Pi_t\| = 1$.

Let now $L_k = \{t \in T_\infty ; |t| = k\}$ and $\Pi_k = \sum_{t \in L_k} \Pi_t$, we get that $\|\Pi_k\| = 1$ and $\|\sum_{t \in L_k} x_t\|^2 = \sum_{t \in L_k} \|x_t\|^2$ for each family $(x_t)_{t \in L_k}$ in JT_∞ such that $\Pi_t x_t = x_t$ for each $t \in L_k$. By duality we get that $\|\sum_{t \in L_k} x_t^*\|^2 = \sum_{t \in L_k} \|x_t^*\|^2$ whenever $\Pi_t^* x_t^* = x_t^*$ $\forall t \in L_k$.

We shall prove the following:

THEOREM IV.2. $B_\infty = \{x^* \in JT_\infty^* ; \lim_k \inf \|\Pi_{L_k}^* x^*\| = 0\}$.

For that we need the following lemma.

LEMMA IV.2. *For every x^* in JT_∞^* and every $\varepsilon > 0$, there exists a full subtree $T_1 \subset T_\infty$ with a finite number of branching points such that $\|x^* - \Pi_{T_1}^* x^*\| \leqslant \varepsilon$.*

Proof. Let $(\varepsilon_t)_{t\in T_\infty}$ be a family of positive real numbers such that $\sum_{t\in T_\infty} \varepsilon_t \leqslant \varepsilon$. Let $t \in T_\infty$ and $S_t = \{s \in T_\infty ; s \geqslant t \text{ and } |s| = |t| + 1\}$. We have:

$$\left\| \sum_{s\in S_t} \Pi_s^* x^* \right\|^2 = \sum_{s\in S_t} \|\Pi_s^* x^*\|^2.$$

Let S_t^1 be a finite subset of S_t such that

$$\sum_{s\in S_t\setminus S_t^1} \|\Pi_s^* x^*\|^2 = \left\| \sum_{s\in S_t\setminus S_t^1} \Pi_s^* x^* \right\|^2 \leqslant \varepsilon_t^2.$$

The construction of T_1 is now clear: for each t, we keep only its successors which are in S_t^1 and we use the same procedure again on each element of S_t^1. Note that the total of the terms eliminated in x^* will have a norm less than $\sum_{t\in T_\infty} \varepsilon_t \leqslant \varepsilon$. The details are left to the interested reader.

Before proving the theorem we shall denote by JT the James-tree space modelled on the tree T whenever T has a finite number of branching points. Note that the usual James-tree space is modelled on the diadic tree but that all the estimates proved in [14] extend trivially to the non-diadic case, and we shall use them freely in the following.

Proof of Theorem IV.2. Let $x^* \in (JT_\infty)^*$ such that $d(x^*, B_\infty) = \delta > 0$. We may find a tree T_1 with a finite number of branching points such that $\|x^* - \Pi_{T_1}^* x^*\| \leqslant \delta/2$. We may consider $x_1^* = \Pi_{T_1}^* x^*$ as an element of JT_1^*. Note that $d(x_1^*, B_1) \geqslant \delta/2$ where $B_1 = [e_t^* ; t \in T_1]$. By applying the results of Stegall and Lindenstrauss [17] to the space JT_1, we can find a branch γ in T_1 (which is also a branch in T_∞) such that $\lim_k x_1^*(e_{\gamma|k}) = \lim_k x^*(e_{\gamma|k}) \neq 0$. It follows that $B_\infty = \{x^* \in JT_\infty^* ; \lim_k \inf \|\Pi_{L_k}^* x^*\| = 0\}$.

COROLLARY IV.2. *Let U be the operator from $l_2(T_\infty)$ into JT_∞ defined by $Ue_t = 2^{-|t|} e_t$ for all $t \in T_\infty$. Then the restriction of U^* to B_∞ is a G_δ-embedding into $l_2(T_\infty)$.*

Proof. Note that U has a dense range, hence U^* is one-to-one. Moreover the ranges of Π_{L_k} and $\Pi_{L_k}^*$ are isometric to l_2. We shall use the same notations for the corresponding projections in $l_2(T_\infty)$. Note now that $G_n = \{y \in U^*(\text{Ball}(JT_\infty^*)); \|\Pi_{L_k}^* y\| \geqslant 2^{-k}/n,\ k \geqslant n\}$ is norm closed and $U^*(\text{Ball}(JT_\infty^*)) \setminus U^*(\text{Ball}(B_\infty)) = \bigcup_n G_n$. Moreover, we get from Lemma II.1 that for any closed subset F of $\text{Ball}(B_\infty)$, we have

$$\text{Ball}(JT_\infty^*) \setminus F = [\text{Ball}(JT_\infty^*) \setminus \text{Ball}(B_\infty)] \cup \bigcup_n K_n$$

where the K_n's are weak*-compact in JT_∞^*. It follows that

$U^*(\mathrm{Ball}(JT_\infty^*))\backslash U^*(F) = (\bigcup_n G_n) \cup (\bigcup_n T^*(K_n))$ is an F_σ since U^* is weak* to weak continuous on JT_∞^*.

PROPOSITION IV.2. *B_∞ fails the (PC) property hence no operator from B_∞ into l_2 is a nice G_δ-embedding.*

Proof. Note that for each $t \in T_\infty$, weak $\mathrm{limit}_{s \in S_t} e_s^* = 0$ since $(e_s^*)_{s \in S_t}$ is isometric to the unit vector basis of l_2. It follows that the set $A = \{e_{t|0}^* + e_{t|1}^* + \cdots + e_{t|k}^*;\ k \in \mathbb{N}, t \in T_\infty \text{ and } |t| \geqslant k\}$ is weakly dense in itself, is contained in $\mathrm{Ball}(B_\infty)$ and doesn't have any point of weak to norm continuity.

Note added in proof. The sequence of compact sets (K_m) that appears in the proofs of Lemmas II.3 and III.2 is not necessarily decreasing hence the statement that $T(\bigcap_m K_m) = \bigcap_m T(K_m)$ is not correct. However, by using the notations of Lemmas II.2 and III.1, we get that for each $\varepsilon > 0$, the sequence $(K_{\alpha,\varepsilon})_\alpha$ is decreasing hence

$$T\Big(\bigcap_{\alpha \leqslant \gamma_\varepsilon} \Big(K_{\alpha,\varepsilon} \cup \Big(\bigcup_{\beta<\alpha} V_{\beta,\varepsilon}\Big)\Big)\Big) = \bigcap_{\alpha \leqslant \gamma_\varepsilon} \Big(T(K_{\alpha,\varepsilon}) \cup \Big(\bigcup_{\beta<\alpha} T(V_{\beta,\varepsilon})\Big)\Big).$$

Since now $B_X \subseteq \bigcap_{\alpha \leqslant \gamma_\varepsilon} (K_{\alpha,\varepsilon} \cup (\bigcup_{\beta<\alpha} V_{\beta,\varepsilon})) \subseteq B_X + \varepsilon B_{X^{**}}$ and since T is a contraction we get the results claimed in Lemmas II.3 and III.2.

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