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Stochastic Processes and their Applications 116 (2006) 370–380
stochastic processes and their applications
www.elsevier.com/locate/spa

Necessary and sufficient condition for comparison theorem of 1-dimensional stochastic differential equations

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Received 12 April 2004; accepted 24 August 2005
Available online 20 September 2005

Abstract

In this paper, we present a new approach to obtain the comparison theorem of two 1-dimensional SDEs with diffusion and jumps. The two equations is treated as one two-dimensional SDE and the comparison requirement is regarded as to keep the solution (X_t^1, X_t^2) within the constraint $K = \{(x_1, x_2);\ x_1 \leqslant x_2\}$. We then apply a new criteria of "viability condition" which is a necessary and sufficient condition to keep the solution to be inside the constraint K.

Keywords: Comparison theorem of SDE; Viability; Viscosity solutions

1. Introduction

We compare the following two one-dimensional stochastic differential equations (SDEs in short) driven by a Brownian motion $(W_t)_{t\geqslant 0}$ and a Poisson process $(N_t)_{t\geqslant 0}$

$$X_s^1 = x_1 + \int_t^s b_1(r, X_r^1)\,\mathrm{d}r + \int_t^s \sigma_1(r, X_r^1)\,\mathrm{d}W_r + \int_t^s \int_Z \gamma_1(r, X_{r-}^1, z)\tilde{N}(\mathrm{d}z\,\mathrm{d}r),$$

$$X_s^2 = x_2 + \int_t^s b_2(r, X_r^2)\,\mathrm{d}r + \int_t^s \sigma_2(r, X_r^2)\,\mathrm{d}W_r + \int_t^s \int_Z \gamma_2(r, X_{r-}^2, z)\tilde{N}(\mathrm{d}z\,\mathrm{d}r),$$

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0304-4149/$ - see front matter
doi:10.1016/j.spa.2005.08.004

where x_1 and x_2 are initial state of these two SDE with initial time $t \in [0, T]$. We are interested in the following problem: to find a necessary and sufficient condition of the coefficients b_i, σ_i, γ_i, $i = 1, 2$, that ensures

$$x_1 \leqslant x_2 \Rightarrow X_s^1 \leqslant X_s^2, \quad \forall s \geqslant t, \ \ P\text{-a.s.}, \ \ \forall t < \infty.$$

Anderson [1], Ikeda and Watanabe [7], Skorokhod [11] and Yamada [12] gave comparison theorems for the solutions of two Itô's stochastic differential equations with the same diffusion coefficients. Yan [13] gave some conclusion about equations driven by general continuous local martingale, continuous increasing process and general increasing process but still based on the same diffusion coefficients. O'Brien [9] studied a comparison theorem for solutions of Itô's equations with different diffusion terms. See also Gal'cuk and Davis [5], X. Mao [8]. Of all those results only sufficient conditions were proved.

In this paper, we propose a new approach to treat this problem. We consider the above two equations as a two-dimensional SDE. In this point of view the above comparison requirement is regarded as a constraint $(X_s^1, X_s^2) \in K = \{(x_1, x_2); x_1 \leqslant x_2\}$. We then apply a new criteria of "viability condition" which is a necessary and sufficient condition to keep the solution to be inside the constraint K. We thus obtain a necessary and sufficient condition (see (15)) of the comparison theorem. Up to our knowledge, this result is new even in the case without jumps, i.e., $\gamma_1 \equiv \gamma_2 \equiv 0$.

In Section 2 the new criteria will be given. Then in Section 3 we will get the sufficient and necessary conditions of the comparison theorem.

This approach can also be applied to multi-dimensional situation (see [6]).

2. A criteria of SDE under state constraint

Let $(\Omega, \mathscr{F}, P)$ be a complete probability space in which two mutually independent processes are defined: $(W_t)_{t\geqslant 0}$ a standard d-dimensional Brownian motion and N a Poisson random measure on $(0, +\infty) \times (Z\backslash\{0\})$, where $Z \subset \mathbf{R}^k$ is equipped with its Borel field $\mathscr{B}_Z$, with the Lévy compensator $\hat{N}(\mathrm{d}t\,\mathrm{d}z) = \mathrm{d}t\, n(\mathrm{d}z)$, i.e., $\{\tilde{N}((0,t] \times A) = (N - \hat{N})((0,t] \times A)\}_{t>0}$ is a $\mathscr{F}_t$-martingale for each $A \in \mathscr{B}_Z$. Here $n(\mathrm{d}z)$ is a positive σ-finite measure satisfying

$$\int_Z n(\mathrm{d}z) < \infty.$$

Let $(\mathscr{F}_t)_{t\geqslant 0}$ be the filtration generated by the above two process and augmented by the P-null sets of $\mathscr{F}$.

We consider the following SDEs with jumps starting from a point $x \in R^n$ at a time $t \geqslant 0$:

$$X_s^{t,x} = x + \int_t^s b(r, X_r^{t,x})\,\mathrm{d}r + \int_t^s \sigma(r, X_r^{t,x})\,\mathrm{d}W_r + \int_t^s \int_Z \gamma(r, X_{r-}^{t,x}, z)\tilde{N}(\mathrm{d}z\,\mathrm{d}r), \tag{1}$$

where b, σ and γ are given continuous coefficients of (t, x):

$$b : [0, \infty) \times R^n \to R^n, \quad \sigma : [0, \infty) \times R^n \to R^n \times d,$$

$$\gamma : [0, \infty) \times R^n \times R^k \to R.$$

We assume that, there exists a sufficiently large constant $\mu > 0$ and a function $\rho : R^k \to R_+$ with

$$\int_Z \rho^2(z) n(\mathrm{d}z) < \infty,$$

such that

(H1) for all $x, x' \in R^n$, $t \in [0, +\infty)$

$$\langle b(t,x) - b(t,x'), x - x' \rangle \leqslant \mu |x - x'|^2,$$

$$|\sigma(t,x) - \sigma(t,x')| \leqslant \mu |x - x'|,$$

$$|b(t,x)| + |\sigma(t,x)| \leqslant \mu(1 + |x|),$$

$$|\gamma(t,x,z) - \gamma(t,x',z)| \leqslant \rho(z)|x - x'|,$$

$$|\gamma(t,x,z)| \leqslant \rho(z)(1 + |x|).$$

Here $\langle \cdot, \cdot \rangle$ and $|\cdot|$ denote, respectively, the Euclidian scalar product and norm. Obviously under the above assumptions there exists a unique strong solution to (1).

Let K be a given closed subset of R^n. We are interested in the following property for SDE (18) in a fixed time interval $[0, T]$:

$$\text{For each } (t,x) \in [0,T) \times K, \quad X_s^{t,x} \in K, \quad \forall s \in [0,T], \text{ a.s.} \tag{2}$$

We will find a necessary and sufficient condition of the coefficients (b, σ, γ) that ensues (2).This corresponds the so-called "viability property" in deterministic control theory.

To this end, we define the following real valued function u:

$$u(t,x) := \mathbf{E}\left[\int_t^T \mathrm{e}^{-C(s-t)} d_K^2(X_s^{t,x})\,\mathrm{d}s + \mathrm{e}^{-C(T-t)} d_K^2(X_T^{t,x})\right], \quad (t,x) \in [0,T] \times \mathbf{R}^n, \tag{3}$$

where $d_K(x)$, $x \in R^n$, denotes the distance function of K:

$$d_K(x) = \inf\{|x - x'| : x' \in K\}.$$

It is a Lipschitz function. In fact we have $|d_K(x) - d_K(x)| \leqslant |x - x'|$, $\forall x,\ x' \in R^n$. Here the constant C is

$$C = 1 + 2\mu + \mu^2 + \int_Z \rho^2(z) n(\mathrm{d}z). \tag{4}$$

It is easy to check that u is continuous in $[0, T] \times R^n$ with quadratic growth in x. Property (2) is equivalent

$$u(t,x) \equiv 0, \quad \forall (t,x) \in [0,T] \times K. \tag{5}$$

It is also well-known that u is the viscosity solution (see [4,2,10,14]) of the following linear parabolic PDE:

$$\begin{cases} \mathscr{L}u(t,x) + \mathscr{B}u(t,x) - Cu(t,x) + d_K^2(x) = 0, & (t,x) \in (0,T) \times R^n, \\ u(T,x) = d_K^2(x), \end{cases} \tag{6}$$

where we denote, for $\varphi \in C^{1,2}([0,T] \times R^n)$,

$$\mathscr{L}\varphi(t,x) = \frac{\partial \varphi(t,x)}{\partial t} + \langle D\varphi(t,x), b(t,x)\rangle + \frac{1}{2}\mathrm{tr}[D^2\varphi(t,x)\sigma\sigma^{\mathrm{T}}(t,x)]$$

and

$$\mathscr{B}\varphi(t,x) := \int_Z [\varphi(t, x+\gamma(t,x,z)) - \varphi(t,x) - \langle D\varphi(t,x), \gamma(t,x,z)\rangle] n(\mathrm{d}z).$$

But it is still not easy to check (5) from (6).

The main idea, introduced in [3] for the situation without jumps, is that the condition $u(t,x) \equiv 0, \forall x \in K$ holds if and only if $d_K^2(x)$ is a viscosity supersolution of the PDE (6). By this we will obtain our necessary and sufficient conditions for the comparison theorem in the next section.

We now give the definition of viscosity solutions for PDE (6). We denote by $\mathrm{UC}_{x,2}([0,T] \times R^n)$ the set of continuous functions in $[0,T] \times R^n$ uniformly continuous in x, uniformly in t, with at most quadratic growth in x.

Definition 2.1. A function $u \in UC_{x,2}([0,T] \times R^n)$ is called a viscosity supersolution (resp., subsolution) of (6) if $u(T,x) \geqslant d_K^2(x)$ (resp., $u(T,x) \leqslant d_K^2(x)$) and for any $\varphi \in C^{1,2}([0,T] \times R^2)$ such that φ is at most quadratic growth in x and at any point $(t,x) \in [0,T] \times \mathbf{R}^2$ at which $u - \varphi$ attains its minimum (resp., maximum),

$$\frac{\partial \varphi}{\partial t} + \mathscr{L}\varphi(t,x) + \mathscr{B}\varphi(t,x) - C\varphi(t,x) + d_K^2(x) \leqslant 0, \quad (\text{resp.}, \geqslant 0). \tag{7}$$

u is called a viscosity solution if it is both viscosity supersolution and subsolution.

It is interesting that the following comparison theorem of PDE, often called "maximum principle" relates closely the comparison theorem in SDE.

Proposition 2.1 (*Comparison Theorem of Integral-PDE*). *We assume* (H1). *Let* $u \in UC_{x,2}([0,T] \times R^n)$ (*resp.*, $v \in UC_{x,2}([0,T] \times R^n)$) *be a viscosity subsolution* (*resp.*, *supersolution*) *of PDE* (6). *Then we have*

$$v(t,x) \geqslant u(t,x), \quad \forall (t,x) \in [0,T] \times R^n.$$

Remark 2.1. This result is mainly due to [10] with a slight modification: the function u, v and the coefficient $d_K^2(\cdot)$ are not linear growth function in x. They are in fact quadratic growth. The proof is also analogous to that of [10], with the following modification: the well-known penalization function is

$$\Phi(t,s,x,y) = u(t,x) - v(s,y) - \frac{\beta}{t} - \frac{1}{2\varepsilon}|x-y|^2 - \delta \mathrm{e}^{\lambda(T-t)}(|x|^4 + |y|^4).$$

In (4.2) of [10] the last term is $-\delta \mathrm{e}^{\lambda(T-t)}(|x|^2 + |y|^2)$. A more general situation was treated in [14].

Proposition 2.2. *We assume* (H1). *Then the following claims are equivalent*:

(i) d_K^2 *is a viscosity supersolution of PDE* (6);
(ii) *The* "*viability property*" (2) *holds*.

Proof. (i) $\Rightarrow$ (ii): Since d_K^2 is a viscosity supersolution. By the above comparison theorem of integral PDE, $d_K^2(x) \geqslant u(t,x)$, $\forall (t,x)$. Since u is nonnegative and $d_K(x)=0$, $\forall x \in K$, thus $u(t,x) \equiv 0$, $\forall x \in K$. Thus (2) holds.

(ii) $\Rightarrow$ (i): For each $(t,x) \in [0,T] \times R^n$, let $\bar{x} \in K$ be such that $d_K(x) = |x-\bar{x}|$ (if $x \in K$ then $\bar{x}=x$). From (ii) we have $X_s^{t,\bar{x}} \in K$, for each $s \in [t,T]$, a.s.. Let $\varphi \in C^{1,2}$ be such that

$$d_K^2(x) - \varphi(t,x) = 0 \leqslant d_K^2(x') - \varphi(t,x'), \quad \forall (t',x') \in [0,T] \times R^n. \tag{8}$$

For each $\varepsilon > 0$, we define the following stopping time:

$$\tau_\varepsilon := \varepsilon \wedge \inf\{s \geqslant t,\ |X_s^{t,x} - x| > \varepsilon\} \wedge \inf\{s \geqslant t,\ |X_s^{t,\bar{x}} - x| > \varepsilon\}.$$

By (8) we have

$$\varphi(\tau_\varepsilon, X_{\tau_\varepsilon}^{t,x}) - \varphi(t,x) \leqslant d_K^2(X_{\tau_\varepsilon}^{t,x}) - d_K^2(x). \tag{9}$$

We apply Itô's formula to $\varphi(\tau_\varepsilon, X_{\tau_\varepsilon}^{t,x})$,

$$\begin{aligned} E[\varphi(\tau_\varepsilon, X_{\tau_\varepsilon}^{t,x}) - \varphi(t,x)] &= E\int_t^{\tau_\varepsilon} [\mathscr{L}\varphi(s, X_s^{t,x}) + \mathscr{B}\varphi(s, X_s^{t,x})]\,\mathrm{d}s \\ &= [\mathscr{L}\varphi(t,x) + \mathscr{B}\varphi(t,x)]E[\tau_\varepsilon - t] + r(\varepsilon) \\ &\leqslant d_K^2(X_{\tau_\varepsilon}^{t,x}) - d_K^2(x), \end{aligned} \tag{10}$$

where $\lim_{\varepsilon \to 0} r(\varepsilon)/E[\tau_\varepsilon - t] = 0$.

Since $X_s^{t,\bar{x}} \in K$, thus $d_K^2(X_{\tau_\varepsilon}^{t,x}) \leqslant |X_{\tau_\varepsilon}^{t,x} - X_{\tau_\varepsilon}^{t,\bar{x}}|^2$. Thus the right-hand-side of (10) is dominated by

$$\begin{aligned} &E[|X_{\tau_\varepsilon}^{t,x} - X_{\tau_\varepsilon}^{t,\bar{x}}|^2] - |x-\bar{x}|^2 \\ &\quad = E\int_t^{\tau_\varepsilon} [2\langle X_s^{t,x} - X_s^{t,\bar{x}}, b(s,X_s^{t,x}) - b(s,X_s^{t,\bar{x}})\rangle \\ &\qquad + |\sigma(s,X_s^{t,x}) - \sigma(s,X_s^{t,\bar{x}})|^2 + \int_Z |\gamma(s,X_s^{t,x},z) - \gamma(s,X_s^{t,\bar{x}},z)|^2 n(\mathrm{d}z)]\,\mathrm{d}s \\ &\quad \leqslant (C-1)E\int_t^{\tau_\varepsilon} |X_s^{t,x} - X_s^{t,\bar{x}}|^2\,\mathrm{d}s, \end{aligned} \tag{11}$$

where $C = 1 + 2\mu + \mu^2 + \int_Z \rho^2(z)n(\mathrm{d}z)$. Since

$$\begin{aligned} E\int_t^{\tau_\varepsilon} |X_s^{t,x} - X_s^{t,\bar{x}}|^2\,\mathrm{d}s &\leqslant (1+1/\varepsilon)E\int_t^{\tau_\varepsilon} |X_s^{t,x} - X_s^{t,\bar{x}} - x - \bar{x}|^2\,\mathrm{d}s \\ &\quad + (1+\varepsilon)E[\tau_\varepsilon - t]|x-\bar{x}|^2 \\ &\leqslant (1+1/\varepsilon)4\varepsilon^2 E[\tau_\varepsilon - t] + (1+\varepsilon)E[\tau_\varepsilon - t]|x-\bar{x}|^2. \end{aligned}$$

This with (10) and (11) implies that

$$\begin{aligned} &[\mathscr{L}\varphi(t,x) + \mathscr{B}\varphi(t,x)]E[\tau_\varepsilon - t] + r(\varepsilon) \\ &\quad \leqslant E[\tau_\varepsilon - t](C-1)[(1+1/\varepsilon)4\varepsilon^2 + (1+\varepsilon)|x-\bar{x}|^2]. \end{aligned}$$

Dividing by $E[\tau_\varepsilon - t]$ and letting $\varepsilon \to 0$, the limit is

$$\mathscr{L}\varphi(t,x) + \mathscr{B}\varphi(t,x) + d_K^2(x) - C\varphi(t,x) \leqslant 0.$$

Thus $d_K(x)$ is a supersolution of (6). The proof is complete. $\square$

3. Comparison theorem: a necessary and sufficient condition

We now apply Proposition 2.2 to our comparison theorem of SDE. Consider the following two one-dimensional SDEs with jumps, defined on $[t,\infty)$:

$$X_s^1 = x_1 + \int_t^s b_1(r, X_r^1)\,\mathrm{d}r + \int_t^s \sigma_1(r, X_r^1)\,\mathrm{d}W_r + \int_t^s \int_Z \gamma_1(r, X_{r-}^1, z)\tilde{N}(\mathrm{d}z\,\mathrm{d}r), \tag{12}$$

$$X_s^2 = x_2 + \int_t^s b_2(r, X_r^2)\,\mathrm{d}r + \int_t^s \sigma_2(r, X_r^2)\,\mathrm{d}W_r + \int_t^s \int_Z \gamma_2(r, X_{r-}^2, z)\tilde{N}(\mathrm{d}z\,\mathrm{d}r), \tag{13}$$

where x_1 and x_2 are the initial conditions of (12) and (13), respectively. We assume that, for $i = 1, 2$, that

(H3.1) b_i, σ_i, γ_i are continuous in (t, x);

(H3.2) for each $x, x' \in R$ and $t \geqslant 0$

$$(x - x')(b_i(t,x) - b_i(t,x')) \leqslant \mu |x - x'|^2,$$

$$|\sigma_i(t,x) - \sigma_i(t,x')| \leqslant \mu |x - x'|,$$

$$|\gamma_i(t,x,z) - \gamma_i(t,x',z)| \leqslant \rho(z)|x - x'|,$$

$$|b_i(t,x)| + |\sigma_i(t,x)| \leqslant \mu(1 + |x|),$$

$$|\gamma_i(t,x,z)| \leqslant \rho(z)(1 + |x|),$$

where μ and $\rho(\cdot)$ are given in Section 2.

The main objective of this paper is to find a necessary and sufficient condition of the above coefficients that ensures

$$x_2 \geqslant x_1 \quad \Rightarrow \quad X_s^2 \geqslant X_s^2, \quad \forall s \in [t, T], \quad P\text{-a.s..}, \quad \forall t \leqslant T. \tag{14}$$

We now assert the main result of this paper:

Theorem 3.1. *We assume* (H3.1) *and* (H3.2). *Then the following conditions are equivalent*:

(a) (14) *holds for SDEs* (12) *and* (13);

(b) *For each* $(t, x) \in [0, T] \times R$, *the coefficients* b_i, σ_i, γ_i, $i = 1, 2$, *satisfy*:

$$\begin{cases} \text{(i) } \sigma_1(t,x) = \sigma_2(t,x); \\ \text{(ii) } b_1(t,x) \leqslant b_2(t,x); \\ \text{(iii) } \gamma_1(t,x,z) = \gamma_2(t,x,z), \ n(\mathrm{d}z)\text{-}a.s.; \\ \text{(iv) } \gamma_1(t,x_1,z) - \gamma_1(t,x_2,z) \leqslant x_2 - x_1, \forall x_1 \leqslant x_2, \ n(\mathrm{d}z)\text{-}a.s. \end{cases} \tag{15}$$

For the situation without jumps, we have

Corollary 3.1. *We assume* (H3.1) *and* (H3.2) *as well as* $\gamma_1 \equiv \gamma_2 \equiv 0$. *Then the following conditions are equivalent*:

(a) (14) *holds for SDEs* (12) *and* (13);

(b) *For each* $(t, x) \in [0, T] \times R$, *the coefficients* b_i, σ_i, $i = 1, 2$, *satisfy*:

$$\begin{cases} \text{(i) } \sigma_1(t,x) = \sigma_2(t,x); \\ \text{(ii) } b_1(t,x) \leqslant b_2(t,x). \end{cases} \tag{16}$$

To prove Theorem 3.1, we first make the following criteria:

Proposition 3.1. *We assume* (H3.1) *and* (H3.2). *Then the following claims are equivalent*:

(i) (14) *holds for SDEs* (12) *and* (13);
(ii) *For each* $(t, x_1, x_2) \in [0, T] \times R \times R$,

$$\begin{aligned} 0 \geqslant\, & 2(x_1 - x_2)^+(b_1(t,x_1) - b_2(t,x_2)) + (\sigma_1(t,x_1) - \sigma_2(t,x_2))^2 1_{\{x_1 > x_2\}}(x) \\ & + (1 - C)((x_1 - x_2)^+)^2 + I_K(x_1, x_2), \end{aligned} \tag{17}$$

where we denote

$$\begin{aligned} I_K(x_1, x_2) = \int_Z & [((x_1 + \gamma_1(t, x_1, z) - x_2 + \gamma_2(t, x_2, z))^+)^2 \\ & - ((x_1 - x_2)^+)^2 - 2(x_1 - x_2)^+(\gamma_1(t, x_1, z) - \gamma_2(t, x_2, z))] n(\mathrm{d}z). \end{aligned}$$

Proof. We trivially set $X_s^{t,x} = (X_s^1, X_s^2)$, $s \geqslant 0$, and treat (12) and (13) as a 2-dimensional SDE defined on $s \in [t, \infty)$:

$$X_s^{t,x} = x + \int_t^s b(r, X_r^{t,x})\,\mathrm{d}r + \int_t^s \sigma(r, X_r^{t,x})\,\mathrm{d}W_r + \int_t^s \int_Z \gamma(r, X_{r-}^{t,x}, z)\tilde{N}(\mathrm{d}z\,\mathrm{d}r), \tag{18}$$

where we denote, for each $t \geqslant 0$ and $x_1, x_2 \in R$,

$$x = (x_1, x_2)^{\mathrm{T}}, \quad \gamma(t, x, z) = (\gamma_1(t, x_1, z), \gamma_2(t, x_2, z))^{\mathrm{T}},$$

$$b(t, x) = (b_1(t, x_1), b_2(t, x_2))^{\mathrm{T}}, \quad \sigma(t, x) = (\sigma_1(t, x_1), \sigma_2(t, x_2))^{\mathrm{T}}.$$

By Assumptions (H3.1) and (H3.2), we have, for each $x, x' \in R^2$ and $t \geqslant 0$,

$$\langle x - x', b(t,x) - b(t,x') \rangle \leqslant \mu |x - x'|^2,$$

$$|\sigma(t,x) - \sigma(t,x')| \leqslant \mu |x - x'|,$$

$$|\gamma(t,x,z) - \gamma(t,x',z)| \leqslant \rho(z) |x - x'|,$$

$$|b(t,x)| + |\sigma(t,x)| \leqslant \mu(1 + |x|),$$

$$|\gamma(t,x,z)| \leqslant \rho(z)(1 + |x|).$$

With this formulation condition (14) is equivalent to the "viability property" (2), where the constraint K is

$$K := \{x = (x_1, x_2)^{\mathrm{T}} \in R^2 : x_1 \leqslant x_2\}.$$

It follows by Proposition 2.2 that (14) is equivalent to that d_K^2 is a viscosity supersolution of the following PDE:

$$\mathscr{L}u(t,x) + \mathscr{B}u(t,x) + d_K^2(t,x) - Cu(t,x) = 0, \quad u(T,x) = d_K^2(x).$$

Since for each $x = (x_1, x_2)^{\mathrm{T}} \in R^2$, we have $d_K^2(x) = ((x_1 - x_2)^+)^2/2$. It is then easy to check that each function $\varphi \in C^{1,2}([0,T] \times R^2)$ such that $d_K^2 - \varphi$ attains its minimum at (t,x) satisfies

$$\frac{\partial \varphi}{\partial t}(t,x) = \frac{\partial d_K^2(x)}{\partial t} = 0, \quad D\varphi(t,x) = D[d_K^2(x)] = \begin{pmatrix} x_1 - x_2 \\ x_2 - x_1 \end{pmatrix} 1_{\{x_1 > x_2\}}(x),$$

$$D_{xx}^2 \varphi(t,x) \in \Theta(x_1, x_2),$$

where $\Theta(x)$ is the following subset of S^2, the space of 2×2 symmetric matrices:

$$\Theta(x_1, x_2) := \left\{ X \in S^2 : X \leqslant 1_{\{x_1 > x_2\}}(x) \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \right\}.$$

We then can easily check that (14) is equivalent to (17). $\square$

Proof of Theorem 3.1. From Proposition 3.1, it suffices to prove that (15) $\Leftrightarrow$ (17).

(17)$\Rightarrow$ (15): For the case $x_1 \leqslant x_2$, (17) becomes

$$\int_Z [\{x_1 + \gamma_1(t, x_1, z) - x_2 - \gamma_2(t, x_2, z)\}^+]^2 n(\mathrm{d}z) \leqslant 0.$$

Thus

$$\gamma_1(t, x_1, z) - \gamma_2(t, x_2, z) \leqslant x_2 - x_1, \quad \forall t, n(\mathrm{d}z)\text{-a.s. in } z. \tag{19}$$

In particular, we have

$$\gamma_1(t,x,z) \leqslant \gamma_2(t,x,z), \quad n(\mathrm{d}z)\text{-a.s. in } z. \tag{20}$$

Now for each $x \in R$ and $\delta \geqslant 0$, by setting $x_2 = x$, $x_1 = x + \delta$ in (17), we have

$$\begin{aligned} 0 \geqslant\; & 2\delta[b_1(t, x+\delta) - b_2(t,x)] + [\sigma_1(t, x+\delta) - \sigma_2(t,x)]^2 + (1-C)\delta^2 \\ & + \int_Z [(\{\delta + \gamma_1(t, x+\delta, z) - \gamma_2(t,x,z)\}^+)^2 - \delta^2 \\ & - 2\delta(\gamma_1(t, x+\delta, z) - \gamma_2(t,x,z))] n(\mathrm{d}z). \end{aligned} \tag{21}$$

By the linear growth conditions of γ_i and b_i in (H3.2), we have

$$|(\gamma_1(t, x+\delta, z) - \gamma_2(t,x,z))| \leqslant \rho(z)(2 + 2|x| + \delta),$$

$$|b_1(t, x+\delta) - b_2(t,x)| \leqslant 2\mu(1 + |x| + |\delta|).$$

Thus, by (21), there exists a constant c_1, independent of x and δ, such that

$$[\sigma_1(t,x+\delta)-\sigma_2(t,x)]^2 \leqslant c_1(1+\delta+|x|)\delta.$$

Letting $\delta \to 0$ yields (i) of (15).

Again from (21) in considering $\sigma_1(t,x) \equiv \sigma_2(t,x)$, we have

$$\begin{aligned}
&2\delta[b_1(t,x+\delta)-b_2(t,x)]\\
&\quad\leqslant \delta^2(n(Z)+C-1)+2\delta\int_Z[\gamma_1(t,x+\delta,z)-\gamma_2(t,x,z)]n(\mathrm{d}z)\\
&\quad\leqslant \delta^2(n(Z)+C-1)+2\delta\int_Z[\gamma_1(t,x+\delta,z)-\gamma_1(t,x,z)]n(\mathrm{d}z)\\
&\quad\leqslant \delta^2(n(Z)+C-1)+2\delta^2\int_Z \rho(z)n(\mathrm{d}z).
\end{aligned}$$

Or

$$2[b_1(t,x+\delta)-b_2(t,x)] \leqslant \delta(n(Z)+C-1)+2\delta\int_Z \rho(z)n(\mathrm{d}z).$$

Thus

$$\begin{aligned}
2[b_1(t,x)-b_2(t,x)] \leqslant\; & 2|b_1(t,x+\delta)-b_1(t,x)|\\
&+\delta(n(Z)+C-1)+2\delta\int_Z \rho(z)n(\mathrm{d}z).
\end{aligned}$$

We let $\delta \to 0$ on both sides. Then (ii) of (15) holds.

We return once more to (21). By (ii) and the Lipschitz conditions of b_1, γ_1,

$$\begin{aligned}
0 &\geqslant 2\delta[b_1(t,x+\delta)-b_2(t,x)]-\delta^2(n(Z)+C-1)\\
&\quad -2\delta\int_Z[\gamma_1(t,x+\delta,z)-\gamma_2(t,x,z)]n(\mathrm{d}z)\\
&\geqslant 2\delta[b_1(t,x+\delta)-b_1(t,x)]-\delta^2(n(Z)+C-1)\\
&\quad -2\delta\int_Z[\gamma_1(t,x+\delta,z)-\gamma_1(t,x,z)]n(\mathrm{d}z)\\
&\quad +2\delta\int_Z[\gamma_2(t,x,z)-\gamma_1(t,x,z)]n(\mathrm{d}z)\\
&\geqslant -\left(2\mu+2\int_Z \rho(z)n(\mathrm{d}z)+(n(Z)+C-1)\right)\delta^2\\
&\quad +2\delta\int_Z[\gamma_2(t,x,z)-\gamma_1(t,x,z)]n(\mathrm{d}z).
\end{aligned}$$

Since δ can be arbitrarily small, we then have

$$\int_Z[\gamma_2(t,x,z)-\gamma_1(t,x,z)]n(\mathrm{d}z) \leqslant 0.$$

This with (20) yields $\gamma_2(t,x,z)-\gamma_1(t,x,z)=0$, $n(\mathrm{d}z)$-a.s. i.e., (iii) holds. The proof of (17) $\Rightarrow$ (15) is complete.

(15)$\Rightarrow$ (17): Given real numbers x and η, we set $x_2 = x$, $x_1 = x + \eta$ and $\gamma_1 \equiv \gamma_2$, $\sigma_1 \equiv \sigma_2$. (17) becomes

$$\begin{aligned}&2\eta^+[b_1(t, x+\eta) - b_2(t,x)] + [\sigma_1(t, x+\eta) - \sigma_1(t,x)]^2 1_{(0,+\infty)}(\eta)\\ &\quad + (1-C)(\eta^+)^2 + I_\eta \leqslant 0,\end{aligned} \tag{22}$$

where we set

$$\begin{aligned}I_\eta := \int_Z &[(\{\eta + \gamma_1(t, x+\eta, z) - \gamma_1(t,x,z)\}^+)^2 - (\eta^+)^2\\ &- 2\eta^+(\gamma_1(t, x+\eta, z) - \gamma_1(t,x,z))]n(\mathrm{d}z).\end{aligned}$$

By (iv) of (15), we have

$$I_\eta = \begin{cases} 0 & \text{if } \eta \leqslant 0,\\ \int_Z [\gamma_1(t, x+\eta, z) - \gamma_1(t,x,z)]^2 n(\mathrm{d}z) & \text{if } \eta > 0.\end{cases}$$

It is clear that (22) hold true for the case $\eta \leqslant 0$. For the case $\eta > 0$, we have

$$2\eta[b_1(t,x+\eta) - b_2(t,x)] \leqslant 2\eta[b_1(t,x+\eta) - b_1(t,x)] \leqslant 2\mu\eta^2,$$

$$[\sigma_1(t,x+\eta) - \sigma_1(t,x)]^2 \leqslant \mu^2\eta^2,$$

$$I_\eta \leqslant \eta^2 \int_Z \rho^2(z) n(\mathrm{d}z).$$

Recall that $C = 1 + 2\mu + \mu^2 + \int_Z \rho^2(z)n(\mathrm{d}z)$. Thus the above three inequalities implies (22) and thus (17). $\square$

Acknowledgment

The author would like to acknowledge the partial support from the Natural Science Foundation of China, Grant no. 10131040.

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