Binary operations and canonical forms for factorial and related models

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ABSTRACT

Binary operations on commutative Jordan algebras are used to extend the grouping of treatments in blocks and the taking of fractional replicates to models where factors have arbitrary number of levels. Up to now these techniques had been restricted to models whose factors have a prime or a power of a prime number of levels. Moreover, the binary operations will enable the use of simple models as building blocks for more complex designs.

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1. Introduction

Factorial designs in which the factors have a prime or power of a prime number of levels enjoy interesting properties. Thus, see for instance [2,5,7], we can group the treatments in blocks to get a better control of experimental error. Moreover, by considering only the treatments in one of the blocks we have a fractional replicate in which there is only a fraction of the treatments corresponding to the combination of factor levels.

The theory yielding these results is based on Galois fields which are available when the number of factor levels is a prime or a power of a prime. To overcome this limitation we will use commutative Jordan algebras, CJA. These algebras are linear spaces constituted by symmetric matrices that commute and containing the square of their matrices. Each CJA A has (see [9]) an unique basis, the principal
basis \( \text{pb}(A) \), constituted by pairwise orthogonal orthogonal projection matrices. As we shall see the use of CJA enables us to present the models we study as associated to CJA. The canonical form of a fixed effects model associated to a CJA \( A \) will be

\[
y = \sum_{j=1}^{w} A_j \zeta_j,
\]

where the matrices \( Q_j = A_j A_j' \), \( j = 1, \ldots, w \), constitute the principal basis of \( A \), the \( \zeta_j \) have mean vectors \( \eta_j \), \( j = 1, \ldots, w + 1 \), and covariance matrices \( \sigma^2_0 I_j \), \( j = 1, \ldots, w + 1 \), and \( \eta_{w+1} = 0 \).

Binary operations on CJAs enable us to consider models obtained:

1. by crossing, i.e., we get a new model whose treatments are all the combinations of the treatments in the initial models;
2. by nesting every treatment in a model inside each treatment of another model.

Moreover, merging factors gives, as we shall see, models associated to sub-algebras. In this way, we will overcome the requirement that the number of levels must be a prime or a power of a prime.

We will start by presenting the main results we use on CJA, binary operations and sub-algebras. Next we consider inference for models associated to CJA. These two sections give us the framework for the study of factorial and related models. We start with prime base factorials before considering their fractional replicates. We will see how, through crossing and nesting, complex models may be derived. Lastly we will consider factor merging which, as we mentioned, leads to models where factors may have an arbitrary number of levels, and the inverse possibility of model terms disaggregation, which leads to models associated to larger algebras.

Our results will give us great flexibility in experiment designing. Then we may start by deciding if we have nesting or not. If we decided there will be nesting we decide which factors will be considered for each nesting tier. For each tier with more than one factor we will have factor crossing. Besides deciding what factors should be considered for each tier we must decide how many levels each of them shall have. The techniques of factor merging and disaggregating models terms give us a great freedom in the choice of the numbers of levels. Applying this approach any tier will be the crossing of prime number of factorials. This lead to the possibility of applying extensively fractional replications thus controlling the size of the final experiment. Moreover, through confounding, the full experiment may be broken up into blocks with a better control of the experimental error.

The flexibility we achieved rests on:

- writing the sub-models in their canonical form:

\[
y = \sum_{j=1}^{w} A_j \eta_j + e.
\]

Then, the matrices \( Q_j = A_j A_j' \), \( j = 1, \ldots, w \), and \( Q_{w+1} = I_n - \sum_{j=1}^{w} Q_j \) constitute the principal basis of the CJA to which the sub-models are associated;
- applying the binary operations corresponding to models crossing and nesting to obtain a CJA associated to the final model; these operations are described in Section 2.

We point out that the sub-models we considered were prime basis factorials and their fractional replicates. For these sub-models, as we show in Sections 4 and 5, the canonical form arises naturally from the initial formulations of such models.

It may be interesting to point out that the range spaces \( R(Q_j) \), \( j = 1, \ldots, w \) and \( R(Q^+) \) of matrices \( Q_j \), \( j = 1, \ldots, w \), and \( Q^+ \) constitute an orthogonal partition of the sample space

\[
\mathbb{R}^n = \bigoplus_{j=1}^{w} R(Q_j) \bigoplus R(Q^+),
\]

where \( \bigoplus \) represents the direct sum of subspaces. Now the \( R(Q_j) \) will be associated to effects and interactions and when \( \eta_j = 0 \) the mean vector \( \mu = \sum_{j=1}^{w} A_j \eta_j \) will span the orthogonal complement.
R(Q_j) of R(Q), j = 1, . . . , w. Thus the hypothesis of absence of effects and/or interactions can naturally be written as

\[ \mathcal{H}_{0j} : \eta_j = 0, \quad j = 1, \ldots, w. \]  

(4)

Moreover, as we shall show, \( \Psi \) is an estimable vector if it may be written as

\[ \Psi = C \sum_{j=1}^{w} A_j \eta_j. \]  

(5)

Thus, the parameters \( \eta_j \) are easy to interpret.

2. Commutative Jordan algebras

Jordan algebras were introduced (see [4]) to provide an algebraic foundation for Quantum Mechanics. Later on these structures were applied (for instance see [8–13]) to carry out linear statistical inference. In these studies Jordan algebras were called quadratic subspaces since they are linear subspaces constituted by square matrices, containing the squares of their matrices. For priority sake we will use their first name. We are interested in commutative Jordan algebras – CJA – where matrices commute. If \( \text{pb}(A) = \{Q_1, \ldots, Q_w\} \) and \( M \in A, M = \sum_{j=1}^{w} a_j Q_j \). When \( M \) is an orthogonal projection matrix – OPM – it is idempotent. Because the \( Q_1, \ldots, Q_w \) are pairwise orthogonal and idempotent, \( a_j = 0 \) or \( a_j = 1, j = 1, \ldots, w. \) With \( C = \{j : a_j = 1\} \), we will have

\[
\begin{aligned}
M &= \sum_{j \in C} Q_j, \\
\text{rank}(M) &= \sum_{j \in C} \text{rank}(Q_j).
\end{aligned}
\]  

(6)

Thus, with \( B \) a sub-algebra of \( A \), the matrices in \( \text{pb}(B) \) will be sums of matrices in \( \text{pb}(A) \). We also see that if a rank one OPM belongs to \( A \), it also belongs to \( \text{pb}(A) \). Namely, if \( \frac{1}{n} J_n = \frac{1}{n} 1_n 1_n' \in A \), we have that \( \frac{1}{n} J_n \in \text{pb}(A) \) and take \( Q_1 = \frac{1}{n} J_n \). We then say that \( A \) is regular. Moreover, if

\[ \sum_{j=1}^{w} Q_j = I, \]  

(7)

we say that \( A \) is complete. Otherwise, we may add

\[ Q_{w+1} = Q^+ = I - \sum_{j=1}^{w} Q_j \]  

(8)

to \( \text{pb}(A) \) to obtain \( \text{pb}(\bar{A}) \), with \( \bar{A} \) the completion of \( A \). A complete and regular sub-CJA will be a principal sub-CJA. Consequently, \( A \) will also be regular and complete.

Given the families \( T_1 \) and \( T_2, T_1 \otimes T_2 \) will be the family of the Kronecker products of the matrices of \( T_1 \) by those of \( T_2 \). Then (see [3]), the CJA \( A_1 \otimes A_2 \) with

\[ \text{pb}(A_1 \otimes A_2) = \text{pb}(A_1) \otimes \text{pb}(A_2) \]  

(9)

will be associated to models obtained crossing models associated to \( A_1 \) to models associated to \( A_2 \). The \( \otimes \) product of principal sub-CJAs gives also principal sub-CJAs.

We point out that (see [3])

\[ A_1 \otimes (A_2 \otimes A_3) = (A_1 \otimes A_2) \otimes A_3. \]  

(10)

This associative property will be useful when we want to cross more than two models. Namely, to obtain through sub-CJA condensation models with factors that, as we shall see, have arbitrary number of levels.

Now, if \( A_1 \) and \( A_2 \) are complete and regular CJAs associated to models, the CJA \( A_1 \ast A_2 \) with

\[ \text{pb}(A_1 \ast A_2) = \left\{ \left( \frac{1}{n_1} J_{n_1} \right) \otimes \left( \frac{1}{n_2} J_{n_2} \right), \ldots, Q_{1,w_1} \otimes \left( \frac{1}{n_2} J_{n_2} \right), I_{n_1} \otimes Q_{2,2}, \ldots, I_{n_1} \otimes Q_{2,w_2} \right\} \]  

(11)
is (see [3]) associated to the model obtained by nesting the treatments of the second model inside the treatments of the first one.

The restricted Kronecker product $\ast$ of CJAs is also (see [3]) associative, which may be useful in deriving complex models.

There is an interesting relation between CJAs and orthogonal matrices. When $\text{pb}([A]) = \{Q_1, \ldots, Q_w\}$, we have

$$Q_j = A_j A_j^\prime, \quad j = 1, \ldots, w$$  \hspace{1cm} (12)

the column vectors of $A_j$ constituting an orthonormal basis for $R(Q_j), j = 1, \ldots, w$. Then

$$P = [A_1 \cdots A_w]$$  \hspace{1cm} (13)

will be an orthogonal matrix associated to $[A]$. There is not an unique orthogonal projection matrix associated to a CJA. With $P_j, j = 1, \ldots, w$, orthogonal matrices, and

$$B_j = A_j P_j, \quad j = 1, \ldots, w$$  \hspace{1cm} (14)

the column vectors of $B_j$ will also constitute an orthonormal basis for $R(Q_j), j = 1, \ldots, w$, so

$$R = [B_1 \cdots B_w]$$  \hspace{1cm} (15)

will also be an orthogonal matrix associated to $[A]$.

This relation between CJAs and orthogonal matrices will be useful when we consider prime basis factorials and their fractional replicates.

In what follows we consider models with $r$ observations per treatment. If, when $r = 1$, the model is associated to the CJA $B$, whatever $r > 1$ the model will (see [3]) be associated to $[A] \ast [A(r)]$, with

$$\text{pb}([A(r)]) = \left\{ \frac{1}{\sqrt{r}} J_r, J_r \right\},$$  \hspace{1cm} (16)

where $J_r = I_r - \frac{1}{r} J_r$. It is interesting to point out that $[A(r)]$ is complete and regular with $\text{dim}([A(r)]) = 2$.

When deriving complex models, we will assume that the simple models we are considering have $r$ replicates. Then, the complex model will initially also have $r = 1$ replicates. The previous observations show how to obtain a CJA associated to the complex model when $r > 1$.

If $\text{pb}([A]) = \{Q_1, \ldots, Q_w\}$, with $Q_j = A_j A_j^\prime, j = 1, \ldots, w$, the matrices in $\text{pb}([A] \ast [A(r)])$ will be

$$Q_j \otimes \left( \frac{1}{\sqrt{r}} J_r \right) = \left( A_j \otimes \left( \frac{1}{\sqrt{r}} J_r \right) \right) \left( A_j^\prime \otimes \left( \frac{1}{\sqrt{r}} J_r \right) \right) = \left( A_j \otimes \left( \frac{1}{\sqrt{r}} J_r \right) \right) \left( A_j^\prime \otimes \left( \frac{1}{\sqrt{r}} J_r \right) \right) = \left( A_j \otimes \left( \frac{1}{\sqrt{r}} J_r \right) \right) \left( A_j^\prime \otimes \left( \frac{1}{\sqrt{r}} J_r \right) \right), \quad j = 1, \ldots, w,$$  \hspace{1cm} (17)

and

$$Q_j^\perp = I_n \otimes J_r = (I_n \otimes T_r)(I_n \otimes T_r)^\prime.$$  \hspace{1cm} (18)

where $T_r$ is obtained deleting the first column equal to $\frac{1}{\sqrt{r}} 1^\prime$ of a $r \times r$ orthogonal matrix.

### 3. Associated models

We will consider fixed effects models with $r$ observations for each of the $\bar{n}$ treatments associated to complete $\text{CJA} \ast [A(r)]$. If $\text{pb}([A]) = \{Q_1, \ldots, Q_w\}$ we will have $\text{pb}([A] \ast [A(r)]) = \{Q_{1'}, \ldots, Q_{w'}\}$ with $Q_{j'} = Q_j \otimes \frac{1}{\sqrt{r}} J_r, j = 1, \ldots, w + 1$, and $Q_{w'+1} = I_{\bar{n}} \otimes J_r = (I_{\bar{n}} \otimes T_r)(I_{\bar{n}} \otimes T_r)^\prime$.

Moreover, if $Q_{j'} = A_j A_j^\prime, j = 1, \ldots, w$, we have $Q_{j'} = A_j A_j^\prime$ with $A_j = A_j \otimes \left( \frac{1}{\sqrt{r}} T_r \right), j = 1, \ldots, w$. We will also have $Q_{w'+1} = A_{w'} A_{w'}^\prime$ with $A_{w'} = I_{\bar{n}} \otimes T_r$.

Initially the model may be written as

$$y = \sum_{j=1}^w A_j \eta_j + e,$$  \hspace{1cm} (19)

where the $\eta_1, \ldots, \eta_w$ are fixed and $e$ is an error vector with null mean vector and variance–covariance matrix $\sigma^2 I_{\bar{n}} (n = \bar{n} r)$. Thus, the mean vector will be
\[ \mu = \sum_{j=1}^{w} A_j \eta_j, \]  

(20)

and the variance–covariance matrix \( \sigma^2 I_n \).

To obtain the canonical form for the model we observe that

\[ I_n = \sum_{j=1}^{w+1} Q_j = \sum_{j=1}^{w+1} A_j A_j', \]  

(21)

so that

\[ y = \sum_{j=1}^{w+1} A_j \zeta_j, \]  

(22)

with

\[ \begin{cases} 
\zeta_j = A_j'y = \eta_j + A_j' \eta, & j = 1, \ldots, w, \\
\zeta_{w+1} = A_{w+1}'y = A_{w+1}' \eta.
\end{cases} \]  

(23)

Thus, the \( \zeta_1, \ldots, \zeta_{w+1} \) will have mean vector \( \eta_1, \ldots, \eta_w, 0 \) and variance–covariance matrix \( \sigma^2 I_{g_j} \) with \( g_j = \text{rank}(A_j) = \text{rank}(Q_j), j = 1, \ldots, w + 1. \)

Thus, the mean vector of \( y \) will be

\[ \mu = \sum_{j=1}^{w} A_j \eta_j = A \eta \]  

(24)

with \( A = [A_1 \cdots A_w] \) and \( \eta = [\eta_1' \cdot \eta_w'] \). Moreover, the orthogonal projection matrices on the space spanned by \( \mu \) will be, see [6],

\[ T = A(A'A)^+ A', \]  

(25)

and

\[ \Psi = C \eta \]  

(26)

is estimable if and only if \( C = WA, \) with LSE (Least Square Estimators) given by

\[ \tilde{\Psi} = C(A'A)^+ A' y = WA(A'A)^+ A' y = WT y. \]  

(27)

Now, we also have \( T = \sum_{j=1}^{w} Q_j = \sum_{j=1}^{w} A_j A_j', \) so

\[ \tilde{\Psi} = W \sum_{j=1}^{w} A_j \tilde{\eta}_j, \]  

(28)

and since \( \Psi = W \sum_{j=1}^{w} A_j \eta_j \) we notice the importance of parameters \( \Psi_j, j = 1, \ldots, w \) and their estimators \( \tilde{\Psi}_j, j = 1, \ldots, w \) in treating estimable vectors.

The cross-covariance matrices of the \( \zeta_j, j = 1, \ldots, w + 1, \) are null so that if we assume the normality of \( y \) the \( \zeta_1, \ldots, \zeta_{w+1} \) will be normal and independent. Thus \( S_j(\eta_j, 0) = \| \zeta_j - \eta_j, 0 \|^2 \) will be the product by \( \sigma^2 \) of a chi-square with \( g_j \) degrees of freedom and non-centrality parameter \( \delta_j(\eta_j, 0) = \frac{1}{\sigma^2} \| \eta_j - \eta_j, 0 \|^2, j = 1, \ldots, w. \) Moreover, \( S_j(\eta_j, 0), j = 1, \ldots, w, \) will be independent from \( S = \| \eta_{w+1, 0} \|^2 \) which will be the product by \( \sigma^2 \) of a central chi-square with \( g = \hat{n}(r - 1) = g_{w+1} \) degrees of freedom. Thus for testing

\[ H_0(\eta_j, 0) : \eta_j = \eta_j, 0, \quad j = 1, \ldots, w, \]  

(29)

we have an \( F \) test with statistic

\[ F_j(\eta_j, 0) = \frac{S_j(\eta_j, 0)}{g_j}, \quad j = 1, \ldots, w. \]  

(30)
The distribution of \( F_j(\eta_{j,0}) \) \( j = 1, \ldots, w \), will be the \( F \) distribution with \( g_j, j = 1, \ldots, w \), and \( g \) degrees of freedom and non-centrality parameter \( \delta_j(\eta_{j,0}) = 0 \) when and only when \( H_0(\eta_{j,0}) \) holds. This test will be strictly unbiased.

The hypothesis \( H_0(\eta_{j,0}) \) generalizes the usual hypothesis
\[
H_0(0) : \eta_j = 0. \tag{31}
\]

This extension is interesting since it leads to a new property of the \( F \) tests. Thus, \( \hat{S}_j = ||\zeta_j - \eta_j||^2 \) is the product by \( \sigma^2 \) of a central chi-square with \( g_j \) degrees of freedom independent from \( S, j = 1, \ldots, w \) and so
\[
\hat{S}_j = \frac{g_j \hat{S}_j}{g_j}. \tag{32}
\]

will have the central \( F \) distribution with \( g_j \) and \( g \) degrees of freedom \( j = 1, \ldots, w \). If the quantile of that distribution for probability \( 1 - q \) is \( \hat{1} = q, j = 1, \ldots, w \).

The first of these inequalities defines a \( 1 - q \) level confidence hypersphere for \( \eta_j \). Now, the \( q \) level \( F \) test for \( H_0(\eta_{j,0}) \) does not reject that hypothesis if and only if \( \eta_{j,0} \) is contained in the \( 1 - q \) level confidence hypersphere, \( j = 1, \ldots, w \). Thus these tests enjoy duality.

Moreover, once the matrices \( A_j = 1, \ldots, w \) are obtained, the analysis of these models is straightforward. So, in the next sections, we will concentrate on the derivation of these matrices.

4. Prime basis factorials

Let us assume that \( N \) factors with \( p \) (prime) levels cross. Usually (for instance, see [5], Chapters 6 and 9) we take \( p = 2 \) or \( p = 3 \). Then we have a \( p^N \) factorial with that number of level combinations. These level combinations will be called treatments.

To obtain a \( CJA \ A(p^N) \) associated to these models we number the factor levels from 0 to \( p - 1 \). This will enable us to work with vector spaces over Galois fields. We put \( [p] = \{0, \ldots, p - 1\} \) and \( [p]_0 = \{1, \ldots, p - 1\} \). Using module \( p \) arithmetic we have the Galois field \( G(p) \) with support \( [p] \). Moreover, the set \( [p]^N \) of dimension \( N \) vectors with components in \( [p] \) will be the support of a dimension \( N \) vector space \( G(p)^N \) over \( G(p) \). The dual \( L = L[p]^N \) of \( G(p)^N \) is constituted by the linear applications
\[
l(x|a) = \left( \sum_{j=1}^{N} a_j x_j \right)_{(p)} \tag{34}
\]

where \( x, a \in G(p)^N \) and \( (p) \) indicates the use of module \( p \) arithmetic. With \( c_1, \ldots, c_u \in [p] \), we put
\[
l(\sum_{i=1}^{u} x_i) = l \left( \sum_{i=1}^{u} (c_i a_i)_{(p)} \right). \tag{35}
\]

Writing \( \phi(a) = l(x|a) \), we define an isomorphism between \( G(p)^N \) and \( L[p]^N \). Then, the \( l(x|a_i), i = 1, \ldots, u \) will be linearly independent if and only if the \( a_1, \ldots, a_u \) are linearly independent. Both \( G(p)^N \) and \( L[p]^N \) have dimension \( N \), and
\[
\#(G(p)^N) = \#(L[p]^N) = p^N. \tag{36}
\]
The null vectors of \( G(p)^N \) and \( L[p]^N \) are \( 0 \) and \( l_0(x) = l(x|0) \).

Putting \( l(x|a_1) \rho l(x|a_2) \) when \( a_2 = (ca_1)_{(p)} \), with \( c \in [p]_0 \), we define an equivalence relation in \( L[p]^N \). Now, \( l_0(x) \) will be isolated in its equivalence class while non null linear applications are grouped in classes with \( p - 1 \) elements. Thus there will be
\[
k_N(p) = \frac{p^N - 1}{p - 1}. \tag{37}
\]
such classes. Each of these classes contains one and only one application whose first non null coefficient is 1. Such linear applications are called reduced and their family is represented by \( C[p]_p \) and \( \#(C[p]_p) = k_N(p) \). The reduced homologue of a linear application is the reduced linear application \( \rho \)-equivalent to it. Applications \( l(x|a_1), \ldots, l(x|a_u) \) are linearly independent if and only if their reduced homologues are linearly independent. Thus, we may assume that a basis \( L = \{l_1, \ldots, l_u\} \) of a linear subspace \( \mathcal{L}_1 = \mathcal{L}_1(L) \) is constituted by reduced applications. The subspace \( \mathcal{L}_1 \) is \( \rho \)-saturated and contains \( p^u \) applications. Thus, with \( \mathcal{L}_{1,r} \) the family of reduced applications belonging to \( \mathcal{L}_1 \), we have \( \#(\mathcal{L}_{1,r}) = k_u(p) \).

Writing \( x_i \in \mathcal{L}_1, x_2 \) when, whatever \( l \in \mathcal{L}_1, l(x_1) = l(x_2) \), we define an equivalence relation in \( G[p]^N \) whose equivalence classes are the blocks

\[
\{l | b \} = \{x : l_i(x) = b ; i = 1, \ldots, u\}. \quad (38)
\]

There are \( p^u \) such blocks, each with cardinal \( p^{N-u} \).

Let us order the reduced applications giving the first indexes to those in \( \mathcal{L}_{1,r} \). If we also order the vectors in \( G[p]^N \) we may define the matrix \( C(l_h) \), with elements

\[
c_{ij}(l_h) = \begin{cases} 1 & \text{when } l_h(x_i) = i - 1; \ i = 1, \ldots, p; \ j = 1, \ldots, p^N. \end{cases} \quad (39)
\]

We now establish

**Lemma 1.** We have \( C(l_h)C(l_h)' = p^{N-1}I_p, h = 1, \ldots, k_N(p) \), and \( C(l_h)C(l_k)' = p^{N-2}T_p \) when \( h \neq k \).

**Proof.** For each column of \( C(l_h) \) we have one element equal to 1, the remaining being null, and for each row of \( C(l_h) \) we have \( p^{N-1} \) elements equal to 1, the remaining being null, so \( C(l_h)C(l_h)' = p^{N-1}I_p, h = 1, \ldots, k_N(p) \). Moreover, \( C(l_h) \) and \( C(l_k) \) will be \( p^{N-2} = \#(\{l_h, l_k|b_1, b_2\}) \) matchings between null non elements on both rows, \( b_1, b_2 = 0, \ldots, p - 1 \), thus \( C(l_h)C(l_k)' = p^{N-2}T_p \) when \( h \neq k \), and the proof is complete. \( \square \)

We may now establish

**Proposition 1.** The matrix

\[
\begin{bmatrix}
\frac{1}{\sqrt{p}}1^{p^N} & A(l_1) & \cdots & A(l_{k_N(p)})
\end{bmatrix},
\]

where \( A(l_h) = \frac{1}{\sqrt{p^N}}C(l_h)T_p, h = 1, \ldots, k_N(p) \), is an orthogonal matrix associated to the CJA \( A(p^N) \) with

\[
\text{pb}(A(p^N)) = \left\{ \frac{1}{p^{N-1}}I_p, Q(l_1), \ldots, Q(l_{k_N(p)}) \right\},
\]

where \( Q(l_h) = A(l_h)A(l_h)' \).

**Proof.** Reasoning as to establish Lemma 1, we get \( C(l_h)1^{p^N} = p^{N-1}1^{p^N} \), thus \( A(l_h)'1^{p^N} = \sqrt{p^{N-1}}T_p1^{p^N} = 0 \), \( h = 1, \ldots, k_N(p) \). Moreover, \( A(l_h)A(l_h)' = \frac{1}{p^{N-1}}T_pC(l_h)C(l_h)'T_p = T_pT_p = I_p \), \( h = 1, \ldots, k_N(p) \). Lastly, \( h \neq k \), \( A(l_h)A(l_k)' = \frac{1}{p^{N-1}}T_pT_p = T_pT_p1^{p^N} = 0_{(p-1) \times (p-1)} \). Thus, \( P(p^N) \) is orthogonal. The rest of the proof is straightforward. \( \square \)

We now order the reduced applications, so that \( l_1, \ldots, l_{k_N(p)} \) take fixed values in the blocks. Thus, if we order the blocks, we may define the matrices \( D(l_h) \) with elements

\[
d_{ij}(l_h) = \begin{cases} 1 & \text{when } l_h(x_i) = i - 1; \ i = 1, \ldots, p; \ j = 1, \ldots, p^u; \ h = 1, \ldots, k_u(p). \end{cases} \quad (40)
\]

Assuming that we have only 1 replicate and representing by \( z \) the vector of block totals, we have

\[
D(l_h)z = C(l_h)y, \quad h = 1, \ldots, k_u(p). \quad (41)
\]
Then, with
\[ B(l_h) = \frac{1}{\sqrt{p^{n-u}}} D(l_h)^T p, \quad h = 1, \ldots, k_u(p), \] (42)
we have
\[ B(l_h) z = \sqrt{p^{N-u}} A(l_h)^T y, \quad h = 1, \ldots, k_u(p). \] (43)
Moreover, we can reason as above to show that the matrix
\[ R(p^{n}) = \left[ \frac{1}{\sqrt{p^{n}}} 1^{p^{n}} \quad B(l_1) \quad \cdots \quad B(l_{k_u(p)}) \right] \] (44)
is orthogonal. Thus, with \( T \) the sum of all observations,
\[ \|z\|^2 - \frac{T^2}{p^{n}} = \sum_{h=1}^{k_u(p)} \|B(l_h)^T z\|^2 = p^{N-u} \sum_{h=1}^{k_u(p)} S(l_h), \] (45)
where
\[ S(l_h) = \|A(l_h)^T y\|^2 = \|Q(l_h) y\|^2, \quad h = 1, \ldots, k_u(p). \] (46)
Thus, with
\[ \begin{bmatrix} A(\mathcal{L}_1) = [A(l_1) \quad \cdots \quad A(l_{k_u(p)})] \\ Q(\mathcal{L}_1) = A(\mathcal{L}_1) A(\mathcal{L}_1)' = \sum_{h=1}^{k_u(p)} Q(l_h)' \end{bmatrix} \] (47)
we have
\[ S(\mathcal{L}_1) = \frac{1}{p^{N-u}} \|z\|^2 - \frac{T^2}{p^{n}} = \|A(\mathcal{L}_1)^T y\|^2 = \sum_{h=1}^{k_u(p)} S(l_h). \] (48)
Moreover, since \( P(p^{n}) \) is orthogonal, we have
\[ \|y\|^2 - \frac{T^2}{p^{n}} = \sum_{h=1}^{k_u(p)} S(l_h) = S(\mathcal{L}_1) + \sum_{h=k_u(p)+1}^{k_u(p)} S(l_h). \] (49)
When we have \( r \) replicates we must take
\[ S(l_h) = \left\| \left( A(l_h) \otimes \frac{1}{\sqrt{r}} 1^r \right) y \right\|^2, \quad h = 1, \ldots, k_u(p) \] (50)
as well as
\[ S(\mathcal{L}_1) = \left\| \left( A(\mathcal{L}_1) \otimes \frac{1}{\sqrt{r}} 1^r \right) y \right\|^2 \] (51)
to get, assuming that the observations are grouped according to treatments,
\[ \|y\|^2 - \frac{T^2}{p^{n}} = \sum_{h=1}^{k_u(p)} S(l_h) + SSE = S(\mathcal{L}_1) + \sum_{h=k_u(p)+1}^{k_u(p)} S(l_h) + SSE, \] (52)
where
\[ SSE = \| (1_{p^n} \otimes T_r) y \|^2 \] (53)
is the usual sum of squares for the error. We then have a first case in which we replace matrices in
the principal basis of a CJA \( A(p^{n}) \) by their sum, thus obtaining the principal basis \( \left\{ \frac{1}{p^{n}} J_{p^{n}}, Q(\mathcal{L}_1), Q_{k_u(p)+1}, \ldots, Q_{k_u(p)} \right\} \) of a sub-CJA \( A(p^{n}/\mathcal{L}_1) \).
In practice, this merging corresponds to the grouping of the treatments in the blocks. A better
control of the experimental errors is then achieved at the price of not being able to consider separately
the \( S(l_h), h = 1, \ldots, k_u(p) \).
The order of \( l \in \mathcal{L}(p)^N \), in the family of reduced applications in \( \mathcal{L}(p)^N \), is the number of its non null coefficients minus 1. If the order is null, the sole non null coefficient will be 1 and the application is related to the effects of the corresponding factor. These applications will be called effects. Otherwise the applications will be factorial interactions between the factors for which they have non null coefficients. Usually, \( \mathcal{L}_1 \) is chosen so that \( \mathcal{L}_1 \) does not contain effects and as few as possible low order factorial interactions. For \( p = 2 \) and \( p = 3 \) this problem has been studied (for instance, see [1]).

This technique of grouping treatments in blocks is known as confounding. The reason for this is, as shown in (41), that the totals of the different levels of \( l \in \mathcal{L}_1 \) are the sums of block totals. Thus, the differences between levels of \( l \in \mathcal{L}_1 \) are confounded with differences between blocks.

We now relate our results with the usual approach for balanced models with \( N \) factors that cross. Given \( D \subseteq F = \{1, \ldots, N\} \), let \( \chi(D) \) be the family of linear applications whose non null coefficient indexes are in \( D \) and \( \chi(D) \) the family of linear reduced applications whose non null coefficient indexes are in \( D \). \( \chi(D) \) is \( \rho \)-saturated. We put

\[
\mathbf{Q}(D) = \sum_{l \in \chi(D)} \mathbf{Q}(l), \quad \emptyset \subset D \subset F,
\]

thus getting the matrices that, with \( \frac{1}{p^N} I_{p^N} \), constitute the principal basis of a CJA \( A(p^N/D) \). This CJA is related to the partitions

\[
\begin{align*}
\|y\|^2 - \frac{T_2}{p^N} &= \sum_{\emptyset \subset D \subset F} S(D) \\
\|y\|^2 - \frac{T_2}{p^N} &= \sum_{\emptyset \subset D \subset F} S(D) + \text{SSE}
\end{align*}
\]

where

\[
S(D) = \sum_{l \in \chi(D)} S(l), \quad \emptyset \subset D \subset F.
\]

These partitions correspond to the usual definition of factors and interactions. When \( \#(D) = 1 \), \( S(D) \) will be the sum of squares for the effects of the corresponding factor and when \( \#(D) > 1 \), \( S(D) \) will be the sum of squares for the interaction between factors with indexes in \( D \). When \( \#(D) > 1 \), the sums of squares for the factorial interactions between the factors with indexes in \( D \) are merged into the sum of squares for their interaction. For an alternative treatment of this subject, see [7], Sections 2.3 and 7.1.

### 4.1. Example 1

Let us consider an example of confounding. We assume that \( N = 4 \), that \( p = 5 \) and that the reduced applications used to define the blocks are

\[
\begin{align*}
l_1(x) &= x_1 + x_2 + x_3 + x_4, \\
l_2(x) &= x_1 + 2x_2 + 3x_3 + 4x_4.
\end{align*}
\]

Then the other reduced linear applications that are confounded will be

\[
\begin{align*}
3(l_1(x) + l_2(x)) &= x_1 + 4x_2 + 2x_3, \\
2(l_1(x) + 2l_2(x)) &= x_1 + 4x_3 + 3x_4, \\
4(l_1(x) + 3l_2(x)) &= x_1 + 3x_2 + 2x_4, \\
4(l_1(x) + 4l_2(x)) &= x_2 + 2x_3 + 3x_4.
\end{align*}
\]

It is interesting to point out that no effects and no first order interaction are confounded and that the "presence" of the four factors in the confounded factorial interactions is balanced since each of them is present in five of the six interactions, each pair in four of them and each triplet in three of them.

Moreover, the system of equations

\[
\begin{align*}
x_1 + x_2 + x_3 + x_4 &= b_1 \quad (= 0, \ldots, 4), \\
x_1 + 2x_2 + 3x_3 + 4x_4 &= b_2 \quad (= 0, \ldots, 4),
\end{align*}
\]

is...
gives
\[
\begin{align*}
x_3 &= (4(b_1 + b_2))_l + 2x_1 + 3x_3, \\
x_4 &= (2(b_1 + 3b_2))_l + 2x_1 + x_2
\end{align*}
\] (58)
so it is easy to obtain the $5^2 = 25$ blocks.

We conclude by pointing that we have 24 = 25 − 1 degrees of freedom for blocks and 6 confounded factorial interactions. Of these 2 have order 3 and 4 have order 2. For an alternative treatment of this subject, see [7], Sections 2.4 and 7.2.

5. Fractional replicates

We now consider only the treatments $x_1, \ldots, x_{pN/u}$ in a chosen block $[L|b]$ in order to have a fractional replicate $\frac{1}{p^r} \times p^N$. We give to the chosen treatments the first $p^{N-u}$ indexes. To study these models we introduce in $\mathcal{L}[p^N]$ an equivalence relation $\rho_{\mathcal{L}_1}$, putting $l \rho_{\mathcal{L}_1} g$ if $g = cl + l_s$, with $c \in [p]_0$ and $l_s \in \mathcal{L}_1$, where $\mathcal{L}_1$ is the linear space of $\mathcal{L}[p]^N$ spanned by $L$. Since $l_s$ will take a fixed value $b$ for all chosen treatments, we will have $g(x_j) = (c(l(x_j) + b)_j(j = 1, \ldots, p^{N-u})$. Thus, with $\mathcal{C}_s(l)$ the sub-matrix of $\mathcal{C}(l)$ constituted by the first $p^{N-u}$ columns, we see that the rows of $\mathcal{C}_s(g)$ are obtained reordering the rows of $\mathcal{C}_s(l)$. Likewise, with
\[
\mathbf{A}_s(l) = \frac{1}{\sqrt{p^{N-u-1}}} \mathbf{C}_s(l) \mathbf{T}_p,
\] (59)
we see that the columns of $\mathbf{A}_s(g)$ are obtained reordering the columns of $\mathbf{A}_s(l)$. Now, when $r = 1$, we have the sums of squares $S_s(l) = \|\mathbf{A}_s(l) \mathbf{y}\|^2$ and, when $r > 1$, $S_s(l) = \|\left(\mathbf{A}_s(l) \otimes \frac{1}{\sqrt{r}} \mathbf{1}^r\right) \mathbf{y}\|^2$ so that, in both cases,
\[
S_s(l) = S_s(g).
\] (60)

This fact leads to the choice in every $\rho_{\mathcal{L}_1}$ equivalence class of a reduced application to which is attributed the difference between the groups of treatments that correspond to the different values taken by the applications. As we saw, if $l \rho_{\mathcal{L}_1} g$, the groups defined by $l$ are the same than those defined by $g$.

Thus, $L$ must be chosen in such a way that effects and as many as possible low order interactions are isolated in their $\rho_{\mathcal{L}_1}$ equivalence classes. For $p = 2$ and $p = 3$, this problem has been studied (for instance, see [1]).

To study confounding in the case of fractional replicates we complete $L_1 = \{l_1, \ldots, l_u\}$ to obtain a basis $\{l_1, \ldots, l_u, \ldots, l_v, \ldots, l_{N-1}\}$ for $\mathcal{L}[p]^N$. Taking $L_2 = \{l_{u+1}, \ldots, l_v\}$ and $L_3 = \{l_{v+1}, \ldots, l_{N-1}\}$ as well as $\mathcal{L}_j = \mathcal{L}(l_j), j = 1, 2, 3$, we have the sub-spaces $\mathcal{L}_1 \oplus \mathcal{L}_2$ and $\mathcal{L}_2 \oplus \mathcal{L}_3$ given by the direct sum of $\mathcal{L}_1$ and $\mathcal{L}_2$ and of $\mathcal{L}_2$ and $\mathcal{L}_3$. With,
\[
\begin{align*}
g_1 &= \sum_{j=1}^{N} a_{1j} l_j, \\
g_2 &= \sum_{j=1}^{N} a_{2j} l_j
\end{align*}
\] (61)
we have $g_1 \rho_{\mathcal{L}_1} g_2$ if and only if $a_{2j} = (ca_{1j})_j(j = u + 1, \ldots, N)$, with $c \in [p]_0$. Let us establish

**Proposition 2.** $\mathcal{L}_1$ is a $\rho_{\mathcal{L}_1}$ equivalence class. Moreover, there are $k_{N-u}(p)$ classes distinct of $\mathcal{L}_1$, each containing $p^u(p - 1)$ applications. The $\rho_{\mathcal{L}_1}$ equivalence classes are $\rho$-saturated and $(\mathcal{L}_1 \oplus \mathcal{L}_2) \setminus \mathcal{L}_1$ is the union of $k_{u-1}(p)$ such classes.

**Proof.** If $g_1 \in \mathcal{L}_1$ we have $g_1 \rho_{\mathcal{L}_1} g_2$ if and only if $g_2 \in \mathcal{L}_1$ and, since when $g_1, g_2 \in \mathcal{L}_1, g_1 \rho_{\mathcal{L}_1} g_2$ we see that $\mathcal{L}_1$ is a $\rho_{\mathcal{L}_1}$ equivalence class. Moreover, if $g_1 = g_{1,1} + g_{1,2}$ and $g_2 = g_{2,1} + g_{2,2}$ with $g_{1,1}, g_{2,1} \in \mathcal{L}_1$ and $g_{1,2}, g_{2,2} \in \mathcal{L}_2$ we have, as we saw, $g_1 \rho_{\mathcal{L}_1} g_2$ if and only if $g_{1,2} \rho g_{2,2}$. So, the $\rho_{\mathcal{L}_1}$ equivalence classes distinct from $\mathcal{L}_1$ will contain one and only one application from $(\mathcal{L}_1 \oplus \mathcal{L}_2)_r$. Thus, there will be
blocks, we have

and, also with no confounding but with if we have

we see that they will contain \(p^u(p - 1)\) applications. Lastly, if \(g_1 \in L_1 \oplus L_2, g_1 \rho_{L_1} g_2\) when and only when \(g_2 \in L_1 \oplus L_2\) will be \(\rho_{L_1}\)-saturated. The rest of the proof is straightforward. □

Let us give the indexes \(1, \ldots, k_u(p)\) \(k_u(p) + 1, \ldots, k_u(p) + k_{v-u}(p): k_u(p) + 1, \ldots, k_u(p) + k_{N-u}(p)\) to the reduced applications in \(L_1[L_2 : L_2 \oplus L_3]\). The remaining reduced applications will receive the indexes from \(k_u(p) + k_{N-u}(p)\) to \(k_N(p)\). If \(\text{mo}(l)\) is a minimum order application \(\rho_1\) equivalent to \(l\), we have

\[
S(\text{mo}(l)) = S(l) \quad l \in (L_1 \oplus L_2)_r.
\]  

(62)

Reasoning as in the preceding section, we show that

\[
P \left( \frac{1}{pu} \times p^N \right) = \left[ \frac{1}{\sqrt{p^{N-u}}} A_u(l) \right] \quad l \in (L_1 \oplus L_2)_r
\]

(63)

is an orthogonal matrix associated to the CJA \(A \left( \frac{1}{pu} \times p^N \right)\) with principal basis constituted by \(\frac{1}{p^{N-u}} J_{p^{N-u}}\) and the \(A_u(l)\). \(A_u(l)\), \(j = k_u(p) + 1, \ldots, k_u(p) + k_{N-u}(p)\). For a more general discussion of this problem see [7].

Putting \(x_1, x_2\) if \(x_1\) and \(x_2\) are chosen treatments and, whatever \(l \in L_2, l(x_1) = l(x_2)\), we will have an equivalence relation defined in \([L_1 | b_1 | \cap L_2 | b_2]\) whose equivalence classes are the sub-blocks

\[
[L_1, L_2 | b_1, b_2] = [L_1 | b_1 | \cap L_2 | b_2].
\]  

(64)

Moreover, we can reason as above to show that, if the vector \(z\) is now the vector of totals of sub-blocks, we have

\[
S(L_2) = \frac{1}{p^{N-u}} \|z\|^2 - \frac{T^2}{p^{N-u}} \sum_{h=k_u(p)+1}^{k_u(p)+k_{v-u}(p)} S(h),
\]

(65)

When \(r = 1\) and no confounding is carried out we have the partition of sums of squares

\[
\|y\|^2 = \frac{T^2}{p^{N-u}} \sum_{h=k_u(p)+1}^{k_u(p)+k_{v-u}(p)} S(h) = \sum_{h=k_u(p)+1}^{k_u(p)+k_{v-u}(p)} S(mo(h)) + SSE.
\]  

(66)

(67)

Besides this, if there is confounding, with

\[
A_u(L_2) = (A_u(L_{k_u(p)+1}), \ldots, A_u(L_{k_u(p)+k_{v-u}(p)}))
\]

(68)

there is a CJA with principal basis constituted by \(\frac{1}{p^r} J_{p^{r-u}}\), \(Q_u(L_2) = A_u(L_2) A_u(L_2)'\) and the \(Q_u(l_h) = A_u(l_h) A_u(l_h)'\), \(h = k_u(p) + k_{v-u}(p) + 1, \ldots, k_u(p) + k_{N-u}(p)\). Then we will also have the partition, when \(r = 1\),

\[
\|y\|^2 = \frac{T^2}{p^{N-u}} = S_u(L_2) + \sum_{h=k_u(p)+k_{v-u}(p)+1}^{k_u(p)+k_{v-u}(p)+1} S_u(l_h) = S_u(L_2) + \sum_{h=k_u(p)+k_{v-u}(p)+1}^{k_u(p)+k_{v-u}(p)+1} S_u(mo(l_h)),
\]

(69)

and, when \(r > 1\),

\[
\|y\|^2 = \frac{T^2}{p^{r-u}} = S_u(L_2) + \sum_{h=k_u(p)+k_{v-u}(p)+1}^{k_u(p)+k_{v-u}(p)+1} S_u(l_h) + SSE = S_u(L_2) + \sum_{h=k_u(p)+k_{v-u}(p)+1}^{k_u(p)+k_{v-u}(p)+1} S_u(mo(l_h)) + SSE.
\]  

(70)
Let \( X_\alpha(D) = X(D) \cap (L_2 \oplus L_3)_\alpha \). Then we have the partitions
\[
\begin{align*}
\|y\|^2 - \frac{1}{p^{N-u}} &= \sum_{\emptyset \subset D \subseteq F} S_\alpha(D) \\
\|y\|^2 - \frac{1}{p^{N-u}} &= \sum_{\emptyset \subset D \subseteq F} S_\alpha(D) + SSE^*
\end{align*}
\]
where
\[
S_\alpha(D) = \sum_{l \in X_\alpha(D)} S_\alpha(l) = \sum_{l \in X_\alpha(D)} S_\alpha(\text{mo}(l)): \emptyset \subset D \subseteq F.
\]
These last partitions correspond to the usual definition of factors and interactions, as was the case of the complete factorials in the preceding sections. If \( P \) is the family of sets \( D \) such that \( X_\alpha(D) \neq \emptyset \), we have the CJA \( A \left( \frac{1}{p^{N-u}} \times p^N/P \right) \) with principal basis constituted by \( \frac{1}{p^{N-u}} J_{p^N-u} \) and the
\[
Q_\alpha(D) = \sum_{l \in X_\alpha(D)} Q_\alpha(l): D \in P.
\]

5.1. Example 2

We now consider an example of fractional replication with \( N = 3 \) and \( p = 5 \), in which we use
\[
l(x) = x_1 + x_2 + 3x_3
\]
to generate the chosen block. Since
\[
x_1 + x_2 + 3x_3 = b \quad (= 0, \ldots, 4)
\]
gives
\[
x_3 = (2b)_5 + 3x_1 + 3x_2.
\]
It is easy to generate that block. Moreover the sole reduced application in \( L_1 \) is \( l(x) \) and there are
\[
k_{3-1}(5) = \frac{5^3 - 1}{5 - 1} = 6
\]
\( \rho_{L_1} \) classes distinct from \( L_1 \). These classes are
\[
\begin{align*}
\{x_1; x_1 + 3x_2 + 4x_3; x_1 + 4x_2 + 2x_3; x_1 + 2x_2 + x_3; x_2 + 3x_3\}, \\
\{x_2; x_1 + 2x_2 + 3x_3; x_1 + 2x_2 + 3x_3; x_1 + 3x_2 + 3x_3; x_1 + 3x_3\}, \\
\{x_2; x_1 + 4x_2 + 4x_3; x_1 + x_2 + 3x_3; x_1 + x_2 + 2x_3; x_1 + x_2 + 2x_3\}, \\
\{x_2; x_1 + 3x_2 + 3x_3; x_1 + x_2 + 4x_3; x_1 + 2x_2 + 4x_3; x_1 + 2x_2 + 4x_3; x_1 + 2x_2 + 4x_3\}, \\
\{x_2; x_1 + 2x_2; x_1 + 2x_3; x_1 + 3x_2 + 2x_3; x_1 + x_3; x_1 + 2x_3\}, \\
\{x_2; x_1 + 3x_2; x_1 + 2x_3; x_1 + 4x_2 + x_3; x_1 + 2x_2 + 4x_3; x_1 + 2x_2 + 4x_3; x_1 + 2x_2 + 4x_3\}, \\
\{x_2; x_1 + 4x_2; x_1 + 2x_3; x_1 + 3x_2; x_1 + x_2 + 3x_3; x_1 + x_2 + 3x_3; x_1 + 4x_3\}.
\end{align*}
\]
We may choose the first application in each of these classes as its representative. In this way, we may test the three effects and three first order interactions one for each pair of factors. Let \( l_{ij}(x) \) be the \( j \)th application in the \( i \)th class with, for instance, \( l_{2,3}(x) = x_1 + 4x_2 + 3x_3 \). It is easy to check that
\[
l_{ij}(x) = c_{ij}(l_{1}(x) + j(x)) \quad j = 2, 3, 4, 5; \quad i = 1, 2, 3, 4, 5, 6
\]
with, for instance, \( c_{2,3} = 2 \). For the treatments in the block \( \{l|b\} \), we have
\[
l_{ij}(x) = c_{ij}(l_{1}(x) + jb), \quad j = 2, 3, 4, 5; \quad i = 1, 2, 3, 4, 5, 6,
\]
which shows how the values of the representative applications determine the values of other reduced applications in the same \( \rho_{L_1} \) class.
6. Crossing and nesting

Let the \( l_{h,1}^e = \text{mo}(l_{h,1}), \ldots, l_{h,k_h}^e = \text{mo}(l_{h,k_h}) \) be the selected applications for the models \( \frac{1}{p_h} \times p_h^N, h = 1, \ldots, z \), their \( \{ \frac{1}{p_h} \times l_{h,1}^e, \ldots, Q_h(l_{h,1}^e), \ldots, Q_h(l_{h,k_h}^e) \} \), \( h = 1, \ldots, z \), the principal basis, and \( [L_{1,1}^e | b_{1,1}] \) the blocks constituted by the chosen treatments, \( h = 1, \ldots, z \). If \( u_h = 0 \), all the treatments are chosen so that \( [L_{1,1}^e | b_{1,1}] = G[p_h]P_h^N \).

When we cross the models the space of the new treatments will be the cartesian product \( \times_{h=1}^z [L_{1,1}^e | b_{1,1}] \), and we have a model associated to the CJA \( \otimes_{h=1}^z A \left( \frac{1}{p_h} \times P_h^N \right) \). With

\[
\Gamma = \{ h : h_i = 0, \ldots, k_i, i = 1, \ldots, z \}
\]

the principal basis of this CJA will be constituted by the \( Q(h) = \otimes_{i=1}^z Q_{i,j}(i,h_j), h \in \Gamma \), \( (80) \)

where \( Q_{i,j}(i,h_j) = \frac{1}{p_h} \times (l_{h,1}^e)_{i,j} \) and \( Q_{i,j}(i,h_j) = Q_{i,j}(i_h,h_j) = 1, \ldots, k_{j,i} \) \( (i,h_j) \), \( i = 1, \ldots, z \).

If we take \( r > 1 \) replicates, we will have the CJA \( \otimes_{h=1}^z A \left( \frac{1}{p_h} \times P_h^N \right) \ast A(r) \).

Taking \( T_{i,0} = I_1 \) and \( T_{i,1} = T_i, h_j = 1, \ldots, k_i, i = 1, \ldots, z, \) we have \( Q(h) = A(h)A(h)' \), \( h \in \Gamma \), \( (81) \)

where

\[
A(h) = \otimes_{i=1}^z A_{i,j}(i,h_j)
\]

\[
= \otimes_{i=1}^z \left( \frac{1}{p_h} \times C_{i,j}(i,h_j)' \right), \quad h \in \Gamma
\]

\[
= \otimes_{i=1}^z \left( C_{i,j}(i,h_j)' \right) \otimes_{i=1}^z (T_i,h_j)
\]

\[
= C(h)'T(h)
\]

(82)

\[
C(h) = \otimes_{i=1}^z C_{i,j}(i,h_j), \quad h \in \Gamma
\]

\[
T(h) = \otimes_{i=1}^z T(i,h_j), \quad h \in \Gamma
\]

(83)

We may point out that if the treatments in the initial models are grouped in blocks \( [L_{1,1}^e | b_{1,1}], h = 1, \ldots, z \), the blocks in the product model will be the cartesian product of the blocks in the original models. If confounding is not used in one or more of the initial models, all the treatments in those models will constitute an unique block. Moreover, we can reason as in Section 4 to define an orthogonal matrix \( P(p_h^N - u_h) \) associated to the confounding. If \( u_h = v_h \) there will be no confounding for the corresponding initial model and \( P(1) = I_1 \). With

\[
P = \otimes_{h=1}^z P(p_h^N - u_h), \quad (84)\]

and \( z \) the vector of block totals we have the sum of squares for blocks given by

\[
S\left( \otimes_{h=1}^z L_{2,h} \right) = \frac{1}{r \prod_{h=1}^z P_h^N - u_h} \| Pz \|^2 - \frac{T^2}{r \prod_{h=1}^z P_h^N - u_h},
\]

(85)
where, as before, $T$ is the grand total. An alternative expression for this sum of squares will be given by

$$S \left( \bigotimes_{h=1}^{z} \mathcal{L}_{2,h} \right) = \| Q \left( \bigotimes_{h=1}^{z} \mathcal{L}_{2,h} \right) y \|^2,$$

(86)

where $Q \left( \bigotimes_{h=1}^{z} \mathcal{L}_{2,h} \right)$ is obtained adding the $\bigotimes_{h=1}^{z} Q_{e}(h)$, where $Q_{e}(h) = \frac{1}{p_{h}^{n_{h} - u_{h}}} J_{p_{h}^{n_{h} - u_{h}}}^{u_{h}}$ or $Q_{e}(l_{h})$, $k_{u_{h}}(p_{h}) < i \leq k_{u_{h}}(p_{h}) + k_{v_{h} - u_{h}}(p_{h})$ with the exception of $\bigotimes_{h=1}^{z} J_{p_{h}^{n_{h} - u_{h}}}^{u_{h}}$. We now have the sub-CJA $A \left( \bigotimes_{h=1}^{z} \left( \frac{1}{p_{h}^{n_{h}}} \times p_{h}^{n_{h}} \right) / \left( \bigotimes_{h=1}^{z} \mathcal{L}_{2,h} \right) \right)$ of $A \left( \bigotimes_{h=1}^{z} \left( \frac{1}{p_{h}^{n_{h}}} \times p_{h}^{n_{h}} \right) \right) = \bigotimes_{h=1}^{z} A \left( \frac{1}{p_{h}^{n_{h}}} \times p_{h}^{n_{h}} \right)$.

When we consider for the initial models the sub-CJA $A \left( \frac{1}{p_{h}^{n_{h}}} \times p_{h}^{n_{h}} / D_{h} \right)$, corresponding to the classic partitions in sums of squares for factors and interactions, for the final model we will have the CJA $A \left( \bigotimes_{h=1}^{z} \left( \frac{1}{p_{h}^{n_{h}}} \times p_{h}^{n_{h}} \right) \times p_{h}^{n_{h}} / D_{h} \right) = \bigotimes_{h=1}^{z} A \left( \frac{1}{p_{h}^{n_{h}}} \times p_{h}^{n_{h}} / D_{h} \right)$.

While we had to discuss a number of details about the crossing of models, the situation of model nesting is much more straightforward. We can apply directly the results of Sections 2 and 3.

6.1. Example 3

We may use the models in the previous two examples, nesting one of them inside the other. This may be done in a straightforward way since the matrices $A$ for the final model may be obtained directly from those of the initial models. It is interesting to point out that crossing the two models would not be convenient since we would be using a second order factorial interaction for selecting the treatments.

In the next section, we give an example in which models crossing is used.

7. Aggregation and disaggregation

A first case of factor merging occurs in prime basis factorials $p^{N}$. If, with $F = \{1, \ldots, N\}$, we have a disjoint partition

$$F = \bigcup_{j=1}^{w} C_{j},$$

(87)

we can merge the factors in each of the $C_{j}, j = 1, \ldots, w$, into a factor with $p^{|C_{j}|}$ levels, $j = 1, \ldots, w$. Each of the levels of one of the new factors will correspond to a combination of levels of the merged factors. With $V \subseteq W = \{1, \ldots, w\}$, let $D(V)$ be the set of reduced applications with at least a non null coefficient for the factors with indexes in each of the $C_{j}$, with $j \in V$ and null coefficients in the remaining $C_{j}$. If $\#(V) = 1, V = \{j\}$ and the only non null coefficients of $l \in D(V)$ will be for factors in $C_{j}$ and either $l$ is the effects of a sub-factor of the $j$th new factor or a factorial interaction between sub-factors of that factor. If $\#(V) > 1, l$ will be a factorial interaction between sub-factors of the new factors with indexes in $V$. Then, we have

$$Q(V) = \sum_{l \in D(V)} Q(l) = A(V)A(V)^{\prime}, \quad V \subseteq W,$$

(88)

where

$$A(V) = [A(l); l \in D(V)],$$

(89)

and we can apply the results in Section 3.

When we merge factors in a fractional replicate, the procedure is the same. The only difference is that we must work with distinct mo($l_{h}$), $h = k_{u}(p) + 1, \ldots, k_{u}(p) + k_{N-u}(p)$, or, if we also carried out confounding, the mo($l_{h}$), $h = k_{u}(p) + k_{v-u}(p) + 1, \ldots, k_{u}(p) + k_{N-u}(p)$. 
Up to now we have merged factors with the same (prime) number of levels. To overcome this limitation we may merge factors in models obtained through crossing, thus obtaining factors with a number of levels that is the product of the number of levels of the factors in the initial models. Now, some or all of the factors to be merged could have been themselves the result of merging, having thus numbers of levels that are powers of prime numbers. In this way we may obtain factors with arbitrary number of levels. The relevant sub-CJA would be obtained using the procedure of condensation described in Section 2. Moreover, part or all of the initial models could be fractional replicates.

Lastly, given a model associated to a CJA \( \mathcal{A} \), we carry out disaggregation when we replace two or more matrices in the principal basis of \( \mathcal{A} \) by sums of pairwise orthogonal OPMs. Usually, this is the last operation to be applied. Thus, once the models to cross and nest are obtained, we carry out these aggregations and afterwards, if such is the case, we carry out disaggregation.

7.1. Example 4

We shall assume that we have two models each with two factors. In the first [second] model the factors have 2 [3] levels. Assuming that we cross these models and that we aggregate the first with the second and the second with the fourth factors we get a model with two factors with six levels each.

The principal basis for the two initial models are \( \{Q_1(l_{0,1}) : Q_1(x_1) ; Q_2(x_2) ; Q_1(x_1 + x_2)\} \) and \( \{Q_1(l_{0,2}) ; Q_1(z_1) ; Q_1(z_1 + z_2) ; Q_1(z_1 + 2z_2)\} \).

In the final model we have a principal basis constituted by the matrices

\[
Q_0 = Q_1(l_{0,1}) \otimes Q_1(l_{0,2}) \otimes \frac{1}{\sqrt{r}} J_r,
\]

\[
Q_1 = Q_1(l_{0,1}) \otimes Q_1(z_1) \otimes \frac{1}{\sqrt{r}} J_r + Q_1(x_1) \otimes Q_1(l_{0,2}) \otimes \frac{1}{\sqrt{r}} J_r + Q_1(x_1) \otimes Q_1(z_1) \otimes \frac{1}{\sqrt{r}} J_r,
\]

\[
Q_2 = Q_1(l_{0,1}) \otimes Q_1(z_2) \otimes \frac{1}{\sqrt{r}} J_r + Q_1(x_2) \otimes Q_1(l_{0,2}) \otimes \frac{1}{\sqrt{r}} J_r + Q_1(x_2) \otimes Q_1(z_2) \otimes \frac{1}{\sqrt{r}} J_r,
\]

\[
Q_{1 \times 2} = Q_1(l_{0,1}) \otimes Q_1(z_1 + z_2) \otimes \frac{1}{\sqrt{r}} J_r + Q_1(l_{0,1}) \otimes Q_1(z_1 + 2z_2) \otimes \frac{1}{\sqrt{r}} J_r
\]

\[
+ Q_1(x_1) \otimes Q_1(z_2) \otimes \frac{1}{\sqrt{r}} J_r + Q_1(x_1) \otimes Q_1(z_1 + z_2) \otimes \frac{1}{\sqrt{r}} J_r + Q_1(x_1)
\]

\[
\otimes Q_1(z_1 + 2z_2) \otimes \frac{1}{\sqrt{r}} J_r + Q_1(x_2) \otimes Q_1(z_1) \otimes \frac{1}{\sqrt{r}} J_r + Q_1(x_2) \otimes Q_1(z_1)
\]

\[
+ z_2) \otimes \frac{1}{\sqrt{r}} J_r + Q_1(x_2) \otimes Q_1(z_1 + 2z_2) \otimes \frac{1}{\sqrt{r}} J_r + Q_1(x_1 + x_2) \otimes Q_1(l_{0,2})
\]

\[
\otimes \frac{1}{\sqrt{r}} J_r + Q_1(x_1 + x_2) \otimes Q_1(z_1) \otimes \frac{1}{\sqrt{r}} J_r + Q_1(x_1 + x_2) \otimes Q_1(z_2) \otimes \frac{1}{\sqrt{r}} J_r
\]

\[
+ Q_1(x_1 + x_2) \otimes Q_1(z_1 + z_2) \otimes \frac{1}{\sqrt{r}} J_r \otimes Q_1(x_1 + x_2) \otimes Q_1(z_1 + 2z_2) \otimes \frac{1}{\sqrt{r}} J_r.
\]

These orthogonal projection matrices have ranks

\[
g_0 = 1 \times 1,
\]

\[
g(1) = 1 \times 2 + 1 \times 1 + 1 \times 2 = 5,
\]

\[
g(2) = 1 \times 2 + 1 \times 1 + 1 \times 2 = 5,
\]

\[
g(1 \times 2) = 1 \times 2 + 1 \times 2 + 1 \times 2 + 1 \times 2 + 1 \times 2 + 1 \times 2 + 1 \times 2 + 1 \times 2 +
\]

\[
+ 1 \times 1 + 1 \times 2 + 1 \times 2 + 1 \times 2 + 1 \times 2 = 25.
\]

Thus the sum of squares for the two six level factors will, as expected, have five degrees of freedom and be given by
\[
S(1) = \|Q(1)y\|^2 = \left\langle \left( A(l_{0,1})' \otimes A(z_1)' \otimes \frac{1}{\sqrt{r}} 1_r \right) y, y \right\rangle \\
+ \left\langle \left( A(x_1)' \otimes A(l_{0,2})' \otimes \frac{1}{\sqrt{r}} 1_r \right) y, y \right\rangle \\
+ \left\langle \left( A(x_1)' \otimes A(z_1)' \otimes \frac{1}{\sqrt{r}} 1_r \right) y, y \right\rangle,
\]
\[
S(2) = \|Q(2)y\|^2 = \left\langle \left( A(l_{0,1})' \otimes A(z_2)' \otimes \frac{1}{\sqrt{r}} 1_r \right) y, y \right\rangle \\
+ \left\langle \left( A(x_2)' \otimes A(l_{0,2})' \otimes \frac{1}{\sqrt{r}} 1_r \right) y, y \right\rangle \\
+ \left\langle \left( A(x_2)' \otimes A(z_2)' \otimes \frac{1}{\sqrt{r}} 1_r \right) y, y \right\rangle.
\]

Likewise we will have the sum of squares \(S(1 \times 2)\) for the interaction with 25 degrees of freedom. While \(S(1)\) and \(S(2)\) are the sum of three terms, \(S(1 \times 2)\) are the sum of thirteen terms. If we assume the models to have fixed effects we can treat globally the hypothesis of absence of effects and interactions. Then with \(S = \|A^t y\|^2\) and \(g = 36(r - 1)\) we will have the \(F\) tests statistics
\[
\mathcal{F}(1) = \frac{36(r-1)S(1)}{5} \\
\mathcal{F}(2) = \frac{36(r-1)S(2)}{5} \\
\mathcal{F}(1 \times 2) = \frac{36(r-1)S(1 \times 2)}{25} 
\]
with 5 and \(g, 5\) and \(g, 25\) and \(g\) degrees of freedom.

We could also consider the hypothesis of absence of effects and interactions (for the six level factors) as the interaction of hypothesis to be tested separately. For instance the hypothesis of absence of effects of the first (six) factors holds if and only if
\[
\|Q(1)y\|^2 = \|(Q(l_{0,1}) \otimes Q(z_1) \otimes \frac{1}{\sqrt{r}} 1_r) y\|^2 + \|(Q(x_1) \otimes Q(l_{0,2}) \otimes \frac{1}{\sqrt{r}} 1_r) y\|^2 + \|(Q(x_1) \otimes Q(z_1) \otimes \frac{1}{\sqrt{r}} 1_r) y\|^2 = 0.
\]

Thus, this hypothesis holds if and only if the
\[
H_{0,1}(1) : \left( A(l_{0,1})' \otimes A(z_1)' \otimes \frac{1}{\sqrt{r}} 1_r \right) \mu = 0,\\
H_{0,2}(1) : \left( A(x_1)' \otimes A(l_{0,2})' \otimes \frac{1}{\sqrt{r}} 1_r \right) \mu = 0,\\
H_{0,3}(1) : \left( A(x_1)' \otimes A(z_1)' \otimes \frac{1}{\sqrt{r}} 1_r \right) \mu = 0,
\]
where \(\mu\) is the mean vector, simultaneously hold, the ranks of of the matrices that define these hypothesis are 2, 1, and 2. So, the corresponding \(F\) tests statistics will be
\[
\mathcal{F}_1(1) = \frac{36(r-1)}{2} \left\langle \left( A(l_{0,1})' \otimes A(z_1)' \otimes \frac{1}{\sqrt{r}} 1_r \right) y, y \right\rangle \\
\mathcal{F}_2(1) = \frac{36(r-1)}{1} \left\langle \left( A(x_1)' \otimes A(l_{0,2})' \otimes \frac{1}{\sqrt{r}} 1_r \right) y, y \right\rangle \\
\mathcal{F}_3(1) = \frac{36(r-1)}{2} \left\langle \left( A(x_1)' \otimes A(z_1)' \otimes \frac{1}{\sqrt{r}} 1_r \right) y, y \right\rangle
\]
so that
\[
\mathcal{F}(1) = \frac{2\mathcal{F}_1(1) + \mathcal{F}_2(1) + 2\mathcal{F}_3(1)}{5}.
\]

Thus, testing separately the sub-hypothesis we may detect significant results that would not be found through global testing.

Let us now assume that we had used the reduced linear applications \(x_1 + x_2\) and \(z_1 + 2z_2\) to generate blocks. For each pair of values \((b_1, b_2), b_1 = 0, 1\) and \(b_2 = 0, 1, 2\) we will have a block. The composition of the blocks will be
The sum of squares for blocks will have 5 degrees of freedom and be given by

\[
\text{SQBl} = \left\| (\mathbf{A}(x_1 + x_2)' \otimes \mathbf{A}(l_{0,2})' \otimes \frac{1}{\sqrt{r}} \mathbf{1}_r) \mathbf{y} \right\|^2 \\
+ \left\| (\mathbf{A}(l_{0,1})' \otimes \mathbf{A}(z_1 + 2z_2)' \otimes \frac{1}{\sqrt{r}} \mathbf{1}_r) \mathbf{y} \right\|^2 \\
+ \left\| (\mathbf{A}(x_1 + x_2)' \otimes \mathbf{A}(z_1 + 2z_2)' \otimes \frac{1}{\sqrt{r}} \mathbf{1}_r) \mathbf{y} \right\|^2,
\]

(94)

these terms being deleted from \(S(1 \times 2)\) which now has only 20 degrees of freedom.

As before, we may test sub-hypothesis for the effects of the two six levels factors and interactions. For the interaction we had 13 sub-hypothesis and now we only have 10.

All the examples of designs of experiments are described in http://pessoa.fct.unl.pt/fmig/papers/laa1.

References