Inequalities for Singular Values and Traces

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ABSTRACT

Let \( A_1, A_2, \ldots, A_m \) be \( n \times n \) matrices over the complex field \( \mathbb{C} \), and \( \alpha_1, \alpha_2, \ldots, \alpha_m \) be positive real numbers. It is proved that if \( \sum_{j=1}^{m} \alpha_j \geq 1 \) and if the \( A_j \)'s are nonnegative definite, then \( |\text{tr} \prod_{j=1}^{m} A_j^{\alpha_j} | \leq \prod_{j=1}^{m} (\text{tr} A_j)^{\alpha_j} \) and equality occurs if and only if (a) for \( \sum_{j=1}^{m} \alpha_j = 1 \), all \( A_j \)'s are scalar multiples of one another, (b) for \( \sum_{j=1}^{m} \alpha_j > 1 \), all \( A_j \)'s are scalar multiples of \( A_1 \) and are of rank 1. This result generalizes many classical inequalities and gives a multivariate version of the recent paper by Magnus (1987). The above inequality can be generalized further: Let \( \sigma_1(C) \geq \sigma_2(C) \geq \cdots \geq \sigma_n(C) \) be singular values of an \( n \times n \) matrix \( C \) over \( \mathbb{C} \). Then for all \( k = 1, 2, \ldots, n \), \( \sum_{j=1}^{k} \sigma_j(\prod_{i=1}^{j} A_j) \leq \sum_{i=1}^{k} \prod_{j=1}^{i} \sigma_i(A_j) \leq \prod_{j=1}^{m} (\sum_{i=1}^{k} \sigma_i(A_j)^{\alpha_j})^{\alpha_j} \). \( \leq \sum_{j=1}^{m} (\sum_{i=1}^{k} \alpha_j)^{\frac{k-1}{\alpha_j}}(\sigma_i(A_j))^{\alpha_j} \).

1. INTRODUCTION

Throughout the paper, \( M_{n \times n} \) will denote the set of all \( n \times n \) matrices over the complex field \( \mathbb{C} \).

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Magnus (1987) obtained the following result:

**Lemma 1.** Let \( p > 1, q = p/(p - 1), \) and \( A \) be a nonzero nonnegative definite (n.n.d.) matrix in \( M_{n \times n} \). The \( \text{tr}(AX) \leq (\text{tr}A^p)^{1/p} \) for all n.n.d. \( X \in M_{n \times n} \) with \( \text{tr} X^q = 1 \), and equality occurs if and only if \( X^q = A^q/\text{tr}A^q \).

As an immediate consequence of Lemma 1, Magnus (1987, Theorem 5) proved the following result of Thompson (1965).

**Lemma 2.** Let \( A_1, A_2 \) be n.n.d. matrices in \( M_{n \times n} \), and let \( \alpha_1, \alpha_2 \) be positive real numbers such that \( \alpha_1 + \alpha_2 = 1 \). Then

\[
\text{tr}(A_1^{\alpha_1}A_2^{\alpha_2}) \leq (\text{tr}A_1)^{\alpha_1}(\text{tr}A_2)^{\alpha_2},
\]

and equality occurs if and only if \( A_1, A_2 \) are scalar multiples of each other.

In this paper, we shall generalize the above result from \( A_1, A_2 \) to \( A_1, A_2, \ldots, A_m \) [Theorem 3(a)]. We then extend Theorem 3(a) to include the \( \alpha_1 \)'s with \( \sum_{j=1}^{m} \alpha_j > 1 \) [Theorem 3(b)]. Theorem 3 is best in that if \( \alpha_1, \alpha_2 > 0 \) with \( \alpha_1 + \alpha_2 < 1 \), then (1.1) may not hold. The inequality in Theorem 3 can be generalized to that for sums and products of the singular values of \( A_1, A_2, \ldots, A_m \) [Theorem 1]. A formula (Theorem 2) that involves only sums of singular values can also be obtained. This theorem gives a matrix version of Minkowski's inequality and generalizes a result of Chen (1990). Among others, we shall use both Lemma 2 and a result of Kiers and Ten Berge (1989) to furnish a proof of Theorem 3. Through various choices of \( \alpha_1, \alpha_2, \ldots, \alpha_m \), we are able to generalize the classical Hölder's inequality and certain mean inequalities for \( y_1, y_2, \ldots, y_m > 0 \) to that for \( n \times n \) n.n.d. matrices \( A_1, A_2, \ldots, A_m \). Theorem 3 can also be used to obtain a multivariate version of the main result in Magnus (1987): \( (\text{tr}A^p)^{1/p} = \max(\text{tr}(AX): X \text{ is n.n.d., } \text{tr} X^q = 1) \).

2. **INEQUALITIES FOR SINGULAR VALUES**

Certain results on majorization will be used. For recent references on majorization and inequalities, we refer the reader to Marshall and Olkin (1979), Bellman (1980), and Wong (1986).

Let \( C \in M_{n \times n} \). Then \( \sigma_1(C) \geq \sigma_2(C) \geq \cdots \geq \sigma_n(C) \) will denote the singular values of \( C \), and \( \sigma(C) \) will denote the column vector \( (\sigma_1(C))_{i=1}^n \) in the \( n \)-dimensional Euclidean space \( \mathbb{R}^n \); the \( x_{[i]} \)'s will denote the rearrangement
of the $x_i$'s with $x_{[1]} \geq x_{[2]} \geq \cdots \geq x_{[n]}$. Let $x = (x_i), y = (y_i) \in \mathbb{R}^n$. Then $x$ is said to be weakly majorized by $y$, written as $x \prec_w y$, if $\sum_{i=1}^k x_{[i]} \leq \sum_{i=1}^k y_{[i]}$ for all $k = 1, 2, \ldots, n$; $x$ is said to be majorized by $y$ if $x \prec_w y$ and if $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$. By a result of Gel'fand and Naimark (1950) [or p. 248 of Marshall and Olkin (1979)], we obtain

$$\sigma(A_1 A_2 \cdots A_m) <_w \sigma(A_1) * \sigma(A_2) * \cdots * \sigma(A_m). \quad (2.1)$$

where each $A_i \in M_{n \times n}$ and $*$ is the Hadamard product (or pointwise product if we view $x \in \mathbb{R}^n$ as a function on $\{1, 2, \ldots, n\}$). By a result of Fan (1951) [or p. 214 of Ando, Horn, and Johnson (1987)],

$$\sigma(A_1 + A_2 + \cdots + A_m) <_w \sigma(A_1) + \sigma(A_2) + \cdots + \sigma(A_m). \quad (2.2)$$

**Theorem 1.** Let $A_1, A_2, \ldots, A_m \in M_{n \times n}$, $\alpha_1, \alpha_2, \ldots, \alpha_m > 0$ with $\sum_{j=1}^n \alpha_j = 1$ and $k \in \{1, 2, \ldots, n\}$. Then

$$\sum_{i=1}^k \sigma_i \left( \prod_{j=1}^m A_j \right) \leq \sum_{i=1}^k \prod_{j=1}^m \sigma_i(A_j) \leq \prod_{i=1}^k \left( \sum_{j=1}^m \sigma_i(A_j) \right)^{1/\alpha_j} \quad \leq \sum_{j=1}^m \left( \sum_{i=1}^k \alpha_j [\sigma_i(A_j)]^{1/\alpha_j} \right).$$

**Proof.** By (2.1),

$$\sum_{i=1}^k \sigma_i \left( \prod_{j=1}^m A_j \right) \leq \sum_{i=1}^k \prod_{j=1}^m \sigma_i(A_j). \quad (2.3)$$

By Hölder's inequality [see e.g. Hardy, Littlewood, and Polya (1952, p. 22)],

$$\sum_{i=1}^k \prod_{j=1}^m \sigma_i(A_j) \leq \prod_{j=1}^m \left( \sum_{i=1}^k [\sigma_i(A_j)]^{1/\alpha_j} \right)^{\alpha_j}. \quad (2.4)$$
By the mean inequality [see e.g. Mitrinovic (1970, p. 100)],

$$\sum_{j=1}^{m} \left( \frac{1}{\alpha_j} \sum_{i=1}^{k} \sigma_i(A_j) \right)^{1/\alpha_j} \leq \sum_{j=1}^{m} \left( \sum_{i=1}^{k} \alpha_j \sigma_i(A_j) \right)^{1/\alpha_j}. \quad (2.5)$$

By (2.3)–(2.5), we obtain the desired result. 

The corresponding result of Chen (1990) is a special case of Theorem 1 with each $\alpha_i = 1/m$.

**Theorem 2.** Let $A_1, A_2, \ldots, A_m \in M_{n \times n}$, $p > 1$, and $k \in \{1, 2, \ldots, n\}$. Then

$$\left( \sum_{i=1}^{k} \left( \sum_{j=1}^{m} \sigma_i(A_j) \right)^{\frac{p}{r}} \right)^{1/p} \leq \left( \sum_{i=1}^{m} \left( \sum_{j=1}^{k} \sigma_i(A_j) \right)^{\frac{p}{r}} \right)^{1/p} \leq \sum_{i=1}^{m} \left( \sum_{j=1}^{k} \sigma_i(A_j) \right)^{\frac{p}{r}}. \quad (2.6)$$

**Proof.** By (2.2), for any increasing convex function $g$ on $[0, \infty)$,

$$(g(\sigma_i(A_1 + A_2 + \cdots + A_m))) \leq (g(\sigma_i(A_1) + \sigma_i(A_2) + \cdots + \sigma_i(A_m))). \quad (2.7)$$

Let $g(t) = t^p$ for $t \geq 0$. Then (2.6) is

$$\sum_{i=1}^{k} \left( \sum_{j=1}^{m} \sigma_i(A_1 + \cdots + A_m) \right)^{p} \leq \sum_{i=1}^{k} \left( \sum_{j=1}^{m} \sigma_i(A_1) + \sigma_i(A_2) + \cdots + \sigma_i(A_m) \right)^{p}. \quad (2.8)$$

By Minkowski’s inequality [see e.g. Marshall and Olkin (1979, p. 459)],

$$\left( \sum_{i=1}^{k} \left( \sigma_i(A_1) + \sigma_i(A_2) + \cdots + \sigma_i(A_m) \right) \right)^{1/p} \leq \sum_{j=1}^{m} \left( \sum_{i=1}^{k} \sigma_i(A_j) \right)^{1/p}. \quad (2.9)$$

By (2.7) and (2.8), we obtain the desired inequality.
Two remarks are in order:

**Remark 1.** By varying $g$ in (2.6), we can obtain various inequalities.

**Remark 2.** If the $A_j$'s in Theorem 2 are n.n.d., then Theorem 2 gives rise to the matrix version of Minkowski's inequality:

$$
\left[ \text{tr} \left( \sum_{j=1}^{m} A_j \right)^p \right]^{1/p} \leq \sum_{j=1}^{m} (\text{tr} A_j_j)^{1/p}.
$$

By Theorem 6 in Section 3 (or even Lemma 1 above), equality occurs in (2.9) if and only if all $A_j$'s are scalar multiples of one another.

3. INEQUALITIES FOR TRACES

We shall now prove the main result of this paper.

**Theorem 3.** Let $A_1, A_2, \ldots, A_m$ be nonzero n.n.d. matrices in $M_{n \times n}$ and $\alpha_1, \alpha_2, \ldots, \alpha_m > 0$.

(a) Suppose that $\sum_{i=1}^{m} \alpha_i = 1$. Then

$$
\left| \text{tr} \left( \prod_{j=1}^{m} A_j^\alpha_j \right) \right| \leq \prod_{j=1}^{m} (\text{tr} A_j)^{\alpha_j},
$$

and equality occurs if and only if all $A_j$'s are scalar multiples of $A_1$.

(b) Suppose that $\sum_{i=1}^{m} \alpha_i > 1$. Then (3.1) holds and equality occurs if and only if all $A_j$'s are scalar multiples of $A_1$ and $r(A_j) = 1$.

**Proof.** (a): For $C = (c_{ij}) \in M_{n \times n}$, $|\text{tr} C| \leq \sum_{i=1}^{n} |c_{ii}| \leq \sum_{i=1}^{n} \sigma_i(C)$. So

$$
\left| \text{tr} \left( \prod_{j=1}^{m} A_j^\alpha_j \right) \right| \leq \sum_{i=1}^{n} \sigma_i \left( \prod_{j=1}^{m} A_j^\alpha_j \right),
$$

(3.2)
By Theorem 1,
\[
\sum_{i=1}^{n} \sigma_i \left( \prod_{j=1}^{m} A_{ij}^{j} \right) \leq \prod_{j=1}^{m} \left\{ \sum_{i=1}^{n} \left[ \sigma_i (A_j^{ij}) \right]^{1/\alpha_j} \right\}^{\alpha_j}.
\] (3.3)

Since \(A_j\) is n.n.d., \(\sigma_i(A_j^{ij}) = \lambda_i(A_j^{ij}) = \lambda_i(A_j)^{\alpha_j}\), where \(\lambda_i(A_j)\) is the \(i\)th largest eigenvalues of \(A_j\). Thus
\[
\prod_{j=1}^{m} \left\{ \sum_{i=1}^{n} \left[ \sigma_i (A_j^{ij}) \right]^{1/\alpha_j} \right\}^{\alpha_j} = \prod_{j=1}^{m} (\text{tr} A_j)^{\alpha_j}.
\] (3.4)

By (3.2), (3.3), and (3.4), we obtain (3.1).

Now for the occurrence of equality in (3.1), let \(j = 1, 2, \ldots, m\) and write
\[
A_j = P_j D_j P_j^*,
\] (3.5)
where \(P_j\) is unitary and \(D_j = (\delta_{ik} \lambda_k(A_j))\) is diagonal. Thus
\[
|\text{tr}(A_1 \alpha_1 A_2 \alpha_2 \cdots A_m \alpha_m)| = \text{tr}(P_m^* P_1^* D_1 \alpha_1 P_2 \alpha_2 D_2 \alpha_2 P_3 \alpha_3 \cdots P_m^* P_m D_m \alpha_m).
\] (3.6)

By a complex version of Theorem 5 of Kiers and Ten Berge (1989),
\[
|\text{tr}(P_m^* P_1 D_1 \alpha_1 P_2 \alpha_2 D_2 \alpha_2 P_3 \alpha_3 \cdots P_m^* P_m D_m \alpha_m)| \leq \text{tr}(D_1 \alpha_1 D_2 \alpha_2 \cdots D_m \alpha_m),
\] (3.7)
and equality occurs (if and) only if
\[
P_m^* P_1 = \pm N_m M_1^*, \quad P_j^* P_{j-1} = N_{j-1} M_j^*, \quad j = 2, 3, \ldots, m,
\] (3.8)
for some unitary matrices \(N_j\), \(M_j\), and \(L_j\) satisfying
\[
N_j C = M_j C = L_j C,
\] (3.9)
where
\[
C = (I_r, 0)', \quad L_j D_j \alpha_j = D_j \alpha_j L_j, \quad r = \min_{1 \leq j \leq m} r(D_j \alpha_j).
\] (3.10)
Since the product of two diagonal matrices is itself diagonal, (1.1) gives

$$\text{tr}(D_1^{a_1}D_2^{a_2} \cdots D_m^{a_m}) \leq \prod_{j=1}^{m} (\text{tr} D_j)^{a_j} = \prod_{j=1}^{m} (\text{tr} A_j)^{a_j},$$

and equality occurs if and only if for any $k = 2, 3, \ldots, m$,

$$D_k = a_k D_1 \quad \text{for some } a_k > 0.$$

So $r = r(D_j)$ for each $j = 1, 2, \ldots, m$. Write

$$N_1 = \begin{pmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{pmatrix}, \quad L_1 = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix},$$

where $N_{11}$ and $L_{11}$ are $r \times r$ matrices. By (3.9) and (3.10),

$$N_{11} = L_{11}, \quad N_{21} = L_{21}.$$

Since $L_1$ commutes with $D_1^{a_1}$, it commutes with

$$D_1 = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix},$$

where $D$ is a nonsingular diagonal matrix. Thus

$$DL_{11} = L_{11}D, \quad L_{12} = 0, \quad L_{21} = 0.$$

By (3.15),

$$L_1 = \begin{pmatrix} L_{11} & 0 \\ 0 & L_{22} \end{pmatrix}.$$

Since $L_1$ is unitary,

$$L_{11}^{*}L_{11} = I_r = L_{11}L_{11}^{*}, \quad L_{22}^{*}L_{22} = I_{n-r} = L_{22}L_{22}^{*}.$$
Since $N_1$ is unitary, $N_1 N_1^* = I_n$ and therefore
\[ N_{11} N_{11}^* + N_{12} N_{12}^* = I_r. \]  
(3.18)

By (3.13), (3.17), and (3.18), $N_{12} N_{12}^* = 0$, whence $N_{12} = 0$. Thus
\[ N_1 = \begin{pmatrix} N_{11} & 0 \\ 0 & N_{22} \end{pmatrix} = \begin{pmatrix} L_{11} & 0 \\ 0 & L_{22} \end{pmatrix}. \]  
(3.19)

By (3.13), (3.15), and (3.19), $N_1 D_1 = D_1 N_1$. By the same argument, we obtain
\[ N_j D_1 = D_1 N_j, \quad M_j D_1 = D_1 M_j. \]  
(3.20)

By (3.5), (3.12), and (3.8),
\[
A_j = a_j P_j D_1 P_j^* \\
= a_j P_1 P_2^* P_3^* \cdots P_{j-1}^* P_j D_1 P_j^* P_{j-1} P_{j-1}^* \cdots P_2 P_2^* P_1 P_1^* \\
= a_j P_1 N_1 M_2^* N_2 M_3^* \cdots N_{j-1} M_j^* D_1 M_j N_j^* \cdots M_3 N_2^* M_2 N_1^* P_1^*.
\]

So by (3.20),
\[
A_j = a_j P_1 D_1 N_1 M_2^* N_2 M_3^* \cdots N_{j-1} M_j^* M_j N_j^* \cdots M_3 N_2^* M_2 N_1^* P_1^*.
\]

Since the $N_j$'s and $M_j$'s are unitary,
\[ A_j = a_j P_1 D_1 P_1^* = a_j A_1. \]  
(3.21)

Thus equality occurs in (3.1) only if (3.21) holds. It is easy to prove that (3.21) implies that equality occurs in (3.1).

(b): Let $\alpha = \sum_{j=1}^{m} \alpha_j$. Then by (a),
\[
\left| \text{tr} \left( \prod_{j=1}^{m} A_j^{\alpha} \right) \right| = \left| \text{tr} \left( \prod_{j=1}^{m} (A_j^{\alpha})^{\alpha_j/\alpha} \right) \right| \leq \prod_{j=1}^{m} \left( \text{tr} A_j^{\alpha_j/\alpha} \right),
\]
and equality occurs if and only if the $A_j^\alpha$'s are scalar multiples of one another.
Note now that for any nonzero n.n.d. matrix $A$ in $M_{n \times n}$,

$$\text{tr} A^\alpha \leq (\text{tr} A)^\alpha,$$  \hspace{1cm} (3.22)

and equality occurs if and only if $A$ is of rank 1. So (3.1) holds, and equality occurs only if all $A_j$'s are scalar multiples of one another and are of rank 1. 

We note that Chen (1988) obtained a special case of Theorem 3(b) where all $\alpha_j = 1$. By Theorem 3 and the arithmetic–geometric-mean inequality [see p. 455 of Marshall and Olkin (1979)], we obtained sharper lower and upper bounds for $\prod_{j=1}^m (\text{tr} A_j)^{\alpha_j}$:

**Theorem 4.** Let $A_1, A_2, \ldots, A_m$ be nonzero n.n.d. matrices in $M_{n \times n}$ and $\alpha_1, \alpha_2, \ldots, \alpha_m$ be positive real numbers.

(a) Suppose that $\sum_{j=1}^m \alpha_j = 1$. Then

$$\left| \text{tr} \left( \prod_{j=1}^m A_j^{\alpha_j} \right) \right| \leq \prod_{j=1}^m (\text{tr} A_j)^{\alpha_j} \leq \sum_{j=1}^m \alpha_j \text{tr} A_j,$$  \hspace{1cm} (3.23)

and equality in the right-hand side occurs if and only if all $\text{tr} A_j$'s are equal; hence equality occurs in the left-hand side and in the right-hand side if and only if all $A_j$'s are equal.

(b) Suppose that $\alpha = \sum_{j=1}^m \alpha_j > 1$. Then

$$\left| \text{tr} \left( \prod_{j=1}^m A_j^{\alpha_j} \right) \right| \leq \prod_{j=1}^m (\text{tr} A_j)^{\alpha_j} \leq \left( \sum_{j=1}^m \frac{\alpha_j}{\alpha} \text{tr} A_j \right)^{\alpha},$$  \hspace{1cm} (3.24)

and equality in the right-hand side occurs if and only if all $\text{tr} A_j$'s are equal; hence equality occurs in the left-hand side and in the right-hand side if and only if all $A_j$'s are equal and are of rank 1.

Note that in (3.24): (a) if each $\alpha_j = 1$, then the inequality in the right-hand side is nothing but the matrix version of the geometric–arithmetic-mean inequality; (b) if $\alpha < 1$, then the inequality in the right-hand side holds; but the inequality in the left-hand side may not hold.
THEOREM 5. Let $i = 1, 2, \ldots, p, A_{i1}, A_{i2}, \ldots, A_{im}$ be nonzero n.n.d. matrices in $M_{n \times n}$, and $\alpha_1, \alpha_2, \ldots, \alpha_m$ be positive numbers such that $\sum_{j=1}^{m} \alpha_j = 1$. Then

$$\sum_{i=1}^{p} \left| \text{tr} \left( \prod_{j=1}^{m} A_{ij} \right) \right| \leq \prod_{j=1}^{m} \left( \sum_{i=1}^{p} \text{tr} A_{ij}^{1/\alpha_j} \right)^{\alpha_j}. \quad (3.25)$$

Proof. Let

$$b_j = \left( \sum_{i=1}^{p} \text{tr} A_{ij}^{1/\alpha_j} \right)^{-1}, \quad B_{ij} = b_j A_{ij}^{1/\alpha_j}. \quad (3.26)$$

Then by Theorem 4,

$$\left| \text{tr} \left( \prod_{j=1}^{m} B_{ij}^{\alpha_j} \right) \right| \leq \sum_{j=1}^{m} \alpha_j \text{tr} B_{ij}. \quad (3.27)$$

By (3.26) and (3.27),

$$\left| \text{tr} \left( \prod_{j=1}^{m} A_{ij} \right) \right| \prod_{j=1}^{m} b_j \leq \sum_{j=1}^{m} \alpha_j b_j \text{tr} A_{ij}^{1/\alpha_j},$$

whence

$$\sum_{i=1}^{p} \left| \text{tr} \left( \prod_{j=1}^{m} A_{ij} \right) \right| \prod_{j=1}^{m} b_j \leq \sum_{i=1}^{p} \sum_{j=1}^{m} \alpha_j b_j \text{tr} A_{ij}^{1/\alpha_j}$$

$$= \sum_{j=1}^{m} \alpha_j b_j \left( \sum_{i=1}^{p} \text{tr} A_{ij}^{1/\alpha_j} \right) = 1;$$

hence (3.25) follows from (3.26). \hfill \Box

Among other things, the proof of our main result, Theorem 3, requires Lemma 2 or, equivalently, Lemma 1, a result of Magnus (1987). In fact we
can use Theorem 3 to generalize Lemma 1:

**THEOREM 6.** Let $A$ be a nonzero n.n.d. matrix in $M_{n \times n}$, $p_1, p_2, \ldots, p_m$ be positive real numbers, and $p = p_1$.

(a) Suppose that $\sum_{i=1}^{m} 1/p_i = 1$. Then for any n.n.d. $X_2, \ldots, X_m$ with each $\text{tr } X_j^{p_j} = 1$,

$$\left| \text{tr}(AX_2X_3 \cdots X_m) \right| \leq \left[ \text{tr}(A^n) \right]^{1/p},$$

and equality occurs if and only if each $X_j^{p_j} = A^n/\text{tr } A^n$.

(b) Suppose that $\sum_{i=1}^{m} 1/p_i > 1$ and $r(A) = 1$. Then the conclusions of (a) still hold.

Theorem 3 can also be derived easily from Theorem 6. So Theorem 6 is just another way of stating Theorem 3. It is clear that Theorem 6 generalizes Lemma 1 and gives a representation of $(\text{tr } A^n)^{1/p}$ in terms of several variables. From this aspect, our paper can be viewed as a multivariate version of the paper by Magnus (1987) and generalizes all results therein.

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