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Circulant Preconditioners for Solving Differential Equations with Multidelays

XIAO-QING JIN* AND SIU-LONG LEI
Faculty of Science and Technology, University of Macau
Macau, P.R. China

YI-MIN WEI†
Department of Mathematics, Fudan University
Shanghai, 200433, P.R. China

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Abstract—We consider the solution of differential equations with multidelays by using boundary value methods (BVMs). These methods require the solution of some nonsymmetric, large and sparse linear systems. The GMRES method with the Strang-type block-circulant preconditioner is proposed to solve these linear systems. If an A_{k_1,k_2}-stable BVM is used, we show that our preconditioner is invertible and the spectrum of the preconditioned matrix is clustered. It follows that when the GMRES method is applied to solving the preconditioned systems, the method would converge fast. Numerical results are given to show the effectiveness of our methods.

Keywords—Differential equation with multidelays, BVM, Block-circulant preconditioner, GMRES method.

1. INTRODUCTION

In this paper, we consider the solution of a differential equation with multidelays

$$\begin{aligned} \mathbf{y}'(t) &= J_n\mathbf{y}(t) + D_n^{(1)}\mathbf{y}(t-\tau_1) + \cdots + D_n^{(s)}\mathbf{y}(t-\tau_s) + \mathbf{f}(t), \qquad && t \ge t_0, \\ \mathbf{y}(t) &= \boldsymbol{\phi}(t), && t \le t_0, \end{aligned} \tag{1}$$

by boundary value methods (BVMs), where $\mathbf{y}(t)$, $\mathbf{f}(t)$, $\boldsymbol{\phi}(t) : \mathbb{R} \to \mathbb{R}^n$; J_n, $D_n^{(1)}, \ldots, D_n^{(s)} \in \mathbb{R}^{n\times n}$, and $\tau_1, \ldots, \tau_s > 0$ are some rational numbers. Such kind of equation appears in many applications [1,2]. The BVMs that we used are relatively new numerical methods for solving ordinary differential equations (ODEs), which is based on the linear multistep formulae, see [1]. The advantage in using BVMs over classical initial value methods (IVMs) comes from the stability properties of BVMs although IVMs, where the system of equations can be solved easily

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by forward recursions, are more efficient than BVMs, see for instance [1]. By applying a BVM, the discrete solution of (1) is given by the solution of a linear system $M\mathbf{y} = \mathbf{b}$.

Recently, Bertaccini in [3] proposed to use BVMs with Krylov subspace methods [4] to solve initial value problems (IVPs) of ODEs. In order to speed up the convergence rate of Krylov subspace methods, he proposed two circulant preconditioners. The use of circulant preconditioners for solving structured linear systems has been studied extensively since 1986, see [5,6]. In [7], Chan, Ng and Jin proposed a new preconditioner called the Strang-type block-circulant preconditioner for solving linear systems from IVPs. The Strang-type preconditioner was also used to solve linear systems from differential-algebraic equations and delay differential equations, see [8,9]. In this paper, we will use the Strang-type preconditioner for solving differential equations with multidelays.

The paper is organized as follows. In Section 2, we recall BVMs and propose the Strang-type block-circulant preconditioner. We introduce the stability properties of BVMs and discuss the invertibility of the Strang-type preconditioner in Section 3. The spectral analysis of our method is given in Section 4 and numerical examples are given in Section 5.

2. BVMS AND STRANG-TYPE PRECONDITIONER

In order to find a reasonable numerical solution for differential equation (1) with multidelays, we require that the solution of (1) is asymptotically stable. We have the following lemma, see [10].

LEMMA 1. *For any $s \geq 1$, if $\eta(J_n) \equiv (1/2)\,\lambda_{\max}(J_n + J_n^\top) < 0$ and*

$$\eta(J_n) + \sum_{j=1}^{s} \left\| D_n^{(j)} \right\|_2 < 0, \tag{2}$$

then the solution of (1) is asymptotically stable.

In the following, for simplicity, we only consider the case of $s = 2$ in (1). The generalization to arbitrary integer $s > 2$ is straightforward. Let

$$h = \frac{\tau_1}{m_1} = \frac{\tau_2}{m_2}$$

be the step size where m_1 and m_2 are positive integers with $m_2 > m_1$ ($\tau_2 > \tau_1 > 0$). By using a BVM with (k_1, k_2)-boundary conditions, we have

$$\sum_{i=0}^{k} \alpha_i \mathbf{y}_{p+i-k_1} = h \sum_{i=0}^{k} \beta_i \left(J_n \mathbf{y}_{p+i-k_1} + D_n^{(1)} \mathbf{y}_{p+i-k_1-m_1} + D_n^{(2)} \mathbf{y}_{p+i-k_1-m_2} + \mathbf{f}_{p+i-k_1} \right), \tag{3}$$

for $p = k_1, \ldots, N-1$, where $k = k_1 + k_2$, and $\{\alpha_i\}$, $\{\beta_i\}$ are coefficients of the given BVM, see [1]. By providing the values

$$\mathbf{y}_{-m_2}, \ldots, \mathbf{y}_{-m_1}, \ldots, \mathbf{y}_0, \qquad \mathbf{y}_1, \ldots, \mathbf{y}_{k_1-1}, \qquad \mathbf{y}_N, \ldots, \mathbf{y}_{N+k_2-1}, \tag{4}$$

equation (3) can be written in a matrix form as

$$M\mathbf{y} = \mathbf{b}, \tag{5}$$

where

$$M = A \otimes I_n - hB \otimes J_n - hC^{(1)} \otimes D_n^{(1)} - hC^{(2)} \otimes D_n^{(2)} \tag{6}$$

with $I_n \in \mathbb{R}^{n \times n}$ being the identity matrix, J_n being the matrix from (1), and "$\otimes$" being the Kronecker product. The vector $\mathbf{y}$ in (5) is defined by

$$\mathbf{y}^T = \left(\mathbf{y}_{k_1}^\top, \mathbf{y}_{k_1+1}^\top, \ldots, \mathbf{y}_{N-1}^\top\right) \in \mathbb{R}^{n(N-k_1)}.$$

The right-hand side $\mathbf{b} \in \mathbb{R}^{n(N-k_1)}$ of (5) depends on $\mathbf{f}$, the boundary values and the coefficients of the method. The A, B, $C^{(1)}$, and $C^{(2)}$ in (6) are Toeplitz matrices in $\mathbb{R}^{(N-k_1)\times(N-k_1)}$ and defined as follows,

$$A = \begin{bmatrix} \alpha_{k_1} & \cdots & \alpha_k & & \\ \vdots & \ddots & \ddots & \ddots & \\ \alpha_0 & \ddots & \ddots & \ddots & \alpha_k \\ & \ddots & & \ddots & \vdots \\ & & \alpha_0 & \cdots & \alpha_{k_1} \end{bmatrix}, \qquad B = \begin{bmatrix} \beta_{k_1} & \cdots & \beta_k & & \\ \vdots & \ddots & \ddots & \ddots & \\ \beta_0 & \ddots & \ddots & \ddots & \beta_k \\ & \ddots & & \ddots & \vdots \\ & & \beta_0 & \cdots & \beta_{k_1} \end{bmatrix},$$

$$C^{(1)} = \begin{bmatrix} \mathbf{O} & & & & \\ \beta_k & \ddots & & & \\ \vdots & \ddots & \ddots & & \\ \beta_0 & \cdots & \beta_k & \ddots & \\ & \ddots & & \ddots & \ddots \\ & & \beta_0 & \cdots & \beta_k & \mathbf{O} \end{bmatrix}, \qquad C^{(2)} = \begin{bmatrix} \mathbf{O} & & & & \\ \beta_k & \ddots & & & \\ \vdots & \ddots & \ddots & & \\ \beta_0 & \cdots & \beta_k & \ddots & \\ & \ddots & & \ddots & \ddots \\ & & \beta_0 & \cdots & \beta_k & \mathbf{O} \end{bmatrix},$$

see [1]. We remark that the first column of $C^{(1)}$ is given by

$$\left(\underbrace{0,\dots,0}_{m_1+k_1-k},\beta_k,\dots,\beta_0,\underbrace{0,\dots,0}_{N-m_1-2k_1-1}\right)^{\top}$$

and the first column of $C^{(2)}$ is given by

$$\left(\underbrace{0,\dots,0}_{m_2+k_1-k},\beta_k,\dots,\beta_0,\underbrace{0,\dots,0}_{N-m_2-2k_1-1}\right)^{\top}.$$

The Strang-type block-circulant preconditioner for (6) is defined as follows:

$$S = s(A)\otimes I_n - hs(B)\otimes J_n - hs\left(C^{(1)}\right)\otimes D_n^{(1)} - hs\left(C^{(2)}\right)\otimes D_n^{(2)}, \tag{7}$$

where $s(E)$ is Strang's circulant preconditioner of matrix E, for $E = A$, B, $C^{(1)}$, and $C^{(2)}$. More precisely, for any given Toeplitz matrix $T_l = [t_{i-j}]_{i,j=1}^{l} = [t_q]$, Strang's circulant preconditioner $s(T_l)$ is a circulant matrix with diagonals given by

$$[s(T_l)]_q = \begin{cases} t_q, & 0 \le q \le \lfloor l/2 \rfloor, \\ t_{q-l}, & \lfloor l/2 \rfloor < q < l, \\ [s(T_l)]_{l+q}, & 0 < -q < l, \end{cases}$$

see [5,6].

3. STABILITY AND INVERTIBILITY

In order to study the invertibility of S, we need to introduce some stability properties of the BVM. We first introduce the characteristic polynomials $\rho(z)$ and $\sigma(z)$ of the BVM which are defined by

$$\rho(z) \equiv \sum_{j=0}^{k} \alpha_j z^j \qquad \text{and} \qquad \sigma(z) \equiv \sum_{j=0}^{k} \beta_j z^j, \tag{8}$$

where $\{\alpha_i\}$, $\{\beta_i\}$ are given by (3). The A_{k_1,k_2}-stability polynomial is defined by

$$\pi(z,q) \equiv \rho(z) - q\sigma(z), \tag{9}$$

where z, $q \in \mathbb{C}$. Let $\mathbb{C}^- \equiv \{q \in \mathbb{C} : \mathrm{Re}(q) < 0\}$ where $\mathrm{Re}(\cdot)$ denotes the real part of a complex number.

DEFINITION 1. *(See [2].) The region*

$$\mathcal{D}_{k_1,k_2} = \{q \in \mathbb{C} : \pi(z,q) \text{ has } k_1 \text{ zeros inside } |z| = 1 \text{ and } k_2 \text{ zeros outside } |z| = 1\}$$

is called the region of A_{k_1,k_2}-stability of a given BVM with (k_1,k_2)-boundary conditions. Moreover, the BVM is said to be A_{k_1,k_2}-stable if $\mathbb{C}^- \subseteq \mathcal{D}_{k_1,k_2}$.

Consider now the equation $\pi(z,q) = 0$. It defines a mapping between the complex z-plane and the complex q-plane. For every $z \in \mathbb{C}$ which is a root of $\pi(z,q)$, (9) provides $q = q(z) = \rho(z)/\sigma(z)$. Let

$$\Gamma \equiv \left\{ q \in \mathbb{C} : q = \frac{\rho\left(e^{i\theta}\right)}{\sigma\left(e^{i\theta}\right)},\ 0 \le \theta < 2\pi \right\}. \tag{10}$$

The Γ is the set corresponding to the roots on the unit circumference and is called the boundary locus. We have the following lemma, see [1].

LEMMA 2. *If a BVM is A_{k_1,k_2}-stable and Γ is defined by (10), then $\mathrm{Re}(q) \ge 0$, for all $q \in \Gamma$.*

Now, we discuss the invertibility of the Strang-type preconditioner S defined by (7). Since any circulant matrix can be diagonalized by the Fourier matrix F, see [5,6], we have

$$s(E) = F^* \Lambda_E F,$$

where $s(E)$ is Strang's circulant preconditioner of Toeplitz matrix E and Λ_E is the diagonal matrix holding the eigenvalues of $s(E)$, for $E = A,\ B,\ C^{(1)}$, and $C^{(2)}$, respectively. Therefore, we obtain

$$S = (F^* \otimes I_n)\left(\Lambda_A \otimes I_n - h\Lambda_B \otimes J_n - h\Lambda_{C^{(1)}} \otimes D_n^{(1)} - h\Lambda_{C^{(2)}} \otimes D_n^{(2)}\right)(F \otimes I_n). \tag{11}$$

Note that the j th-block of

$$\Lambda_A \otimes I_n - h\Lambda_B \otimes J_n - h\Lambda_{C^{(1)}} \otimes D_n^{(1)} - h\Lambda_{C^{(2)}} \otimes D_n^{(2)}$$

is given by

$$S_j = [\Lambda_A]_{jj} I_n - h[\Lambda_B]_{jj} J_n - h\left[\Lambda_{C^{(1)}}\right]_{jj} D_n^{(1)} - h\left[\Lambda_{C^{(2)}}\right]_{jj} D_n^{(2)},$$

for $j = 1, 2, \ldots, N - k_1$. Therefore, we need to prove that

$$S_j = [\Lambda_A]_{jj} I_n - h[\Lambda_B]_{jj} J_n - h\left[\Lambda_{C^{(1)}}\right]_{jj} D_n^{(1)} - h\left[\Lambda_{C^{(2)}}\right]_{jj} D_n^{(2)},$$

are invertible, for $j = 1, 2, \ldots, N - k_1$. Let $w_j = e^{2\pi i j/N-k_1}$, where $i \equiv \sqrt{-1}$. We have

$$[\Lambda_A]_{jj} = \frac{\rho(w_j)}{w_j^{k_1}}, \qquad [\Lambda_B]_{jj} = \frac{\sigma(w_j)}{w_j^{k_1}},$$

$$[\Lambda_{C^{(1)}}]_{jj} = \beta_k w_j^{-m_1-k_1+k} + \cdots + \beta_0 w_j^{-m_1-k_1} = \frac{\sigma(w_j)}{w_j^{m_1+k_1}},$$

and

$$[\Lambda_{C^{(2)}}]_{jj} = \beta_k w_j^{-m_2-k_1+k} + \cdots + \beta_0 w_j^{-m_2-k_1} = \frac{\sigma(w_j)}{w_j^{m_2+k_1}},$$

where $\rho(z)$ and $\sigma(z)$ are defined as in (8), see [7]. Therefore,

$$S_j = \frac{1}{w_j^{m_2+k_1}} \left[w_j^{m_2} \left(\rho(w_j) I_n - h\sigma(w_j) J_n - h w_j^{-m_1} \sigma(w_j) D_n^{(1)} \right) - h\sigma(w_j) D_n^{(2)} \right].$$

We then have the following theorem.

THEOREM 1. *If the BVM with (k_1, k_2)-boundary conditions is A_{k_1,k_2}-stable and (2) holds, then for arbitrary $\theta \in \mathbb{R}$, the matrix*

$$e^{im_2\theta}\left(\rho\left(e^{i\theta}\right)I_n - h\sigma\left(e^{i\theta}\right)J_n - he^{-im_1\theta}\sigma\left(e^{i\theta}\right)D_n^{(1)}\right) - h\sigma\left(e^{i\theta}\right)D_n^{(2)}$$

is invertible. It follows that the Strang-type preconditioner S, defined by (7), is also invertible.

PROOF. Suppose that there exist $\mathbf{x} \in \mathbb{C}^n$ with $\|\mathbf{x}\|_2 = 1$ and $\theta \in \mathbb{R}$, such that

$$\left[e^{im_2\theta}\left(\rho\left(e^{i\theta}\right)I_n - h\sigma\left(e^{i\theta}\right)J_n - he^{-im_1\theta}\sigma\left(e^{i\theta}\right)D_n^{(1)}\right) - h\sigma\left(e^{i\theta}\right)D_n^{(2)}\right]\mathbf{x} = 0.$$

Then,

$$\mathbf{x}^*\left[\rho\left(e^{i\theta}\right)I_n - h\sigma\left(e^{i\theta}\right)J_n - he^{-im_1\theta}\sigma\left(e^{i\theta}\right)D_n^{(1)} - he^{-im_2\theta}\sigma\left(e^{i\theta}\right)D_n^{(2)}\right]\mathbf{x} = 0,$$

i.e.,

$$\rho\left(e^{i\theta}\right) - h\mathbf{x}^*J_n\mathbf{x}\sigma\left(e^{i\theta}\right) - he^{-im_1\theta}\mathbf{x}^*D_n^{(1)}\mathbf{x}\sigma\left(e^{i\theta}\right) - he^{-im_2\theta}\mathbf{x}^*D_n^{(2)}\mathbf{x}\sigma\left(e^{i\theta}\right) = 0.$$

We, therefore, have

$$\begin{aligned}\rho\left(e^{i\theta}\right) - &\left(h\mathbf{x}^*J_n\mathbf{x} + he^{-im_1\theta}\mathbf{x}^*D_n^{(1)}\mathbf{x} + he^{-im_2\theta}\mathbf{x}^*D_n^{(2)}\mathbf{x}\right)\sigma\left(e^{i\theta}\right)\\ &= \pi\left(e^{i\theta}, h\left(\mathbf{x}^*J_n\mathbf{x} + e^{-im_1\theta}\mathbf{x}^*D_n^{(1)}\mathbf{x} + e^{-im_2\theta}\mathbf{x}^*D_n^{(2)}\mathbf{x}\right)\right) = 0,\end{aligned}$$

where $\pi(z, q)$ is given by (9). Thus,

$$h\left(\mathbf{x}^*J_n\mathbf{x} + e^{-im_1\theta}\mathbf{x}^*D_n^{(1)}\mathbf{x} + e^{-im_2\theta}\mathbf{x}^*D_n^{(2)}\mathbf{x}\right) \in \Gamma,$$

where Γ is the boundary locus defined by (10). Since the BVM is A_{k_1,k_2}-stable, from Lemma 2, we know that

$$\mathrm{Re}\left(\mathbf{x}^*J_n\mathbf{x} + e^{-im_1\theta}\mathbf{x}^*D_n^{(1)}\mathbf{x} + e^{-im_2\theta}\mathbf{x}^*D_n^{(2)}\mathbf{x}\right) \geq 0.$$

By Cauchy-Schwarz inequality, we have

$$\mathrm{Re}\left(e^{-im_1\theta}\mathbf{x}^*D_n^{(1)}\mathbf{x}\right) \leq \left|\mathbf{x}^*D_n^{(1)}\mathbf{x}\right| \leq \|\mathbf{x}\|_2\left\|D_n^{(1)}\mathbf{x}\right\|_2 \leq \left\|D_n^{(1)}\right\|_2\|\mathbf{x}\|_2 = \left\|D_n^{(1)}\right\|_2,$$

similarly, $\mathrm{Re}(e^{-im_1\theta}\mathbf{x}^*D_n^{(2)}\mathbf{x}) \leq \|D_n^{(2)}\|_2$, and

$$\eta(J_n) = \max_{\|\mathbf{x}\|_2=1}\mathrm{Re}\left(\mathbf{x}^*J_n\mathbf{x}\right) \geq \mathrm{Re}\left(\mathbf{x}^*J_n\mathbf{x}\right).$$

Thus, we have

$$\eta(J_n) + \left\|D_n^{(1)}\right\|_2 + \left\|D_n^{(2)}\right\|_2 \geq 0,$$

which is a contradiction to (2). Therefore, the matrix

$$e^{im_2\theta}\left(\rho\left(e^{i\theta}\right)I_n - h\sigma\left(e^{i\theta}\right)J_n - he^{-im_1\theta}\sigma\left(e^{i\theta}\right)D_n^{(1)}\right) - h\sigma\left(e^{i\theta}\right)D_n^{(2)}$$

is invertible and it follows by (11) that the Strang-type preconditioner S is also invertible. ∎

4. SPECTRAL ANALYSIS

In this section, we discuss the convergence rate of preconditioned Krylov subspace methods with the Strang-type block-circulant preconditioner. It is well known that the convergence rate of Krylov subspace methods is closely related to the spectrum of the preconditioned matrix $S^{-1}M$. By noting that

$$S^{-1}M = I + S^{-1}(M - S),$$

one can easily prove the following result for the spectrum of preconditioned matrix. We, therefore, omit its proof.

THEOREM 2. *Let M be given by (6) and S be given by (7). Then, we have*

$$S^{-1}M = I_{n(N-k_1)} + L,$$

where $I_{n(N-k_1)} \in \mathbb{R}^{n(N-k_1)\times n(N-k_1)}$ is the identity matrix and L is a low rank matrix with

$$\operatorname{rank}(L) \leq (2k + m_1 + m_2 + 2k_1 + 2)n.$$

Now, we discuss the convergence property of Krylov subspace methods for solving the preconditioned system $S^{-1}M\mathbf{y} = S^{-1}\mathbf{b}$. We note that in [6], the following lemma was proved.

LEMMA 3. *Let $A = I + L$ where I is the identity matrix. If Krylov subspace methods are applied to solving the linear system $A\mathbf{x} = \mathbf{b}$, then the methods will converge in at most* $\operatorname{rank}(L) + 1$ *iterations in exact arithmetic.*

By combining Theorem 2 and Lemma 3, we have the following corollary.

COROLLARY 1. *When Krylov subspace methods are applied to solving the preconditioned system*

$$S^{-1}M\mathbf{y} = S^{-1}\mathbf{b},$$

the methods will converge in at most

$$(2k + m_1 + m_2 + 2k_1 + 2)n + 1 = \mathcal{O}(n)$$

iterations in exact arithmetic.

We observe from Corollary 1 that if the step size $h = \tau_1/m_1 = \tau_2/m_2$ is fixed, the number of iterations for convergence of Krylov subspace methods, when applied to solving the preconditioned system $S^{-1}M\mathbf{y} = S^{-1}\mathbf{b}$, will be independent of N, and therefore, is independent of the length of the interval that we considered. We should emphasize that numerical examples in the next section show a much faster convergence rate than that predicted by the estimate provided by Corollary 1.

5. NUMERICAL TESTS

In this section, we illustrate the efficiency of our preconditioner by solving the following problems. All the experiments were performed in MATLAB 6.1. We used the MATLAB-provided M-file "gmres" to solve the preconditioned systems. In our tests, the zero vector is the initial guess and the stopping criterion is

$$\frac{\|\mathbf{r}_q\|_2}{\|\mathbf{r}_0\|_2} < 10^{-6},$$

where $\mathbf{r}_q$ is the residual after the q^{th} iteration.

EXAMPLE 1. Consider

$$\begin{aligned} \mathbf{y}'(t) &= J_n\mathbf{y}(t) + D_n^{(1)}\mathbf{y}(t-0.5) + D_n^{(2)}\mathbf{y}(t-1), && t \geq 0, \\ \mathbf{y}(t) &= (\sin t, 1, \dots, 1)^{\top}, && t \leq 0, \end{aligned}$$

where

$$J_n = \begin{bmatrix} -10 & 2 & & & \\ 2 & \ddots & \ddots & & \\ 1 & \ddots & \ddots & \ddots & \\ & \ddots & \ddots & \ddots & 2 \\ & & 1 & 2 & -10 \end{bmatrix}, \qquad D_n^{(1)} = \frac{1}{n} \begin{bmatrix} 2 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{bmatrix},$$

and

$$D_n^{(2)} = \frac{1}{n} \begin{bmatrix} 2 & 1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & 2 \end{bmatrix}.$$

EXAMPLE 2. Consider

$$\begin{aligned} \mathbf{y}'(t) &= J_n \mathbf{y}(t) + D_n^{(1)} \mathbf{y}(t-0.5) + D_n^{(2)} \mathbf{y}(t-1), && t \geq 0, \\ \mathbf{y}(t) &= (1, 1, \ldots, 1)^\top, && t \leq 0, \end{aligned}$$

where

$$J_n = \begin{bmatrix} -8 & 3 & 1 & & \\ 3 & \ddots & \ddots & \ddots & \\ 1 & \ddots & \ddots & \ddots & 1 \\ & \ddots & \ddots & \ddots & 3 \\ & & 1 & 3 & -8 \end{bmatrix} \quad \text{and} \quad D_n^{(1)} = D_n^{(2)} = \begin{bmatrix} 0 & -1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & 1 & 0 \end{bmatrix}.$$

Example 1 is solved by using the third-order generalized backward differentiation formulae (GBDF) and Example 2 is solved by using the fifth-order generalized Adams method (GAM) for $t \in [0, 4]$. In practice, we do not have the boundary values $\mathbf{y}_1, \ldots, \mathbf{y}_{k_1-1}$ and $\mathbf{y}_N, \ldots, \mathbf{y}_{N+k_2-1}$ provided in (4). Instead of giving the above values, $k_1 - 1$ initial additional equations and k_2 final additional equations are given. The equations of the GAM and the GBDF with the corresponding additional equations can be found in [1]. We remark that after introducing the additional equations, the matrices A, B, $C^{(1)}$, and $C^{(2)}$ in (6) are Toeplitz matrices with small rank perturbations. By neglecting the small rank perturbations, we can also construct the Strang-type preconditioner (7).

Table 1 lists the number of iterations required for convergence of the GMRES method with different preconditioners. In the table, I means no preconditioner is used and S denotes the Strang-type block-circulant preconditioner defined as in (7). Besides, T and P denote Chan's and Bertaccini's block-circulant preconditioners, respectively. We remark that for a Toeplitz matrix $A = [t_{i-j}]_{i,j=1}^{l} = [t_q]$, the diagonals of Chan's circulant preconditioner $c(A)$ are defined by

$$[c(A)]_q = \left(1 - \frac{q}{l}\right) t_q + \frac{q}{l}\, t_{q-l}, \qquad q = 0, \ldots, l-1,$$

see [11]. Thus, Chan's block-circulant preconditioner for (6) is defined as

$$T \equiv c(A) \otimes I_n - hc(B) \otimes J_n - hc\left(C^{(1)}\right) \otimes D_n^{(1)} - hc\left(C^{(2)}\right) \otimes D_n^{(2)}.$$

Similarly, the diagonals of Bertaccini's circulant preconditioner $p(A)$ for $A = [t_{i-j}]_{i,j=1}^{l} = [t_q]$ are defined by

$$[p(A)]_q = \left(1 + \frac{q}{l}\right) t_q + \frac{q}{l}\, t_{q-l}, \qquad q = 0, \ldots, l-1,$$

Table 1. Number of iterations for Example 1 (left) and Example 2 (right).

n	m	I	S	T	P
12	10	50	9	9	9
	20	93	12	13	13
	40	177	16	18	18
	80	349	22	23	24
24	10	52	9	11	12
	20	97	11	12	13
	40	185	15	16	17
	80	367	19	22	23
48	10	53	12	13	13
	20	98	14	15	16
	40	189	14	16	17
	80	378	17	19	20

n	m	I	S	T	P
12	10	49	8	10	11
	20	88	8	10	11
	40	168	7	10	11
	80	328	6	10	11
24	10	50	10	12	15
	20	90	9	12	15
	40	170	9	12	15
	80	330	9	12	15
48	10	52	13	14	20
	20	91	12	14	20
	40	171	12	15	20
	80	332	11	15	20

see [3], and therefore, Bertaccini's block-circulant preconditioner for (6) is defined as

$$P \equiv p(A) \otimes I_n - hp(B) \otimes J_n - hp\left(C^{(1)}\right) \otimes D_n^{(1)} - hp\left(C^{(2)}\right) \otimes D_n^{(2)}.$$

From Table 1, we note that the number of iterations for convergence with a block-circulant preconditioner is much less than that with no preconditioner. The performance of the Strang-type preconditioner is better than that of other preconditioners in these examples.

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