Wavelet Regularization with Error Estimates on a General Sideways Parabolic Equation

CHU-LI FU, CHUN-YU QIU AND YOU-BIN ZHU
Department of Mathematics, Lanzhou University
Lanzhou, 730000, P.R. China
fuchuli@lzu.edu.cn

(Received May 2002; accepted June 2002)

Abstract—A wavelet regularization method for a general sideways parabolic equation is given. Some sharp stability estimates are also provided. © 2003 Elsevier Science Ltd. All rights reserved.

Keywords—Sideways parabolic equation, Ill-posed problem, Regularization, Meyer wavelet, Error estimate.

1. INTRODUCTION

This paper is concerned with the following sideways parabolic equation in the quarter plane [1]:

\[ \begin{align*}
    u_t &= a(x)u_{xx} + b(x)u_x + c(x)u, & x > 0, \quad t > 0, \\
    u(1,t) &= g(t), & t \geq 0, \\
    u(x,0) &= 0, & x \geq 0.
\end{align*} \tag{1.1} \]

Here \( a, b, \) and \( c \) are given functions such that for some \( \lambda, \Lambda > 0, \)

\[ \begin{align*}
    \lambda &< a(x) < \Lambda, \\
    c(x) &< 0, & x \in \mathbb{R}^+, \\
    a(\cdot) &\in C^2(\mathbb{R}^+), \\
    b(\cdot) &\in C^1(\mathbb{R}^+), \\
    c(\cdot) &\in C(\mathbb{R}^+).
\end{align*} \tag{1.2} \]

We want to know \( u(x,t) \) for \( 0 \leq x < 1; \) this is a severely ill-posed problem [1]. Several authors have dealt with the case of heat equation with constant coefficients [2–4]. Numerical methods have been developed also for more general equations [3,5], but, in most cases, the stability theory and convergence proofs have not been generalized accordingly. This paper remedies this by a new wavelet regularization method.

As we consider the problem in \( L^2(\mathbb{R}) \) with respect to variable \( t, \) we extend \( u(x,\cdot), g(\cdot), f(\cdot) := u(0,\cdot), \) and other functions appearing in the paper to be zero for \( t < 0. \) The notations \( \| \cdot \|, (\cdot, \cdot) \)

The subject is supported by the National Natural Science Foundation of China (No. 10271050) and the Natural Science Foundation of Gansu Province (No. ZS021-A25-001-Z).

0893-9659/03/$ - see front matter © 2003 Elsevier Science Ltd. All rights reserved. Typeset by AMS-TEX
PII: S0893-9659(03)00024-7
denote $L^2$-norm and scalar product, respectively, and $\hat{h}(\xi) = (1/\sqrt{2\pi}) \int_{-\infty}^{\infty} e^{-i\xi t} h(t)\,dt$ is the Fourier transform of $h(t)$. The corresponding direct problem with (1.1) is
\begin{align}
  u_t &= a(x)u_{xx} + b(x)u_x + c(x)u, \quad x > 0, \quad t > 0, \\
  u(0,t) &= f(t), \quad t \geqslant 0, \quad f(\cdot) \in L^2(\mathbb{R}), \\
  u(x,0) &= 0, \quad x \geqslant 0.
\end{align}
(1.3)

For the uniqueness of solution, we require $\|u(x,\cdot)\|$ be bounded. The following conclusions can be found in [1].

**Lemma 1.1.** Let $v(x,\xi)$ be the solution of the following boundary value problem:
\begin{align}
  i\xi v(x,\xi) &= a(x)v_{xx} + b(x)v_x + c(x)v, \quad x > 0, \quad \xi \in \mathbb{R}, \\
  v(0,\xi) &= 1, \\
  \lim_{x \to \infty} v(x,\xi) &= 0, \quad \xi \neq 0.
\end{align}
(1.4)
For $\xi = 0$, we require $v(x,0)$ be bounded as $x$ tends to $\infty$. Suppose that problem (1.3) has a solution $u$, then
\begin{align}
  \hat{u}(x,\xi) &= v(x,\xi)\hat{f}(\xi), \quad x > 0.
\end{align}
(1.5)

**Note.** From (1.5), we know if problem (1.3) has a solution, then
\begin{align}
  \hat{g}(\xi) &= v(1,\xi)\hat{f}(\xi), \\
  \hat{u}(x,\xi) &= \frac{v(x,\xi)}{v(1,\xi)} \hat{g}(\xi).
\end{align}
(1.6)

**Lemma 1.2.** There exist constants $c_1, c_2$, such that for $x \in [0,1]$ and $|\xi|$ large enough, say $|\xi| \geqslant \xi_0$,
\begin{align}
  c_1 e^{-A(x)\sqrt{|\xi|}/2} \leqslant |v(x,\xi)| \leqslant c_2 e^{-A(x)\sqrt{|\xi|}/2},
\end{align}
(1.8)
where $A(x) = \int_0^x \frac{1}{1/\sqrt{\sqrt{3}x}}\,ds$. Moreover, for $x \in [0,1]$, the right-hand side in (1.8) is valid for all $\xi \in \mathbb{R}$ with another constant $c_2$.

**Lemma 1.3.** If the boundary value problem
\begin{align}
  a(x)v_{xx} + b(x)v_x + c(x)v &= 0, \\
  v(0) &= 1, \quad v(x)|_{x \to \infty} \text{ bounded},
\end{align}
(1.9)
has a unique solution, then there exist constants $c'_1, c'_2$ such that
\begin{align}
  c'_1 e^{-A(1)\sqrt{|\xi|}/2} \leqslant |v(1,\xi)| \leqslant c'_2 e^{-A(1)\sqrt{|\xi|}/2}, \quad \forall \xi \in \mathbb{R}.
\end{align}
(1.10)

### 2. Regularization and Error Estimates

Let $\varphi(t), \psi(t)$ be Meyer scaling and wavelet functions, respectively, then from [6] we know $\text{Supp} \psi = [-4/3, 4/3]$, $\text{Supp} \hat{\psi} = [-8/3, 8/3] \cup [-2/3, 2/3]$, and $\psi_{jk}(t) := 2^{-j} \varphi\left(2^{-j}t - k\right)$, $j, k \in \mathbb{Z}$ constitute an orthonormal basis of $L^2(\mathbb{R})$ and
\begin{align}
  \text{Supp} \hat{\psi}_{jk}(\xi) &= \left[-\frac{8}{3}\pi 2^j, -\frac{2}{3}\pi 2^j\right] \cup \left[\frac{2}{3}\pi 2^j, \frac{8}{3}\pi 2^j\right], \quad k \in \mathbb{Z}.
\end{align}
(2.1)

The multiresolution analysis (MRA) $\{V_j\}_{j \in \mathbb{Z}}$ of Meyer wavelet is generated by
\begin{align}
  V_j &= \{\varphi_{jk} : k \in \mathbb{Z}\}, \quad \varphi_{jk} := 2^{j/2} \varphi\left(2^{j/2}t - k\right), \quad j, k \in \mathbb{Z}, \\
  \text{Supp} \varphi_{jk}(\xi) &= \left[-\frac{4}{3}\pi 2^j, \frac{4}{3}\pi 2^j\right], \quad k \in \mathbb{Z}.
\end{align}
(2.2)
The orthogonal projection of a function \( g \in L^2(\mathbb{R}) \) on space \( V_j \) is given by \( P_J g := \sum_{k \in \mathbb{R}} (g, \varphi_{Jk}) \varphi_{Jk} \), while \( Q_J g := \sum_{k \in \mathbb{R}} (g, \psi_{Jk}) \psi_{Jk} \) denotes the projection on wavelet space \( W_j \) with \( V_{j+1} = V_j \oplus W_j \).

It is easy to see from (2.2) and (2.1) that
\[
P_J g(\xi) = 0, \quad \text{for } |\xi| \geq \frac{4}{3} \pi 2^j, \tag{2.3}
\]
\[
Q_J g(\xi) = 0, \quad \text{for } j > J \text{ and } |\xi| < \frac{4}{3} \pi 2^j. \tag{2.4}
\]

Since \( (I - P_J)g = \sum_{j \geq J} Q_J g \) and from (2.4), we know
\[
((I - P_J)g)(\xi) = \sum_{j \geq J} Q_J g(\xi), \quad \text{for } |\xi| < \frac{4}{3} \pi 2^J. \tag{2.5}
\]

**Lemma 2.1.** (See [7].) Let \( \{V_j\}_{j \in \mathbb{Z}} \) be Meyer’s MRA and suppose \( J \in \mathbb{N}, \ r \in \mathbb{R} \). Then for all \( g \in V_J \), we have
\[
\|D^k g\|_{H^r} \leq C 2^{(J-1)k} \|g\|_{H^r}, \quad k \in \mathbb{N}, \tag{2.6}
\]
where \( C \) is a positive constant and \( D^k = \frac{d^k}{dt^k} \).

Let \( T_\varepsilon \) be the operator: \( g(t) \mapsto u(x, t) \) for \( 0 \leq x < 1 \) defined by (1.7), i.e.,
\[
\tilde{T_\varepsilon} g(\xi) = \hat{u}(x, \xi) = \frac{u(x, \xi)}{u(1, \xi)} \hat{g}(\xi), \quad 0 \leq x < 1, \tag{2.7}
\]
where \( u(x, t) \) is the solution of problem (1.1). We can prove the following.

**Lemma 2.2.** Suppose problem (1.9) has a unique solution, \( \{V_j\}_{j \in \mathbb{Z}} \) are Meyer’s MRA, \( J \in \mathbb{N}, 0 < x < 1, r \in \mathbb{R} \). Then for all \( g(t) \in V_J \), we have
\[
\|T_\varepsilon g\|_{H^r} \leq C \exp \left\{ 2^{(J-1)/2}(A(1) - A(x)) \right\} \|g\|_{H^r}. \tag{2.8}
\]

**Proof.** For convenience, we will denote different constants appearing in the proof by same \( C \).

Note that
\[
e^{(A(1) - A(x))\sqrt{|\xi|}/2} \leq 2\sqrt{2} \left| \cosh \left( \sqrt{|\xi|} (A(1) - A(x)) \right) \right|, \tag{2.9}
\]
and from (2.7), (1.8), (1.10), (2.9), and Hölder inequality, we know that for \( g \in V_J, J \in \mathbb{N}, r \in \mathbb{R} \) holds
\[
\|T_\varepsilon g\|_{H^r} = \left( \int_{-\infty}^{\infty} \left| \tilde{T_\varepsilon} g(\xi) \right|^2 (1 + |\xi|^2)^r d\xi \right)^{1/2} = \left( \int_{-\infty}^{\infty} \left| \frac{u(x, \xi)}{u(1, \xi)} \hat{g}(\xi) \right|^2 (1 + |\xi|^2)^r d\xi \right)^{1/2}
\]
\[
\leq C \left( \int_{-\infty}^{\infty} e^{(A(1) - A(x))\sqrt{|\xi|}/2} \left| \frac{u(x, \xi)}{u(1, \xi)} \hat{g}(\xi) \right|^2 (1 + |\xi|^2)^r d\xi \right)^{1/2}
\]
\[
\leq C \left( \int_{-\infty}^{\infty} \left| \cosh \left( \sqrt{|\xi|} (A(1) - A(x)) \right) \hat{g}(\xi) \right|^2 (1 + |\xi|^2)^r d\xi \right)^{1/2}
\]
\[
= C \left( \int_{-\infty}^{\infty} \left( \sum_{k=0}^{\infty} \frac{(A(1) - A(x))^{2k}}{(2k)!} (i\xi)^k \hat{g}(\xi) \right)^2 (1 + |\xi|^2)^r d\xi \right)^{1/2}
\]
\[
\leq C \sum_{k=0}^{\infty} \frac{(A(1) - A(x))^{2k}}{(2k)!} \left( \int_{-\infty}^{\infty} \left| (i\xi)^k \hat{g}(\xi) \right|^2 (1 + |\xi|^2)^r d\xi \right)^{1/2}
\]
\[
\leq C \sum_{k=0}^{\infty} \frac{(A(1) - A(x))^{2k}}{(2k)!} \|D_k g\|_{H^r} \leq C \sum_{k=0}^{\infty} \frac{(A(1) - A(x))^{2k}}{(2k)!} 2^{(J-1)k} \|g\|_{H^r}
\]
\[
= C \cosh \left( 2^{(J-1)/2}(A(1) - A(x)) \right) \|g\|_{H^r} \leq C \exp \left\{ 2^{(J-1)/2}(A(1) - A(x)) \right\} \|g\|_{H^r}.
\]
Let \( g(\cdot), g_m(\cdot) \) be exact and measured data, respectively, which satisfy
\[
\|g - g_m\|_{H^r} \leq \varepsilon, \quad \text{for some } r \leq 0. \tag{2.10}
\]

Since \( g_m \) belongs, in general, to \( L^2(\mathbb{R}) \subset H^r(\mathbb{R}) \) for \( r \leq 0 \), so \( r \) should not be positive. We also need an additional condition \( f(t) := u(0, t) \in H^s(\mathbb{R}) \) for some \( s \geq r \), and
\[
\|f\|_{H^r} \leq M. \tag{2.11}
\]

Letting \( T_{x,J} := T_x P_J \), we can show it approximates \( T_x \) in a stable way for an appropriate choice of \( J \in \mathbb{N} \) depending on \( \varepsilon \) and \( M \).

**Theorem 2.1.** Suppose problem (1.9) has a unique solution, then for every fixed \( J \in \mathbb{N} \), problem (1.1) with data \( g \) in \( V_J \) is well-posed. Suppose (2.10),(2.11) hold, then the problem of calculating \( T_{x,J} g_m \) is stable. Furthermore, with
\[
J^* := \left\lfloor \log_2 \left( 2 \left( \frac{1}{A(1)} \ln \left( \frac{M}{\varepsilon} \right) \left( \ln \left( \frac{M}{\varepsilon} \right)^{-2(s-r)} \right)^2 \right) \right)^{1/2} \right\rfloor, \tag{2.12}
\]
where \( \lfloor a \rfloor \) denotes the largest integer less than or equal to \( a \in \mathbb{R} \), then
\[
\|T_x g - T_{x,J} g_m\|_{H^r} \leq \left( C + (c_2 + c_2' C) A(1)^{2(s-r)} \left( \frac{\ln (M/\varepsilon)}{\ln (M/\varepsilon) + \ln (M/\varepsilon)^{-2(s-r)}} \right)^{2(s-r)} \right) \epsilon M^{1-(A(\varepsilon)/A(1))} \epsilon (A(\varepsilon)/A(1)) \left( \ln \left( \frac{M}{\varepsilon} \right)^{-2(s-r)(1-(A(\varepsilon)/A(1)))} \right), \tag{2.13}
\]
where \( C, c_2, c_2' \) are the constants appearing in (2.8), (1.8), (2.11), respectively.

**Proof.** \( \|T_x g - T_{x,J} g_m\|_{H^r} \leq \|T_x g - T_{x,J} g\|_{H^r} + \|T_{x,J} (g - g_m)\|_{H^r} \).

\[
\|T_{x,J} (g - g_m)\|_{H^r} = \|T_{x,J} P_J (g - g_m)\|_{H^r} \leq C \exp \left\{ 2^{(J-1)/2} (A(1) - A(x)) \right\} \|P_J (g - g_m)\|_{H^r} \leq C \exp \left\{ 2^{(J-1)/2} (A(1) - A(x)) \right\} \varepsilon.
\]

Note that from (1.6),(2.3), we know
\[
\|T_x g - T_{x,J} g\|_{H^r} = \|T_x (I - P_J) g\|_{H^r} = \left( \int_{-\infty}^{\infty} \frac{|v(x, \xi)|}{v(1, \xi)} ((I - P_J) g)'(\xi) \left( 1 + |\xi|^2 \right)^{r} d\xi \right)^{1/2} \leq \left( \int_{|\xi| > (4/3) \pi 2^J} \frac{|v(x, \xi)|}{v(1, \xi)} \hat{g}(\xi)^2 \left( 1 + |\xi|^2 \right)^{r} d\xi \right)^{1/2} + \left( \int_{|\xi| < (4/3) \pi 2^J} \frac{|v(x, \xi)|}{v(1, \xi)} ((I - P_J) g)'(\xi) \left( 1 + |\xi|^2 \right)^{r} d\xi \right)^{1/2} := I_1 + I_2.
\]

From (1.6),(1.8), we know
\[
I_1 \leq \left( \int_{|\xi| > (4/3) \pi 2^J} \frac{1}{v(1, \xi)} \hat{f}(\xi)^2 \left( 1 + |\xi|^2 \right)^{r} d\xi \right)^{1/2} \leq c_2 2^{-(J-s-r)} \exp \left\{ -A(\pi) 2^{J/2} \right\} \|f\|_{H^s}.
\]
From (2.5), (2.7), (2.8), and noting that \( Q_J g \in W_J \subset V_{J+1} \), we have

\[
I_2 = \left( \int_{|\xi| < (4/3)2^J} \frac{|v(\xi, \xi)|}{v(1, \xi)} |Q_J g(\xi)|^2 \left( 1 + |\xi|^2 \right)^{-1/2} d\xi \right)^{1/2} \leq \|T_x Q_J g\|_{H^r} \\
\leq C \exp \left\{ 2^{J/2}(A(1) - A(x)) \right\} \|Q_J g\|_{H^r}.
\]

Note that from (1.6) and (1.10), it is easy to show

\[
\|Q_J g\|_{H^r} \leq c_2 2^{-J(s-r)} \exp \left\{ -A(x)2^{J/2} \right\} \|f\|_{H^s},
\]

and so

\[
I_2 \leq c_2 2^{-J(s-r)} \exp \left\{ -A(x)2^{J/2} \right\} \|f\|_{H^s}.
\]

Hence,

\[
\|T_x g - T_x J^* g_m\|_{H^r} \leq C \exp \left\{ 2^{(J-1)/2}(A(1) - A(x)) \right\} \varepsilon \\
+ (c_2 + c_3) 2^{-J(s-r)} \exp \left\{ -A(x)2^{J/2} \right\} \|f\|_{H^s}.
\]

Note the representation of \( J^* \) and by a simple computation using (2.14), inequality (2.13) can be obtained. The proof is completed.

**Remark 2.1.** Taking \( s = r = 0 \), we can obtain an \( L^2 \)-estimate

\[
\|T_x g - T_x J^* g_m\|_{L^2} \leq (1 + c_2 + c_3) CM^{1-(A(x)/A(1))} \varepsilon(A(x)/A(1))
\]

\[
:= C_1 M^{1-(A(x)/A(1))} \varepsilon(A(x)/A(1)).
\]

Especially, when \( a(x) \equiv 1 \), then \( A(1) = 1 \), \( A(x) = x \). So

\[
\|T_x g - T_x J^* g_m\|_{L^2} \leq C_1 M^{1-x} \varepsilon^x.
\]

This result has been obtained by several authors by using different regularization methods for the standard sideways heat equation \([2-4]\) and this estimate is at least close to optimal or it is "order optimal" \([4]\) for the sideways heat equation.

**Remark 2.2.** From (2.15),(2.16), we know when \( x \to 0^+ \), the accuracy of regularized solutions become all progressively lower. At \( x = 0 \), they merely imply that the errors are bounded by \( C_1 M \). But if we take \( s - r > 0 \), then from (2.13), we know the speed of convergence of the regularized solution is faster, and at \( x = 0 \),

\[
\|T_0 g - T_0 J^* g_m\|_{H^r} = \|f - T_0 J^* g_m\|_{H^r}
\]

\[
\leq \left( C + (c_2 + c_3) A(1)2^{(s-r)} \left( \frac{\ln (M/\varepsilon)}{\ln (M/\varepsilon) + \ln (\ln (M/\varepsilon))^{-2(s-r)}} \right) \right)^{2(s-r)}
\]

\[
\cdot M \left( \frac{\ln M}{\varepsilon} \right)^{-2(s-r)} \to 0, \quad \text{as } \varepsilon \to 0.
\]

**REFERENCES**