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Reciprocal Relations in Oscillations of Dissipative Systems

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Abstract

The equations of the dynamic response of discrete dissipative systems at arbitrary load are considered within the theory of time analysis. The structure of the kinematic parameters of a reaction is analyzed and it is shown that the constituent matrices of movements, velocities and accelerations of the nodes of the system are symmetrical which is considered as a specific case of reciprocal relations in an elastic dissipative system. On the basis of the properties of matrices, the more general reciprocal relations are proved, where in addition to the familiar principle of reciprocity of virtual work there are reciprocal relations in the form of a product of mass forces and velocities and in the form of a product of mass forces and accelerations. The theoretical calculations are illustrated by the example.

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Keywords: Dissipative system, reciprocity, movement, velocity, acceleration, oscillation, mass, force, symmetry, relation, matrix, vector, equation

1. Introduction

Reciprocal relations in elastic dissipative systems for the first time in his research were obtained by Rayleigh who introduced the scattering function for this purpose [1]. He accepted periodic forces of a harmonic type as an external action on the nodes of a discrete dissipative system (DDS) and he considered steady-state oscillations that followed the harmonic law as well. Thus, all the forces involved had the same period of vibration and were in the same phase.

It should also be noted that in the construction of his evidence Rayleigh relied on the analysis of the potential and kinetic energies of a DDS, which are homogeneous quadratic functions of generalized coordinates and velocities,

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correspondingly.

One more approach was used by Professor P.L. Pasternak [2, 3] where he introduced a scheme of derivations of reciprocal relations based on a clear algebraic rendering of the principle of reciprocity. According to this approach, the property of reciprocity is peculiar to any system of n linear equations with n unknowns, possessing a symmetrical structure of coefficients.

This algebraic approach, which does not involve the potential and kinetic energy of a DDS, is the basis for proving the theorem of reciprocity [4] due to the development of a new mathematical tool to perform the theory of time analysis for a DDS [5]. Derivation of the theory is connected with the analysis and solution of a matrix quadric equation (MQE), which is a characteristic of one to a homogeneous differential equation of movement of a DDS.

Research into the calculation of structures to static and dynamic effects on the basis of reciprocal relations were carried out by Russian scientists [6-12] and foreign scientists [13-21].

This article highlights the development of ideas given in [5] as to reciprocity, as well as extending the understanding of the theorem of reciprocity in dissipative systems under arbitrary dynamic forces.

2. Features of response equations of a discrete dissipative

The equation of movement of a DDS and its corresponding characteristic MQE are the following:

$$M\ddot{Y}(t) + C\dot{Y}(t) + KY(t) = P(t), \tag{1}$$

$$MS2 + CS + K = 0, \tag{2}$$

where $M = \mathrm{diag}\,(m_1, \ldots, m_n)$, $C = C^{\mathrm{T}}$, $K = K^{\mathrm{T}} \in M_n(R)$ are correspondingly matrices of mass, damping and stiffness of the system; $Y(t) = [y_j(t)]$, $P(t) = [p_j(t)]$ $(j = 1, \ldots, n)$ are vectors of movements and external load; $S \in M_n(\boldsymbol{C})$ is the matrix of internal dynamic characteristics; n is the number of degrees of freedom of a DDS.

The transition from (Eq. 1) to (Eq. 2) is made with the help of a fundamental matrix $\Phi(t) = e^{St}$ of the homogeneous differential equation corresponding to (Eq. 1): $\Phi(t)$ is a fundamental matrix of the homogeneous equation in (Eq. 1) then and only then when the matrix S meets the MQE (Eq. 2).

In [5] it is shown, that at finite solubility of the MQE its solutions are set into a root pair:

$$S_{1,2} = M^{-1}(-C + W \pm U)/2, \tag{3}$$

where $W = -W^{\mathrm{T}}$ and $U = U^{\mathrm{T}}$. Fundamental relations are true for matrix roots (Eq. 3):

$$S^{\mathrm{T}}U = US, \quad \Phi(t)^{\mathrm{T}}U = U\Phi(t), \tag{4}$$

which are determine the fact of the symmetry of the matrices-multipliers US, $U\Phi(t)$.

At an insignificant dissipation of an elastic system, which meets the requirements of structural oscillations, for matrices W and U in (Eq. 3) we have the following conditions:

$$W = \mathrm{Re}W, \quad U = i\mathrm{Im}U = 2iM\,\mathrm{Im}S \quad (i = \sqrt{-1}\,). \tag{5}$$

whence it follows that $S_1 = S$, $S_2 = \overline{S}$ are complex-conjugate matrices.

Based on the properties (Eq. 3) – (Eq. 5), there opens the possibility of developing the complete set of equations of the dynamic response of DDS at forced oscillations [5]:

$$Y(t) = 2\,\mathrm{Re}\{U^{-1}\int_{t_0}^{t} \Phi\,(t-\tau)^{\mathrm{T}}\,P(\tau)\,\mathrm{d}\tau\}, \tag{6}$$

$$\dot{Y}(t) = 2\,\mathrm{Re}\{SU^{-1}\int_{t_0}^{t} \Phi\,(t-\tau)^{\mathrm{T}}\, P(\tau)\,\mathrm{d}\tau\}. \tag{7}$$

$$\ddot{Y}(t) = 2\,\mathrm{Re}\{S^2U^{-1}\int_{t_0}^{t} \Phi\,(t-\tau)^{\mathrm{T}}\, P(\tau)\,\mathrm{d}\tau\} + M^{-1}P(t). \tag{8}$$

In closed form these equations make it possible to determine the movements (Eq. 6), velocities (Eq. 7) and accelerations (Eq. 8) of nodes of the DDS. (Eq. 6) is the nontrivial matrix form of the Duhamel integral, the integrand of which in contrast to the equivalents of this integral [22] has the matrix $\Phi(t)$ of a homogeneous equation corresponding to (Eq. 1).

3. Reciprocal relations in a discrete dissipative system

Suppose arbitrary external forces $p_j(t)$, which are characterized by vector $P(t)$, act in nodes of the elastic DDS:

$$P(t) = f(t)P_0, \tag{9}$$

where $f(t)$ is a dimensionless scalar function of time t, which is determining the law of the load application; $P_0 = [p_{0j}]$ $(j = 1, \ldots, n)$ is the amplitude vector. Then (Eq. 6) – (Eq. 8) take the form:

$$Y(t) = D(t)P(t), \quad \dot{Y}(t) = V(t)P(t), \quad \ddot{Y}(t) = A(t)P(t), \tag{10}$$

where

$$D(t) = 2\,\mathrm{Re}\{U^{-1}\int_{t_0}^{t} \Phi\,(t-\tau)^{\mathrm{T}} f(\tau)\,\mathrm{d}\tau\} f(t)^{-1}, \quad V(t) = 2\,\mathrm{Re}\{\,SU^{-1}\int_{t_0}^{t} \Phi\,(t-\tau)^{\mathrm{T}} f(\tau)\,\mathrm{d}\tau\} f(t)^{-1}, \tag{11}$$

$$A(t) = B(t) + M^{-1}, \quad B(t) = 2\,\mathrm{Re}\{S^2U^{-1}\int_{t_0}^{t} \Phi\,(t-\tau)^{\mathrm{T}} f(\tau)\,\mathrm{d}\tau\}\, f(t)^{-1}. \tag{12}$$

The matrix functions $D(t) = [\delta_{jk}(t)]$, $V(t) = [v_{jk}(t)]$, $A(t) = [a_{jk}(t)]$ $(j, k\ \ 1, \ldots, n)$ have the property of symmetry. To prove this fact let us consider the auxiliary matrix function $\Phi(t-\tau)U^{-1}(S^k)^{\mathrm{T}}$, where k is any integer number. Using Eq. (4) it can be shown that this function is of a class of symmetrical matrices: $S^kU^{-1}\Phi(t-\tau)^{\mathrm{T}} = \Phi(t-\tau)U^{-1}(S^k)^{\mathrm{T}}$.

For $k = 0$, 1 and 2 the expressions of these matrices are the same as for the integrands in (Eq. 11), (Eq. 12) to an accuracy of scalar factor. While analyzing it is necessary to consider the fact that the matrices U^{-1}, SU^{-1}, S^2U^{-1} are constants, so they can be made under the integral sign. Consequently

$$D(t) = D(t)^{\mathrm{T}}, \quad V(t) = V(t)^{\mathrm{T}}, \quad A(t) = A(t)^{\mathrm{T}}. \tag{13}$$

Analysis of the expressions (Eq. 11), (Eq. 12) shows that the symmetrical structure of the matrices is determined by the inner properties of the dissipative system and does not depend on the character of external action. It is not too difficult to see that matrix elements (Eq. 13) are kinematic parameters of the system such as the movements, velocities and accelerations of the nodes of the DDS correspondingly. It follows from (Eq. 10) at unit exposures on the nodes of the DDS.

Under symmetry (Eq. 13) the expression in (Eq. 10) make it possible to obtain more general reciprocal relations along with Betti's principle. For that purpose let us consider a new group of forces acting on the nodes of the DDS and introduced by the vector $P(t)' = [p_j(t)']$ $(j = 1, \ldots, n)$. According to the law of load application (Eq. 9) characterized by the scalar function $f(t)$, we have the following: $P(t)' = f(t)P_0'$, где $P_0' = [p_{0j}']$ $(j = 1, \ldots, n)$. Then (Eq. 10) takes the form:

$$Y(t)' = D(t)P(t)', \quad \dot{Y}(t)' = V(t)P(t)', \quad \ddot{Y}(t)' = B(t)P(t)'. \tag{14}$$

New values of vectors of movement, velocity and acceleration correspond to this group of forces: $Y(t)' = [y_j(t)']$, $\dot{Y}(t)' = [\dot{y}_j(t)']$, $\ddot{Y}(t)' = [\ddot{y}_j(t)']$ $(j = 1, \dots, n)$.

For each pair of expressions in (Eq. 10) and (Eq. 14), having the same symmetrical matrices (Eq. 13), scalar equations of the following types are performed:

$$Y(t)^{\mathrm{T}}P(t)' = Y(t)'^{\mathrm{T}}P(t), \quad \dot{Y}(t)^{\mathrm{T}}P(t)' = \dot{Y}(t)'^{\mathrm{T}}P(t), \quad \ddot{Y}(t)^{\mathrm{T}}P(t)' = \ddot{Y}(t)'^{\mathrm{T}}P(t). \tag{15}$$

These equations are expressing the law of reciprocity in an elastic DDS.

Let us show the correctness of the given dependencies (Eq. 15) for the first relation. In (Eq. 10) we perform the operation of transposition: $Y(t)^{\mathrm{T}} = P(t)^{\mathrm{T}}D(t)$, and in (Eq. 14) we perform reverse conversion expressing vector $P(t)'$ through $Y(t)'$: $P(t)' = D(t)^{-1}Y(t)'$. After termwise multiplication of the left and right parts of these expressions as a series to a column we have the desired result.

The obtained relation expresses the equality of virtual work (Betti's principle), performed in a dissipative system by the group of forces (vectors $P(t)$, $P(t)'$) at the corresponding total of movement (vectors $Y(t)'$, $Y(t)$).

The physical meaning of the second and the third expressions in (Eq. 15) is less straightforward. The second correlation expresses the reciprocity of possible values, introduced in the form of the multiplication of groups of forces ($P(t)$, $P(t)'$), acting in the nodes of the elastic DDS by the sum total of mass velocities ($\dot{Y}(t)'$, $\dot{Y}(t)$). The third correlation expresses the reciprocity of possible values in the form of the multiplication of the same groups of forces by the sum total of mass acceleration ($\ddot{Y}(t)'$, $\ddot{Y}(t)$).

Correlations (Eq. 13) should be considered as particular theorems of reciprocity. At unit exposure on the nodes of the DDS by generalized forces: $p_j(t) = 1{\cdot}f(t)$, $p_k(t)' = 1{\cdot}f(t)$ from the first correlation we obtain the known Maxwell's reciprocal theorem [3]:

$$\delta_{jk}(t) = \delta_{kj}(t) \; (j, k = 1, \dots, n). \tag{16}$$

From the second and the third correlations we obtain the principles of the reciprocity of velocities and accelerations correspondingly:

$$v_{jk}(t) = v_{kj}(t), \quad a_{jk}(t) = a_{kj}(t) \quad (j, k = 1, \dots, n). \tag{17}$$

The reciprocal correlation for velocity in (Eq. 17) is obtained in [4] by a different way. As for the reciprocal correlation for acceleration in (Eq. 17), the authors recognise the fact that it is a new one and is obtained for the first time.

Description of the reciprocity principle for acceleration:

The acceleration ajk(t) of the j-th mass towards the j-th unit force, induced in elastic DDS by the action of k-th unit force pk(t) = 1·f(t), equals to the acceleration akj(t) of k-th mass towards k-th unit force, induced by the action of j-th unit force pj(t) = 1·f(t).

Thus, reciprocal relations (Eq. 15) in an elastic DDS are obtained as a result of proving the symmetry of the matrices in (Eq. 11), (Eq. 12), expressing partial reciprocal laws (Eq. 16), (Eq. 17) in a dissipative system. In additions the conditions for matrix symmetry in (Eq. 13) are the expression of more profound properties (Eq. 3) – (Eq. 5), exposed by analysis of a characteristic MQE (Eq. 2), containing matrices M, C, K of a symmetrical structure.

4. Example calculation

Let us illustrate the reciprocal principle (Eq. 16), (Eq. 17) in an elastic DDS by the example of an analysis of the forced oscillations of a two-storeyed space reinforced concrete frame (Fig. 1 *a*) under the influence of vibration forces at scalar function value in (Eq. 9) $f(t) = \sin(\vartheta t + \varphi)$.

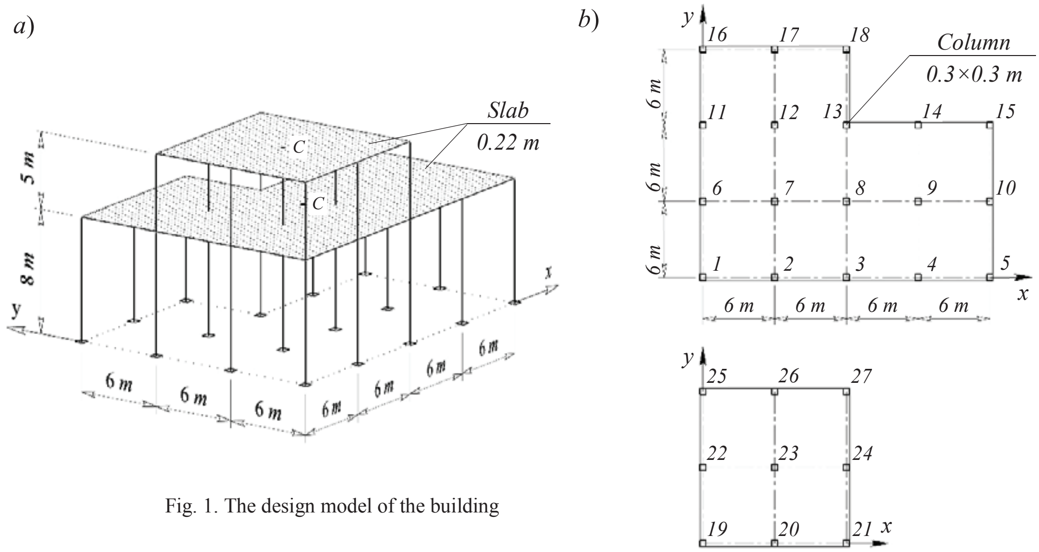

Fig. 1. The design model of the building

The column grid step is 6×6 m (Fig. 1 *b*), the cross section of the columns is 300×300 mm, the thickness of the floor slabs is 220 mm. The columns of framing storeys are rigidly braced with floor slabs. The design dynamic model of the frame has 33 degrees of freedom:

- the first six degrees of freedom are connected with the horizontal oscillations of the floor slabs. Within a storey the slab is considered to be a completely rigid disk with three degrees of freedom (two translational movements of the centre of gravity of slabs (points C1 and C2) towards axes x, y and a rotary movement of a disk in respect to a vertical axis, passing through the shear centre of elastic constraints);

- in a vertical direction of oscillations the slab is considered as an elastic strained body. The discrete model of the frame is developed by means of laying slabs into unit cells (square and rectangular). Point masses are concentrated on the levels of storeys in tie points of weightless columns to floor slabs.
  Accounting of the internal friction is adopted in accordance with the model of non-proportional damping [5].
  Fig. 2, 3 give design models of the frame with two variants of the unit load (*a*) and oscillograms of the kinematic parameters for the frame reactions: movements (*b*), velocities (*c*) and accelerations (*d*).

The first variant (Fig. 2 *a*) has a unit load of momentum type $p_6(t) = 1\cdot\sin(\vartheta\cdot t + \varphi)$ in the slab of the second storey (point C_2) with parameters of a vibration exposure: $\vartheta = 70\ s^{-1}$, $\varphi = 0$. Fig. 2 (*b*, *c*, *d*) shows oscillograms of the movements (*b*), velocities (*c*) and accelerations (*d*) of the centre of gravity of the first floor slab (point C_1) along axis ξ ($\xi \parallel y$): $y_2(t)$, $\dot{y}_2(t)$, $\ddot{y}_2(t)$.

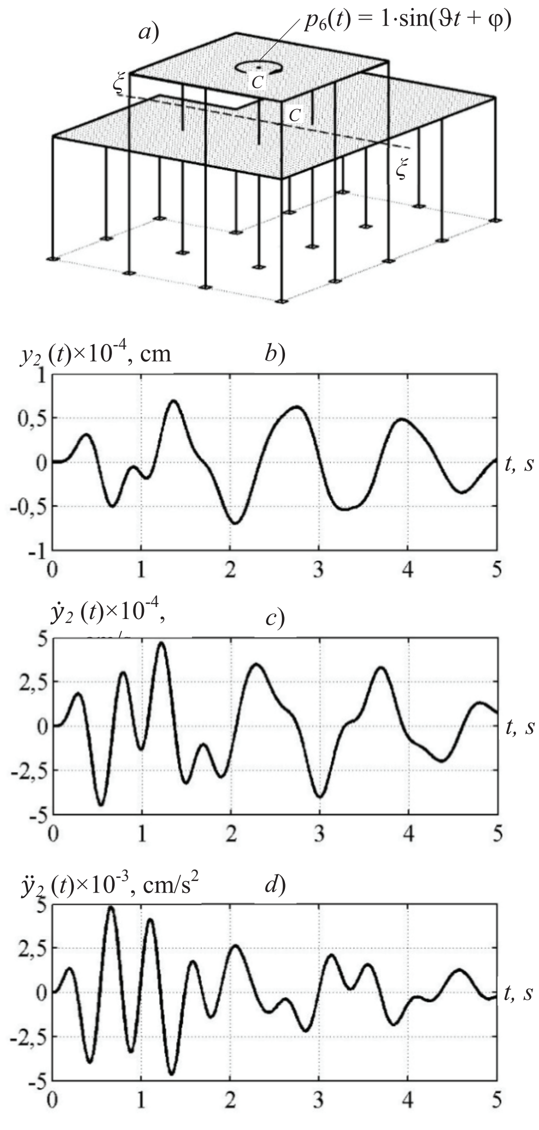

Fig. 2. Design model of the frame by the first variant of loading (*a*) and oscillograms of the dynamic response parameters towards the axis ξ: *b* – movements; *c* – velocities; *d* –

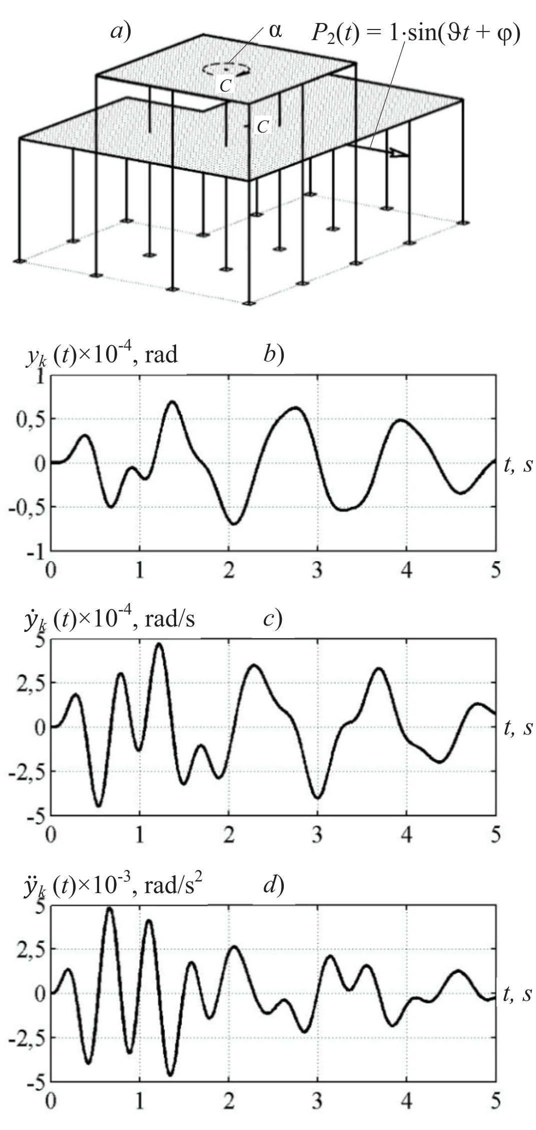

Fig. 3. Design model of the frame by the second variant of loading (*a*) and oscillograms of the dynamic response parameters α in the slab of the second storey: *b* – angles of rotation; *c* – angular velocities; *d* – angular accelerations

In the second variant (Fig. 3 *a*) the unit force $p_2(t) = 1\cdot\sin(\vartheta\cdot t + \varphi)$ with the same parameters of a vibration exposure acts on a slab of the first floor along axis ξ. Fig. 3 (*b*, *c*, *d*) shows oscillograms of rotation angles, velocities and accelerations for the slab of the second floor: $y_6(t)$, $\dot{y}_6(t)$, $\ddot{y}_6(t)$.

The identical type of oscillograms in figures 2 and 3 prove the correctness of theorems (16), (17).

In the case that the parameters of a scalar function $f(t)$ in one of the set of forces $P(t)$ are different from the parameters of function $f(t)$ in other sets of forces $P(t)'$, then relations (Eq. 15) – (Eq. 17) are not performed. To prove this idea let us consider the design model of the frame, loaded in accordance with the diagram of the first variant by singular momentum $p_6(t) = 1 \cdot \sin(\vartheta t + \varphi)$ (Fig. 2 *a*). Oscillograms of movements (*a*), velocities (*b*) and accelerations (*c*) of the center of gravity of the first storey slab along axis ξ (Fig. 4) are built at values of initial phases $\varphi = 0, \pi/2, \pi$. These data indicate that at coinciding values of initial phase $\varphi = 0$ (Fig. 3) reciprocal relations of kinematic parameters (Eq. 16), (Eq. 17) are performed. For different values φ, when initial phases in the first and second variants differ from one another, there is no reciprocity of kinematic parameters.

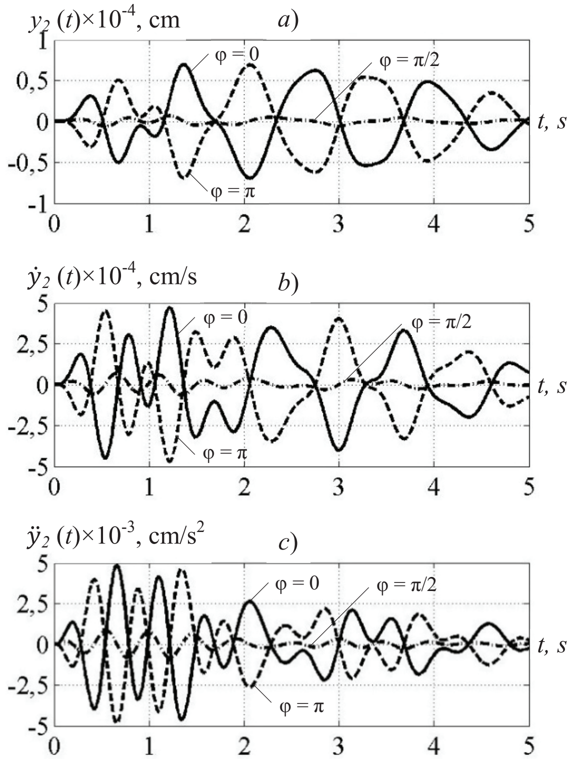

Fig. 4. Oscillograms of kinematic parameters of the dynamic response towards the axis ξ at 3 values of initial phase φ: *a* – movements; *b* – velocities; *c* – accelerations

5. Conclusions

The general method of proving reciprocal relations in any DDS, the movement of which is described by (Eq. 1), is given. Matrices of the movements, velocities and accelerations determining the kinematic parameters of the dynamic response of a DDS at arbitrary load are built. The fact of the symmetry of these matrices being an expression of the specific laws of reciprocity in an elastic dissipative system is stated. Simultaneously we obtain general reciprocal relations in DDS (Eq. 15).

The presence of the last two reciprocal relations in (Eq. 15) shows that the laws of reciprocity in dynamic systems are outside the scope of Betti's principle. The extension of the scope of application of these laws opens up the possibility of analyzing oscillating systems with complex damping and dynamic effects.

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