NORTH-HOLLAND

On the Powers of Matrices in Bottleneck / Fuzzy Algebra*

Katarína Cechlárová
Department of Geometry and Algebra
P. J. Šafárik University
Košice, Slovakia

Submitted by Richard A. Brualdi

ABSTRACT

Powers of square matrices under the operations $\oplus = \max$ and $\otimes = \min$ are studied. We show that the powers of a given matrix stabilize if and only if its orbits stabilize for each starting vector and prove a necessary and sufficient condition for this property using the associated graphs of the matrix. Applications of the obtained results to several special classes of matrices (including circulants) are given.

1. INTRODUCTION

An algebraic structure $(\mathscr{B}, \oplus, \otimes)$ where $\mathscr{B}$ is a bounded linearly ordered set with the upper bound denoted by **1** and the lower by **0**, $\oplus = \max$, and $\otimes = \min$ is called by some authors a *bottleneck algebra* (see e.g. [2]), by others a *fuzzy algebra* (e.g. [8, 9]). The important property of the operations $\oplus$ and $\otimes$ is that they are both idempotent; moreover, for every $a, b \in \mathscr{B}$

$$a \oplus b \in \{a, b\} \quad \text{and} \quad a \otimes b \in \{a, b\}.$$

*This research was carried out during a visit of the author to Birmingham University, financed under the European Community's Action for Cooperation in Sciences and Technology with Central and Eastern European countries, Proposal No. 10342. The author would like to express her gratitude for this sponsorship.

LINEAR ALGEBRA AND ITS APPLICATIONS 246:97–111 (1996)

655 Avenue of the Americas, New York, NY 10010
0024-3795/96/$15.00
SSDI 0024-3795(94)00338-E

The set of all n-tuples (n-vectors) over $\mathscr{B}$ will be denoted by $\mathscr{B}_n$, and the set of all square matrices of order n by $\mathscr{B}(n, n)$. The operations $\oplus$ and $\otimes$ are extended to vectors and matrices in the usual way, i.e., for $A, B \in \mathscr{B}(n, n)$ we have $A \otimes B = C \in \mathscr{B}(n, n)$, where

$$c_{ij} = \sum_{k=1}^{n} {}^{\oplus} a_{ik} \otimes b_{kj}.$$

If we create the sequence of powers A, $A^2 = A \otimes A$, $A^3 = A^2 \otimes A, \dots$ of a given matrix A, then no entry different from the entries in the original A can be obtained. Hence although the sequence of powers is infinite, it contains only a finite number of different members. That means that at some point a repetition must occur, resulting in some periodic behavior or stabilization.

This phenomenon has already been observed in [10], where a condition for stabilization of compact matrices has been derived. In [8] and [9] it was shown that the period of a square matrix divides $[n]$, the least common multiple of the integers $1, 2, \dots, n$, and an algorithm for computing this period was given. Some other works concentrate on periodicity and stabilization of powers of special matrices; one of the most recent papers is [7], where a sufficient condition for stabilization of powers of circulant matrices is proved.

All the above papers use purely algebraic methods. However, for computations in extremal algebraic structures like the bottleneck/fuzzy algebra, the language of graph theory has been used since, say, the 1960s. It has proved to be useful in both directions: matrix operations can be used for designing some graph-theoretical algorithms, and graphs shed light on some questions formulated for matrices. From an extensive literature on this topic let us mention [2, 3, 11].

The aim of the present paper is to introduce digraphs into the study of powers of matrices over bottleneck/fuzzy algebra and use them to derive a necessary and sufficient condition for stabilization of power sequences. In the final section we illustrate this theory by its application to some special classes of matrices.

2. A REVIEW OF GRAPH-THEORETICAL NOTIONS

A *digraph* is a pair $\mathscr{G} = (V, H)$ where V is a finite set, called the *node set*, and H is a subset of $V \times V$, called the *arc set*. $\mathscr{G}' = (V', H')$ is a *subgraph* of $\mathscr{G}$ if $V' \subseteq V$ and $H' \subseteq H$. If each arc (i, j) is assigned a *weight*

$c(i, j)$ (sometimes called its *capacity*), then the digraph is called a *weighted digraph*. A sequence of nodes

$$p = (i_0, i_1, \dots, i_m) \tag{1}$$

is called a *path* if for all $j = 1, \dots, m$ the pair $(i_{j-1}, i_j) \in H$; for brevity we shall also say that the arcs (i_{j-1}, i_j) are *on the path* p. If all the nodes on a path are different, the path is called *elementary*. If $i_0 = i_m$, then the path is a *cycle*; if all nodes (except of the first and the last one) on a cycle are different, then the cycle is *elementary*. The *length* of a path or a cycle is equal to the number of arcs on it and denoted by $l(p)$. Now, for the present theory the following obvious fact is crucial:

LEMMA 1. *Every path of length at least n in a digraph on n nodes contains a cycle.*

A digraph that does not contain any cycle is called *acyclic*. If for each pair of nodes u, v in $\mathcal{G}$ there is a path from u to v and a path from v to u in $\mathcal{G}$, then $\mathcal{G}$ is called *strongly connected*. A maximal strongly connected subgraph of a given digraph is its *strongly connected component* (SCC for short). Every SCC can contain either several nodes (in that case it must contain at least one cycle) or a single node u; and in the latter case we shall call it an *acyclic* SCC if the loop (u, u) is not its arc.

The greatest common divisor (gcd for short) of all cycle lengths in a non-acyclic digraph $\mathcal{G}$ is called the *period* of $\mathcal{G}$. Note that in digraphs it makes no difference whether we consider all cycles or only the elementary ones (see [1]).

Finally, if $\mathcal{G}$ is a weighted digraph and p a path in $\mathcal{G}$ of the form (1), then the number

$$c(p) = c(i_0, i_1) \otimes c(i_1, i_2) \otimes \cdots \otimes c(i_{m-1}, i_m)$$

is called the *capacity* of the path p.

Denote the set of all integers by $\mathbb{N}$, the set $1, 2, \dots, n$ by N; and suppose that a square matrix A of order n is given. We shall define two kinds of digraphs connected with A. First, the *associated digraph* of A, denoted by $\mathcal{G}(A)$, is the complete weighted digraph on the node set N with each arc (i, j) assigned the capacity a_{ij}. Conversely, for a given weighted digraph $\mathcal{G}$ the corresponding matrix can be created in the usual way.

EXAMPLE 1. As an illustration, $\mathscr{G}(A)$ for the matrix

$$A = \begin{pmatrix} 0.1 & 0.8 & 0 & 0.7 \\ 0.5 & 0.1 & 0.2 & 0.5 \\ 0 & 0.4 & 0.1 & 0.2 \\ 0.7 & 0 & 0.4 & 0.5 \end{pmatrix}$$

is given in Figure 1.

If a value h is given together with the matrix $A \in \mathscr{B}(n, n)$, then the *associated threshold digraph* $\mathscr{G}(A, h) = (V, H(h))$ is defined by $V = N$ and $(i, j) \in H(h)$ if and only if $a_{ij} \geqslant h$. It can be easily seen that $\mathscr{G}(A, h)$ is a subgraph of $\mathscr{G}(A, h')$ for $h' \leqslant h$, because as the value of the threshold decreases, some new arcs can be added, but none will disappear. So $\mathscr{G}(A, h)$ consists of n isolated nodes for h greater that the maximum entry in A, and as soon as h is less than or equal to the minimum entry of A, $\mathscr{G}(A, h)$ is a complete digraph with loops. We shall imagine decreasing the threshold later too, so when we say "the first nontrivial threshold digraph" we mean the threshold digraph $\mathscr{G}(A, h)$ for h equal to the maximum entry in A.

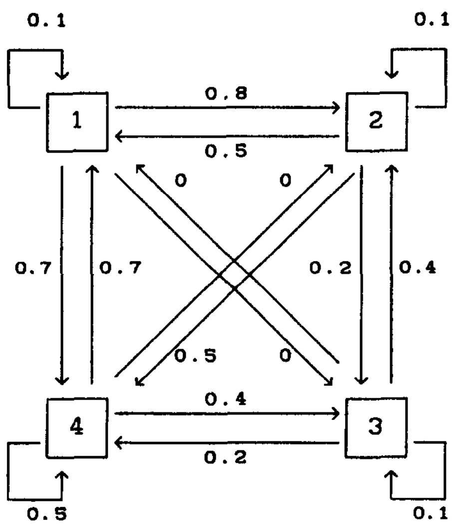


FIG. 1.

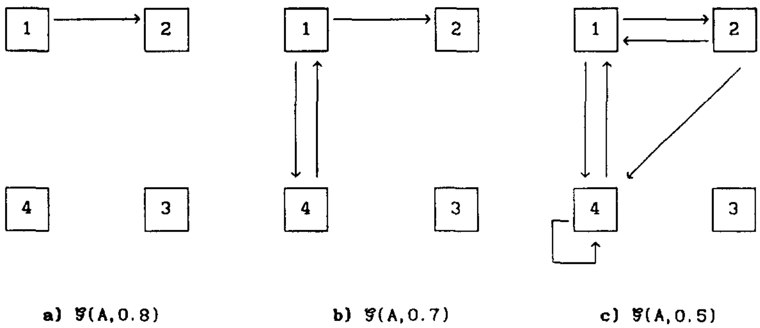


FIG. 2.

EXAMPLE 1 (Continued). In Figure 2(a), (b), (c) the threshold digraphs $\mathscr{G}(A, 0.8)$, $\mathscr{G}(A, 0.7)$, and $\mathscr{G}(A, 0.5)$ for the above matrix A are given.

Now we shall examine the connection between the entries in powers of A and the paths in the associated digraph. First of all, the entry a_{ij} can be viewed as the capacity of the (unique) path of length 1 from node i to node j. The entries of A^2 are of the form

$$a_{ij} = \sum_{k=1}^{n} {}^{\oplus} a_{ik} \otimes a_{kj},$$

which can be recognized as the *maximum capacity of a path of length* 2 beginning at node i and ending at node j. By induction, for A^m we get

$$a_{ij}^{m} = \sum_{k=1}^{n} {}^{\oplus} a_{ik}^{m-1} \otimes a_{kj}, \tag{2}$$

where a_{ik}^{m-1} is the maximum capacity of a path of length $m-1$ from node i to node k. Now, each path p of length m that begins at node i and ends at node j can be split into a path of length $m-1$ from i to some intermediate node k and then a single arc from k to j. Clearly, for p to have the maximum capacity, it is necessary that such a partition will result in a maximum capacity path of length $m-1$ from i to k. We see that (2) computes the maximum of such partitions over all possible intermediate nodes k, showing that a_{ij}^m is the *maximum capacity of a path of length* m from i to j.

LEMMA 2. *For a given matrix* $A \in \mathscr{B}(n, n)$ *the following conditions are equivalent*:

(*i*) $a_{ij}^m \geqslant h$.

(*ii*) *In the threshold digraph* $\mathscr{G}(A, h)$ *there exists a path of length* m *from node* i *to node* j.

If, moreover, $m \geqslant n$, *then these conditions are equivalent with*:

(*iii*) *In the threshold digraph* $\mathscr{G}(A, h)$ *there is a path from* i *to* j *containing a cycle.*

Proof. If $a_{ij}^m \geqslant h$, then there exists a path p from node i to node j with length m and capacity at least h. But, due to the properties of the operation $\otimes$, this means that for each arc (k, l) on p we have $c(k, l) = a_{kl} \geqslant h$. However, this is exactly the condition for the pair (k, l) to be in the arc set of $\mathscr{G}(A, h)$; therefore p itself is the sought path.

Conversely, any path p in $\mathscr{G}(A, h)$ corresponds to the same path in $\mathscr{G}(A)$, but here its capacity is at least h, since p contains only arcs of weight at least h. So any path p from i to j in $\mathscr{G}(A, h)$ with length m will ensure $a_{ij}^m \geqslant h$.

For the equivalence with (iii) simply use Lemma 1. ∎

3. ORBITS OF A MATRIX AND THEIR INTERPRETATION IN DIGRAPHS

Let a matrix $A \in \mathscr{B}(n, n)$ and a vector $b \in \mathscr{B}_n$ be given.

DEFINITION 1. The sequence of vectors $x(0), x(1), x(2), \dots, x(k), \dots$ where

$$x(0) = b,$$

$$x(k+1) = A \otimes x(k)$$

is called the *orbit* of the matrix A generated by the vector b. We shall denote it by $\mathscr{O}rb(A, b)$.

DEFINITION 2. The orbit $\mathcal{O}\!\mathcal{b}(A, b)$ is said to *stabilize* if there exists an integer k_0 such that for $k \geqslant k_0$ we have $x(k+1) = x(k)$. $\mathcal{O}\!\mathcal{b}(A, b)$ *oscillates* if it does not stabilize but there exists two integers k_0 and $t > 1$ such that $x(k+t) = x(k)$ for each $k \geqslant k_0$. The smallest t with this property is called the *period* of the orbit.

EXAMPLE 2. If we take the matrix A from Example 1 and the starting vector $b = (0.5, 0.4, 0.8, 0.3)^T$, then we obtain the following orbit:

$$x(0) = \begin{pmatrix} 0.5 \\ 0.4 \\ 0.8 \\ 0.3 \end{pmatrix}, \qquad x(1) = \begin{pmatrix} 0.4 \\ 0.5 \\ 0.4 \\ 0.5 \end{pmatrix},$$

$$x(2) = \begin{pmatrix} 0.5 \\ 0.5 \\ 0.4 \\ 0.5 \end{pmatrix}, \qquad x(3) = \begin{pmatrix} 0.5 \\ 0.5 \\ 0.4 \\ 0.5 \end{pmatrix}, \dots,$$

which clearly stabilizes; whereas for $b = (0.7; 0; 0; 0)^T$ we get

$$x(0) = \begin{pmatrix} 0.7 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad x(1) = \begin{pmatrix} 0.1 \\ 0.5 \\ 0 \\ 0.7 \end{pmatrix}, \qquad x(2) = \begin{pmatrix} 0.7 \\ 0.5 \\ 0.4 \\ 0.5 \end{pmatrix},$$

$$x(3) = \begin{pmatrix} 0.5 \\ 0.5 \\ 0.4 \\ 0.7 \end{pmatrix}, \qquad x(4) = \begin{pmatrix} 0.7 \\ 0.5 \\ 0.4 \\ 0.5 \end{pmatrix},$$

and we see that the period of this orbit is 2.

We can argue, as in the beginning of this paper, that each orbit must either stabilize or oscillate, since no new entries are generated in the process. Moreover, in [2] it was shown that for each matrix A there is at least one nonzero starting vector b causing $\mathcal{O}\!\mathcal{b}(A, b)$ to stabilize, in which case the vector we arrive at is clearly an eigenvector of the matrix A.

DEFINITION 3. A matrix $A \in \mathcal{B}(n, n)$ is called *strongly stable* if $\mathcal{O}\!\mathcal{b}(A, b)$ stabilizes for each $b \in \mathcal{B}_n$.

THEOREM 1. *A matrix $A \in \mathscr{B}(n, n)$ is strongly stable if and only if the powers of A stabilize.*

Proof. The sufficient condition is implied by the definition of the orbit, since $x(k) = A^k \otimes b$. For the converse implication notice that powers of A are nothing but matrices whose columns are in fact orbits, generated by columns of A as starting vectors. Hence if each orbit stabilizes, the powers of A will eventually stabilize too. ∎

Let us now look at the interpretation of orbits in the associated digraph. First express

$$x(m)_i = (A^m \otimes b)_i = \sum_{j=1}^{n}{}^{\oplus} a_{ij}^m \otimes b_j. \tag{3}$$

We already know that a_{ij}^m is the maximum capacity of a path of length m from node i to node j. The vector b can be viewed as assigning capacities to nodes; let us call the value b_j the *terminal capacity at node* j. Then, since in (3) the maximum is taken over all terminal nodes j of the paths, we can interpret this as the *maximum capacity of a terminated path of length* m, *starting at node* i. That means, that the capacity of a terminated path is computed by multiplying the capacity of the path itself by the capacity of the node it terminates at. The capacity of a terminated path p will be denoted by $\operatorname{ct}(p)$.

Now take a fixed $i \in N$ and denote

$$h = \limsup_{k \to \infty} \{x(k)_i\}.$$

That means that h is the maximum number that appears in the sequence $\{x(k)_i\}_{k=1}^{\infty}$ infinitely many times, or that for each $k_0 \in \mathbb{N}$ there exists an integer $k \geqslant k_0$ such that $x(k)_i = h$. The significance of the value h in the associated digraphs is summarized in the following lemma, whose proof follows easily from the definitions.

LEMMA 3. *Let $A \in \mathscr{B}(n, n)$, $b \in \mathscr{B}_n$, and $i \in N$ be given. The following conditions are equivalent:*

(i) $h = \limsup_{k \to \infty}\{x(k)_i\}$;

(ii) *h is the greatest value such that for every $k_0 \in \mathbb{N}$ there is $k \geqslant k_0$ such that $\mathscr{G}(A)$ contains a path p starting at i with $l(p) = k$ and* $\operatorname{ct}(p) \geqslant h$;

(iii) *h is the greatest value such that for each $k_0 \in \mathbb{N}$, $\mathscr{G}(A, h)$ contains a path p with length $l(p) \geqslant k_0$ beginning at i and ending at some node j with $b_j \geqslant h$;*

(iv) *h is the greatest value such that in $\mathscr{G}(A, h)$ there is a path p from i to some node j with $b_j \geqslant h$ such that p contains a cycle.*

In the proof of the main theorem we shall need the following well-known statement of number theory:

LEMMA 4. *If $c_1, c_2, \ldots, c_m$ are integers with $\gcd(c_1, c_2, \ldots, c_m) = 1$, then there exists $k_0 \in \mathbb{N}$ such that each integer $k \geqslant k_0$ can be expressed as a nonnegative linear combination of $c_1, c_2, \ldots, c_m$, i.e., there exist nonnegative integers $\alpha_1, \alpha_2, \ldots, \alpha_m$, such that $k = \alpha_1 c_1 + \alpha_2 c_2 + \cdots + \alpha_m c_m$.*

DEFINITION 4. A digraph $\mathscr{G}$ is *strongly stable* if each strongly connected component of $\mathscr{G}$ either is acyclic or has period 1.

THEOREM 2. *A matrix $A \in \mathscr{B}(n, n)$ is strongly stable if and only if each threshold digraph $\mathscr{G}(A, h)$ is strongly stable.*

Proof. For the "if" implication, fix b and i, take $h = \limsup_{k \to \infty}\{x(k)_i\}$, and look at the digraph $\mathscr{G}(A, h)$. Due to Lemma 3, there is a path p beginning at i and ending at some node j with $b_j \geqslant h$ such that p contains a cycle, say $\mathscr{C}$. Look at the SCC $\mathscr{G}'$ of $\mathscr{G}(A, h)$ containing $\mathscr{C}$. Now $\mathscr{G}(A, h)$ is strongly stable by the assumption; therefore $\mathscr{G}'$ contains cycles $\mathscr{C}_1, \mathscr{C}_2, \ldots, \mathscr{C}_m$ with lengths $c_1, c_2, \ldots, c_m$ such that $\gcd(c_1, c_2, \ldots, c_m) = 1$. As they are all in the same SCC, it is possible to pick a path, say p', that starts at i, meets each of the cycles $\mathscr{C}_1, \mathscr{C}_2, \ldots, \mathscr{C}_m$, and ends at j. Then, by Lemma 4, there exists $k_0 \in \mathbb{N}$ such that one can find in $\mathscr{G}(A, h)$ a path from i to j of length k for any $k \geqslant k_0$, traversing the cycles $\mathscr{C}_1, \mathscr{C}_2, \ldots, \mathscr{C}_m$ suitable numbers of times. That means that $x(k)_i \geqslant h$ for all $k \geqslant k_0$. However, the same cannot be said about any $h' > h$, since h was chosen to be the greatest value appearing in the sequence $\{x(k)_i\}_{k=1}^{\infty}$ infinitely many times. Therefore $\{x(k)_i\}_{k=1}^{\infty}$ stabilizes for each i, implying that $\{x(k)\}_{k=1}^{\infty}$ stabilizes too.

For the "only if" part suppose that the orbit $\mathscr{O}\!\mathscr{r}(A, b)$ stabilizes for each starting vector b, but $\mathscr{G}(A, h)$ is not strongly stable for some $h > 0$. Then there is a SCC $\mathscr{G}'$ of $\mathscr{G}(A, h)$, containing a cycle, but such that the gcd of all the cycle lengths in $\mathscr{G}'$ is $d > 1$. Now take an arbitrary node i from $\mathscr{G}'$, and define a starting vector by $b_i = h$, $b_j = 0$ for all the remaining nodes.

Using $h > 0$, it is easy to see that to get $x(k)_i = h$ for some $k \in \mathbb{N}$, it is necessary and sufficient that there be a cycle $\mathscr{C}$ (not necessarily elementary) of length k from i to i in $\mathscr{G}(A, h)$. In fact that means that $\mathscr{C}$ is in $\mathscr{G}'$. But since d divides the lengths of all such cycles, d divides k too. We can conclude that $x(k)_i = h$ will occur infinitely many times, but only for k a multiple of d. Therefore $\mathscr{O}\!\mathscr{b}(A, b)$ does not stabilize. ■

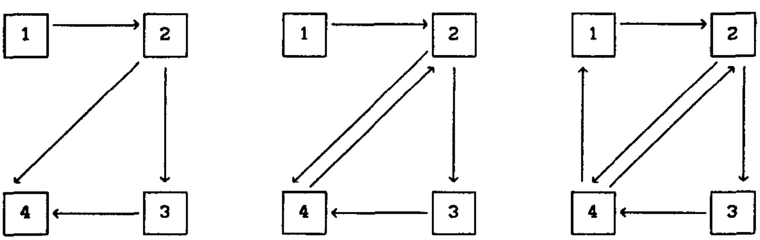
EXAMPLE 3. For the matrix A from Example 1 we conclude that it is not strongly stable by looking at its threshold digraphs. $\mathscr{G}(A, 0.7)$ contains a strong component formed by nodes 1 and 4, which consists of the only cycle of length 2, and therefore its period is 2.

The associated threshold digraphs $\mathscr{G}(A, 0.7)$, $\mathscr{G}(A, 0.5)$, and $\mathscr{G}(A, 0.4)$ for

$$A = \begin{pmatrix} 0 & 0.7 & 0 & 0 \\ 0 & 0 & 0.7 & 0.7 \\ 0 & 0 & 0 & 0.7 \\ 0.4 & 0.5 & 0 & 0 \end{pmatrix}$$

are given in Figure 3. Notice that all the other threshold digraphs for this matrix are either equal to one of those, or complete, or containing no arcs, therefore it is sufficient to examine the above three.

$\mathscr{G}(A, 0.7)$ is acyclic. $\mathscr{G}(A, 0.5)$ separates into two SCCs. The one containing only node 1 is acyclic, while the one formed by nodes 2, 3, 4 contains two elementary cycles: (2, 4) and (2, 3, 4). Their lengths are comprime; therefore $\mathscr{G}(A, 0.5)$ is strongly stable. $\mathscr{G}(A, 0.4)$ is strongly connected itself, with the lengths of its elementary cycles equal to 2, 3, 4, hence strongly stable.



$\mathscr{G}(A, 0.7)$ $\mathscr{G}(A, 0.5)$ $\mathscr{G}(A, 0.4)$

FIG. 3.

Therefore we can conclude that A is strongly stable, which also means that its powers will stabilize.

4. APPLICATIONS TO SPECIAL CLASSES OF MATRICES

DEFINITION 5. A matrix $A \in \mathscr{B}(n, n)$ is called *upper triangular* if each entry over the main diagonal is not less than any entry below the main diagonal, i.e.,

$$\min\{a_{ij};\, i \leqslant j\} \geqslant \max\{a_{ij};\, i > j\}.$$

Note that this definition includes the classical upper triangular matrices, i.e. such that below the main diagonal there are only zero entries.

THEOREM 3. *The powers of an upper triangular matrix always stabilize.*

Proof. Notice that if any cycle containing at least two nodes appears in a threshold digraph $\mathscr{G}(A, h)$ of an upper triangular matrix A, then this means that some entry a_{ij} below the main diagonal of the matrix has been involved as the capacity of an arc on this cycle. But due to the definition of upper triangular matrices, that means that all the main-diagonal entries fulfill $a_{kk} \geqslant a_{ij} \geqslant h$. Therefore $\mathscr{G}(A, h)$ contains all the loops, and each SCC in it has period 1. ■

DEFINITION 6. A matrix $A \in \mathscr{B}(n, n)$ is *symmetric* if $a_{ij} = a_{ji}$ for all $i, j \in N$.

Notice that all the associated digraphs of a symmetric matrix are symmetric; therefore we can consider just their undirected versions, and the condition for stabilization becomes simpler.

THEOREM 4. *A symmetric matrix A is strongly stable if and only if in each undirected threshold graph $\mathscr{G}(A, h)$ each SCC either contains an odd cycle or is an isolated node.*

DEFINITION 7. A matrix $A \in \mathscr{B}(n, n)$ is called a *circulant matrix*, or simply a *circulant*, if it is of the form

$$A = \begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_{n-1} & a_n \\ a_n & a_1 & a_2 & \cdots & a_{n-2} & a_{n-1} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ a_2 & a_3 & a_4 & \cdots & a_n & a_1 \end{pmatrix}.$$

The set of entries with the same index is called a *stripe*; the entries a_i form the ith stripe. In the associated digraph $\mathscr{G}(A)$ each stripe i defines a set of arcs of the form $(k, k + i - 1)$ for $k = 1, 2, \dots, n$; obviously all the numbers here are considered modulo n. We shall say that the *span* of an arc in the ith stripe is $i - 1$. As was observed already in [6], that means that these arcs fall into a set of disjoint cycles, all with the same length equal to $n/\gcd(n, i - 1)$ for $i = 2, \dots, n$ and 1 for the first stripe.

EXAMPLE 4. As an illustration, consider the circulant

$$A = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ a_6 & a_1 & a_2 & a_3 & a_4 & a_5 \\ a_5 & a_6 & a_1 & a_2 & a_3 & a_4 \\ a_4 & a_5 & a_6 & a_1 & a_2 & a_3 \\ a_3 & a_4 & a_5 & a_6 & a_1 & a_2 \\ a_2 & a_3 & a_4 & a_5 & a_6 & a_1 \end{pmatrix}.$$

The corresponding digraphs for its individual stripes are given in Figure 4. From these pictures we can make the following observations, which we later generalize. If $a_1 > \max\{a_2, \dots, a_6\}$, then the first nontrivial threshold digraph consists of six loops; therefore it consists of six SCCs, each with period 1. For a lower value of the threshold, some more arcs will appear, causing some SCCs to merge, but the period of none of them will increase. Therefore in this case the powers of the circulant will stabilize. However, if e.g. $a_2 > \max\{a_1, a_3, \dots, a_6\}$, then the first nontrivial threshold digraph consists of a single cycle of length 6; it has therefore a single SCC with period 6. Thus the powers of A will oscillate.

Denote $J(A) = \{j;\ a_j = \max\{a_1, a_2, \dots, a_n\}\}$ for a given circulant $A \in \mathscr{B}(n, n)$.

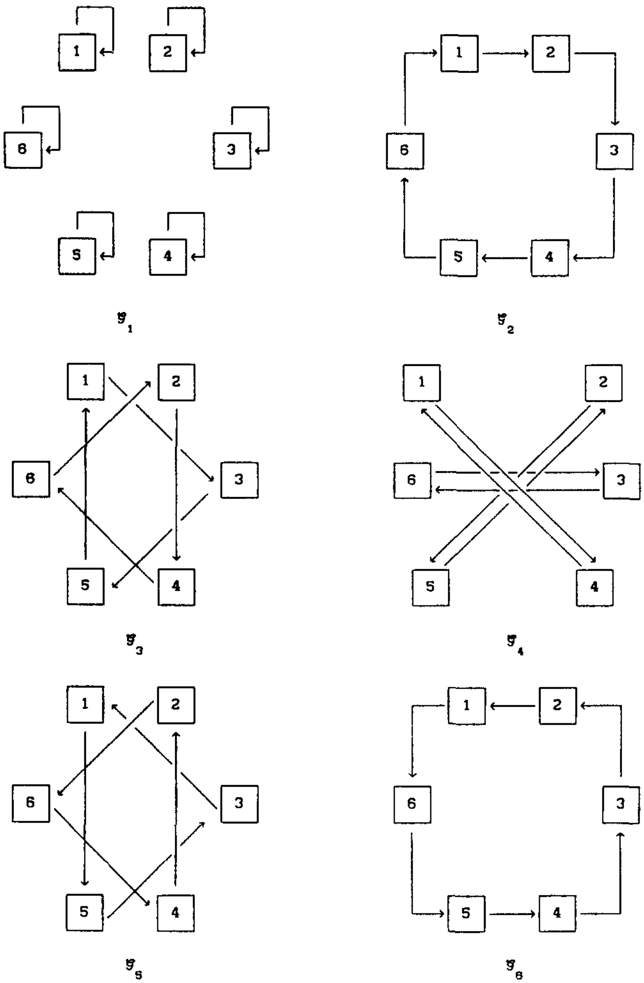


FIG. 4.

THEOREM 5. *If* $1 \in J(A)$ *for a given circulant* $A \in \mathscr{B}(n, n)$, *then A is strongly stable. If* $\gcd(n, J(A)) = d > 1$, *then the powers of A oscillate.*

Proof. The first assertion follows from the above. For the second one it is sufficient to show that the period of the first threshold digraph $\mathscr{G}$ for A is equal to $d > 1$. Let $J(A) = \{i_1, i_2, \ldots, i_k\}$ and $1 \notin J(A)$. Then $\mathscr{G}$ contains arcs with spans $i_1 - 1, i_2 - 1, \ldots, i_k - 1$. To get a cycle in $\mathscr{G}$ we must return to the starting node, traversing only arcs with these spans. This means that, adding the spans of the arcs on a cycle together, we must get a multiple of n, or

$$0 \equiv m_1(i_1 - 1) + m_2(i_2 - 1) + \cdots + m_k(i_k - 1) \pmod{n}, \tag{4}$$

where $m_1, m_2, \ldots, m_k$ are the numbers of arcs on the cycle, chosen from strips $i_1, i_2, \ldots, i_k$. Equation (4) is also equivalent to the following congruence:

$$m_1 + m_2 + \cdots + m_k \equiv m_1 i_1 + m_2 i_2 + \cdots + m_k i_k \pmod{n}. \tag{5}$$

Notice that $m_1 + m_2 + \cdots + m_k$ is the length of the obtained cycle. Elementary number theory now implies that if $i_1, i_2, \ldots, i_k$ have a common divisor d with n, then all the cycle lengths in $\mathcal{G}$ will be divisible by d, and the proof is complete. ∎

REMARK. The results presented in this paper have been recently generalized by M. Gavalec to an arbitrary period d. A polynomial algorithm for computing the period of power sequences was also given (see [4] and [5]).

REFERENCES

1 E. M. Arkin, C. H. Papadimitriou, and M. Yannakakis, Modularity of cycles and paths in graphs, *J. Assoc. Comput. Mach.* 38(2):255–274 (1991).
2 K. Cechlárová, Eigenvectors in bottleneck algebra, *Linear Algebra Appl.* 175:63–73 (1992).
3 R. A. Cuninghame-Green, *Minimax Algebra*, Lecture Notes in Econom. and Math. Systems 166, Springer-Verlag, Berlin, 1979.
4 M. Gavalec, Periodicity of matrices and orbits in fuzzy algebra, *Tatra Mt. Math. Publ.* 6(1995), 35–46.
5 M. Gavalec, An algorithm for computing the period of a fuzzy matrix, Preprint, Technical Univ. Košice, 1994.
6 Garfinkel, Minimizing wallpaper waste, *Oper. Res.* 25:741–751 (1977).
7 R. Hemasinha, N. R. Pal, and J. C. Bezdek, Iterates of fuzzy circulant matrices, Preprint, 1993.
8 Li Jian-Xin, Periodicity of powers of fuzzy matrices (finite fuzzy relations), *Fuzzy Sets and Systems* 48:365–369 (1992).

9 Li Jian-Xin, An upper bound on indices of finite fuzzy relations, *Fuzzy Sets and Systems* 49:317–321 (1992).
10 M. G. Thomasson, Convergence of powers of a fuzzy matrix, *J. Math. Anal. Appl.* 57:476–480 (1977).
11 U. Zimmermann, *Linear and Combinatorial Optimization in Ordered Algebraic Structures*, Ann. Discrete Math. 10, North Holland, Amsterdam, 1981.

Received 13 April 1994; final manuscript accepted 17 October 1994