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# Extensional realizability

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**Abstract**

Two straightforward "extensionalisations" of Kleene's realizability are considered; denoted $\mathbf{r}_e$ and $\mathbf{e}$. It is shown that these realizabilities are not equivalent. While the $\mathbf{r}_e$-notion is (as a relation between numbers and sentences) a subset of Kleene's realizability, the $\mathbf{e}$-notion is not. The problem of an axiomatization of $\mathbf{e}$-realizability is attacked and one arrives at an axiomatization over a conservative extension of arithmetic, in a language with variables for finite sets. A derived rule for arithmetic is obtained by the use of a $\mathbf{q}$-variant of $\mathbf{e}$-realizability; this rule subsumes the well-known Extended Church's Rule. The second part of the paper focuses on toposes for these realizabilities. By a relaxation of the notion of partial combinatory algebra, a new class of realizability toposes emerges. Relationships between the various realizability toposes are given, and results analogous to Robinson and Rosolini's characterization of the effective topos, are obtained for a topos generalizing $\mathbf{e}$-realizability.

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## 0. Introduction

In many accounts of Kleene's realizability, the analogy with the Brouwer–Heyting–Kolmogorov proof interpretation is stressed. However, if one reads this interpretation (in the case of implication) as: "a proof of an implication $A \rightarrow B$ is an operation which assigns proofs of $B$ to proofs of $A$", there is a problem with extensionality in the case of nested implications.

A Kleene realizer for $(A \rightarrow B) \rightarrow C$ codes an operation which assigns, to *codes* of operations for $A \rightarrow B$, a realizer for $C$; but two different codes for the "same" (in some sense) operation may well be sent to different realizers.

"Extensional realizability" is a modification of Kleene's original definition, where a notion of "$x$ and $y$ are equivalent as realizers of $A$" is built in; it is then required that


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realizers of $A \to B$ code operations which send equivalent realizers of $A$ to equivalent realizers of $B$.

There are at least two ways to do this:

1. One may define, for every formula $A$, a partial equivalence relation $\sim_A$ on the set of Kleene realizers of $A$ by recursion on $A$; I say that $x$ $\mathbf{r}_e$-realizes $A$ (abbreviated $x\,\mathbf{r}_e\,A$) if $x \sim_A x$.

2. One may simultaneously define, by recursion on $A$, the set of realizers of $A$ and an equivalence relation $=_A$ on that set. I call this notion **e**-realizability; $x$ **e**-realizes $A$ $(x\,\mathbf{e}\,A)$ iff $x =_A x$.

These two ways resemble the two constructions of an extensional type structure out of the structure **HRO** of hereditarily recursive operations: giving $\mathbf{HRO}^E$ and **HEO**, respectively (see [18] for details).

Inductive definitions for the two approaches are presented in Section 1. The second approach was first given by Beeson ([2] and [3], both with a mistake in the clause for implication, though) with an interpretation of Martin–Löf's type theory in mind. There is also an application in [15], and in [7], 6.3.

It will be shown that $\mathbf{r}_e$- and **e**-realizability are not equivalent as interpretations of intuitionistic arithmetic **HA**. The proof rests on a lemma which has another interesting corollary: the open schema

$$A \to \exists x(x\,\mathbf{e}\,A)$$

is not **e**-realizable. This failure of "idempotency" of **e**-realizability makes it impossible to prove a characterization result of the kind

$$\mathbf{HA} \vdash \exists x(x\,\mathbf{e}\,A) \;\Leftrightarrow\; \mathbf{HA} + F \vdash A \tag{1}$$

(for an axiom or axiom scheme $F$), in a straightforward way like Troelstra's characterization of Kleene's realizability [17]. For there, he used

$$\mathbf{HA} + F \vdash A \leftrightarrow \exists x(x\,\mathbf{r}\,A) \tag{2}$$

for arbitrary *formulas* $A$, to derive (1) (here $F$ was the schema $\mathrm{ECT}_0$, and $x\,\mathbf{r}\,A$ means $x$ realizes $A$ in Kleene's sense).

However, I shall obtain a characterization of **e**-realizability over a conservative extension $\mathbf{HA}^\alpha$ of $\mathbf{HA}$ + Markov's Principle MP. More precisely, $\mathbf{HA}^\alpha$ has variables $\alpha$ of a new sort, and I define the notion "$\alpha$ realizes $A$" for formulas $A$ in the extended language. This definition will be idempotent for arithmetical formulas (formulas in the language of **HA**), and we obtain

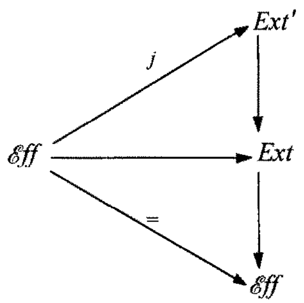
$$\mathbf{HA}^\alpha + ECT^\alpha \vdash A \leftrightarrow \exists\alpha(\alpha \text{ realizes } A)$$

for some scheme $ECT^\alpha$ and arithmetical formulas $A$, as well as

$$\mathbf{HA}^\alpha \vdash \exists\alpha(\alpha \text{ realizes } A) \;\Leftrightarrow\; \mathbf{HA} + \mathrm{MP} \vdash \exists x(x\,\mathbf{e}\,A)$$

In Section 2 I turn to proof-theoretical aspects of **e**-realizability. By a suitable "**q**-variant" of **e**-realizability, a derived rule of **HA** is obtained which subsumes the well-known extended Church's rule.

Section 3 deals with toposes generalizing notions of extensional realizability in the same way as Hyland's "effective topos" $\mathscr{E}ff$ [9] generalizes Kleene's realizability. The first description of a topos generalizing **e**-realizability was given by Pitts [13]. I call this topos $Ext$ and explain some of its internal logic. There is also a topos $Ext'$ which generalizes $\mathbf{r}_e$-realizability. We have a commutative diagram of geometric morphisms between these toposes of the form




In this diagram, $j$ is an *open inclusion*, i.e. there is a subobject $U$ of 1 in $Ext'$ such that $\mathscr{E}ff$ is equivalent to the slice topos $Ext'/U$ and $j^*$ is, modulo this equivalence, the pullback functor $Ext' \to Ext'/U$.

The topos $Ext$ seems rather hard to analyse. However, there is another topos (the construction of which mirrors the extension $\mathbf{HA}^\alpha$ of **HA** in Section 1) into which $Ext$ embeds; and this embedding preserves the logic of all finite types over the natural numbers. This new topos $\mathscr{A}$ is somewhat easier to handle because the construction is similar to that of $\mathscr{E}ff$ and several results about $\mathscr{E}ff$ have their counterparts for $\mathscr{A}$; in particular, the results in [16, 4]. I show that the topos $\mathscr{A}$ is the *exact completion* of the category of $\neg\neg$-separated objects of the effective topos (for definitions see Section 3).

Behind the construction of $\mathscr{A}$ there is a generalization of the notion of "partial combinatory algebra" (pca), called $\leqslant$ *-pca*, which I think may be independently interesting.

## 1. Definitions and basic properties

### *1.1. Nonequivalence of two notions of extensional realizability*

In the following definition, the notions $x \sim_A y$, $x\,\mathbf{r}_e\,A$, $x =_A y$ and $x\,\mathbf{e}\,A$ are defined, for numbers $x, y$ and arithmetical formulas $A$. These notions will also be taken as arithmetical formulas themselves.

I write $x \bullet y$ for the outcome, if any, of the computation of the $x$th Turing machine with input $y$; $\downarrow$ means "defined" so $x \bullet y\downarrow$ is equivalent to $\exists z T(x, y, z)$ where

$T$ is Kleene's predicate. $\langle\cdot,\cdot\rangle$, $(\cdot)_0$, $(\cdot)_1$ are primitive recursive functions such that $\langle(x)_0,(x)_1\rangle = x, (\langle x,y\rangle)_0 = x$ and $(\langle x,y\rangle)_1 = y$.

The notation $x\ \mathbf{r}\ A$ means that $x$ realizes $A$ in Kleene's sense, or the formula expressing this in arithmetic.

**Definition 1.1.** (1) Define for every formula $A$ the formula $x \sim_A y$, where $x, y$ are variables which do not occur in $A$:

$$x \sim_{t=s} y \equiv x = y \wedge t = s$$

$$x \sim_{A\wedge B} y \equiv (x)_0 \sim_A (y)_0 \wedge (x)_1 \sim_B (y)_1$$

$$x \sim_{A\vee B} y \equiv ((x)_0 = (y)_0 = 0 \wedge (x)_1 \sim_A (y)_1) \vee ((x)_0 \neq 0 \wedge (y)_0 \neq 0 \wedge (x)_1 \sim_B (y)_1)$$

$$x \sim_{A\to B} y \equiv x\ \mathbf{r}\ A \to B \wedge y\ \mathbf{r}\ A \to B \wedge \forall w w'(w \sim_A w' \to x \bullet w \sim_B y \bullet w')$$

$$x \sim_{\neg A} y \equiv \forall w \neg(w\ \mathbf{r}\ A)$$

$$x \sim_{\forall n A(n)} y \equiv \forall n(x \bullet n\downarrow \wedge\ y \bullet n\downarrow \wedge\ x \bullet n \sim_{A(n)} y \bullet n)$$

$$x \sim_{\exists n A(n)} y \equiv (x)_0 = (y)_0 \wedge (x)_1 \sim_{A((x)_0)} (y)_1$$

I write $x\ \mathbf{r}_e\ A$ for $x \sim_A x$.

(2) Define simultaneously by recursion on $A$, the formulas $x\ \mathbf{e}\ A$ and $x =_A y$ (again, $x, y$ do not occur in $A$):

$$x\ \mathbf{e}\ t = s \equiv t = s$$

$$x =_{t=s} y \equiv x = y \wedge t = s$$

$$x\ \mathbf{e}\ A \wedge B \equiv (x)_0\ \mathbf{e}\ A \wedge (x)_1\ \mathbf{e}\ B$$

$$x =_{A\wedge B} y \equiv (x)_0 =_A (y)_0 \wedge (x)_1 =_B (y)_1$$

$$x\ \mathbf{e}\ A \vee B \equiv ((x)_0 = 0 \wedge (x)_1\ \mathbf{e}\ A) \vee ((x)_0 \neq 0 \wedge (x)_1\ \mathbf{e}\ B)$$

$$x =_{A\vee B} y \equiv ((x)_0 = (y)_0 = 0 \wedge (x)_1 =_A (y)_1) \vee ((x)_0 \neq 0 \wedge (y)_0 \neq 0 \wedge (x)_1 =_B (y)_1)$$

$$x\ \mathbf{e}\ A \to B \equiv \forall y y'(y =_A y' \to x \bullet y\downarrow \wedge\ x \bullet y'\downarrow \wedge\ x \bullet y =_B x \bullet y')$$

$$x =_{A\to B} y \equiv x\ \mathbf{e}\ A \to B \wedge y\ \mathbf{e}\ A \to B \wedge \forall w(w\ \mathbf{e}\ A \to x \bullet w =_B y \bullet w)$$

$$x\ \mathbf{e}\ \neg A \equiv \forall w \neg(w\ \mathbf{e}\ A)$$

$$x =_{\neg A} y \equiv \forall w \neg(w\ \mathbf{e}\ A)$$

$$x\ \mathbf{e}\ \forall n A(n) \equiv \forall n(x \bullet n\downarrow \wedge\ x \bullet n\ \mathbf{e}\ A(n))$$

$x =_{\forall nA(n)} y \equiv \forall n(x \bullet n\downarrow \wedge y \bullet n\downarrow \wedge x \bullet n =_{A(n)} y \bullet n)$

$x \, \mathbf{e} \, \exists nA(n) \equiv (x)_1 \, \mathbf{e} \, A((x)_0)$

$x =_{\exists nA(n)} y \equiv (x)_0 = (y)_0 \wedge (x)_1 =_{A((x)_0)} (y)_1$

Some obvious consequences of Definition 1.1 are that $\sim_A$ and $=_A$ are symmetric and transitive relations, that $x \sim_A x$ implies $x \, \mathbf{r} \, A$ and that $x =_A x$ is equivalent to $x \, \mathbf{e} \, A$.

A difference between the notions $x \, \mathbf{r}_e \, A$ and $x \, \mathbf{e} \, A$ that presents itself immediately, is in the clause for implication (from which the one for negation follows). Using classical logic, it is easy to see that $A \vee \neg A$ is **e**-realizable for sentences $A$; not so for $\mathbf{r}_e$-realizability. Classically again, it is true that $A \vee \neg A \vee \neg\neg A$ is $\mathbf{r}_e$-realizable for any sentence $A$: if $A$ and $\neg A$ are not $\mathbf{r}_e$-realizable, then $A$ must be Kleene-realizable, $\sim_{\neg A}$ is the empty relation and $\neg\neg A$ is $\mathbf{r}_e$-realizable.

We shall see that all three possibilities do in fact occur.

First, let us record:

**Proposition 1.2** (Soundness). *If* **HA** $\vdash A$ *then* **HA** $\vdash \exists x(x \, \mathbf{r}_e \, A)$ *and* **HA** $\vdash \exists y(y \, \mathbf{e} \, A)$.

**Proof.** A routine induction on **HA** $\vdash A$. Examples of this kind of proof abound in [18]. □

For the next lemma, recall that an *almost negative* formula is a formula built up from formulas of form $\exists y(t = s)$ using only the connectives $\wedge, \rightarrow, \forall$.

**Lemma 1.3.** *Let, for almost negative formulas $A$, $\psi_A$ be the p-term (i.e. a "term" built up using $\bullet$, so it is not always defined) from* [18], 3.2.11. *Then*:

$$\mathbf{HA} \vdash A \rightarrow \psi_A\downarrow \wedge \psi_A \, \mathbf{r}_e \, A \wedge \psi_A \, \mathbf{e} \, A$$

$$\mathbf{HA} \vdash \exists y(y \, \mathbf{r}_e \, A) \rightarrow A$$

$$\mathbf{HA} \vdash \exists y(y \, \mathbf{e} \, A) \rightarrow A$$

**Proof.** Trivial. □

**Lemma 1.4.** *The following sentence of* **HA** *is neither* $\mathbf{r}_e$*- nor* **e***-realizable*:

$$\forall e[\forall x \exists y(\neg\neg\exists z T(e,x,z) \rightarrow T(e,x,y))$$
$$\rightarrow \exists v \forall x \exists u(T(v,x,u) \wedge (\neg\neg\exists y T(e,x,y) \rightarrow T(e,x,U(u))))]$$

(*Here $T$ is Kleene's predicate, and $U$ the result extracting function.*)

**Proof.** The proof is similar for both realizabilities; I give it for **e**-realizability. The reasoning is informal; but can be carried out in **HA**+**MP**. Let $A$ denote the sentence in the statement of the lemma, and suppose for contradiction that $w \, \mathbf{e} \, A$. Then $w$ codes a total recursive function. I remark:

(i) If $e$ codes the empty function, then $\Lambda f.((w \bullet e) \bullet f)_0$ is an effective operation of type 2 (i.e. sends codes for the same total recursive function to the same number), for every code of a total recursive function will realize $\forall x \exists y(\neg\neg\exists z T(e,x,z) \to T(e,x,y))$, and these realizers are equivalent if they code the same function.

(ii) If $k$ realizes $\forall x \exists y(\neg\neg\exists z T(e,x,z) \to T(e,x,y))$, then $((w \bullet e) \bullet k)_1$ realizes the formula

$$\forall x \exists u (T(((w \bullet e) \bullet k)_0, x, u) \wedge (\neg\neg\exists y T(e,x,y) \to T(e,x,U(u))))$$

which is equivalent to an almost negative formula, and therefore holds. So in this case we always have

$$\forall x[((w \bullet e) \bullet k)_0 \bullet x\downarrow \wedge (\neg\neg\exists y T(e,x,y) \to T(e,x,((w \bullet e) \bullet k)_0 \bullet x))]$$

Using the recursion theorem we can find a code $e$ for a partial recursive function of three variables such that

$$e \bullet (k,n,x) \simeq \begin{cases} \text{undefined} & \text{if not } T(n,n,x) \\ & \text{if } T(n,n,x): \\ \text{undefined} & \text{if } ((w \bullet S^2_1(e,k,n)) \bullet \Lambda x.0)_0 \bullet x \text{ is undefined} \\ 0 & \text{if } ((w \bullet S^2_1(e,k,n)) \bullet \Lambda x.0)_0 \bullet x \text{ is defined and not } \\ & T(S^2_1(e,k,n),x,((w \bullet S^2_1(e,k,n)) \bullet \Lambda x.0)_0 \bullet x) \\ U[((w \bullet S^2_1(e,k,n)) \bullet \Lambda x.0)_0 \bullet x)] + 1 & \text{else.} \end{cases}$$

Again some remarks:

(iii) If $T(n,n,x)$, then $((w \bullet S^2_1(e,k,n)) \bullet \Lambda x.0)_0 \bullet x$ is always defined. For if not, $S^2_1(e,k,n)$ would code the empty function, and see (i) and (ii) above.

(iv) If $T(n,n,x)$, then never $T(S^2_1(e,k,n),x,((w \bullet S^2_1(e,k,n)) \bullet \Lambda x.0)_0 \bullet x)$. For were this the case we would have

$$S^2_1(e,k,n) \bullet x = U[((w \bullet S^2_1(e,k,n)) \bullet \Lambda x.0)_0 \bullet x)],$$
$$e \bullet (k,n,x) = U[((w \bullet S^2_1(e,k,n)) \bullet \Lambda x.0)_0 \bullet x)] + 1,$$

which is contradictory.

Again using the recursion theorem, with $e$ as just defined, we take a code $k$ for a partial recursive function of two variables, such that

$$k \bullet (n,x) \simeq \begin{cases} 0 & \text{if not } T(n,n,x), \\ \langle \mu z.T(S_1^2(e,S_1^1(k,n),n),x,z), \Lambda x.0\rangle & \text{else.} \end{cases}$$

Then $S_1^1(k,n)$ always realizes

$$\forall x \exists y(\neg\neg\exists z T(S_1^2(e,S_1^1(k,n),n),x,z) \rightarrow T(S_1^2(e,S_1^1(k,n),n),x,y)).$$

Furthermore, if $n \bullet n$ is undefined then $S_1^1(k,n)$ codes the constant zero function and $S_1^2(e,S_1^1(k,n),n)$ the empty function, so

$$((w \bullet S_1^2(e,S_1^1(k,n),n)) \bullet S_1^1(k,n))_0 = ((w \bullet S_1^2(e,S_1^1(k,n),n)) \bullet \Lambda x.0)_0.$$

If $n \bullet n$ is defined, say $T(n,n,x)$, then (see remark (ii))

$$((w \bullet S_1^2(e,S_1^1(k,n),n)) \bullet S_1^1(k,n))_0 \bullet x$$

is defined, and

$$T(S_1^2(e,S_1^1(k,n),n),x,((w \bullet S_1^2(e,S_1^1(k,n),n)) \bullet S_1^1(k,n))_0 \bullet x)$$

holds. By remarks (iii) and (iv) we have that

$$((w \bullet S_1^2(e,S_1^1(k,n),n)) \bullet \Lambda x.0)_0 \bullet x$$

is defined and *not*

$$T(S_1^2(e,S_1^1(k,n),n),x,((w \bullet S_1^2(e,S_1^1(k,n),n)) \bullet \Lambda x.0)_0 \bullet x).$$

Therefore, in this case

$$((w \bullet S_1^2(e,S_1^1(k,n),n)) \bullet S_1^1(k,n))_0 \neq ((w \bullet S_1^2(e,S_1^1(k,n),n)) \bullet \Lambda x.0)_0.$$

(Note, that both sides are always defined!)
This gives us a decision procedure for the question: "is $n \bullet n$ defined?", and the contradiction is obtained. □

**Corollary 1.5.** $\mathbf{r}_e$- *and* **e**-*realizability are not equivalent.*

**Proof.** For the sentence $A$ of Lemma 1.4 we clearly have that $\neg A$ is **e**-realizable. However, the sentence $A$ is an instance of Church's thesis $\mathrm{CT}_0$ so Kleene-realizable; it follows that $\neg\neg A$ is $\mathbf{r}_e$-realizable. These facts are provable in **HA**+MP, where MP denotes Markov's principle.

**Corollary 1.6.** *The open schema*

$$B \rightarrow \exists x(x\,\mathbf{e}\,B)$$

*is not* **e**-*realizable.*

**Proof.** Take for $B$ the formula $\forall x \exists y(\neg\neg\exists z T(e,x,z) \rightarrow T(e,x,y))$. Then $\exists v(v \mathbf{e}\ B)$ is equivalent to

$$\exists v \forall x \exists u(T(v,x,u) \wedge (\neg\neg\exists z T(e,x,z) \rightarrow T(e,x,U(u)))$$

and apply Lemma 1.4. □

### *1.2. Characterization of* e*-realizability*

As hinted in the Introduction, Corollary 1.6 blocks the way to a straightforward characterization result for **e**-realizability. I now present an extension of **HA** over which **e**-realizability can be characterized.

**Definition 1.7.** The theory $\mathbf{HA}^\alpha$ is an extension of **HA** in a 2-sorted language. Variables of the extra sort are denoted $\alpha, \beta, \gamma, \ldots$ There is an extra non-logical symbol $\in$ and the new atomic formulas are of form $t \in \alpha$ and $\alpha = \beta$.

$\mathbf{HA}^\alpha$ has the following extra axioms (besides those of **HA**, and induction for the full extended language):

(1) $\neg\neg\exists n(n \in \alpha)$,
(2) $\forall n(\neg\neg n \in \alpha \rightarrow n \in \alpha)$,
(3) $\forall \alpha\beta(\alpha = \beta \leftrightarrow \forall n(n \in \alpha \leftrightarrow n \in \beta))$
(4) $\forall \alpha\beta(\forall nm(n \in \alpha \wedge m \in \beta \rightarrow n \bullet m\downarrow) \rightarrow \exists \gamma \forall k(k \in \gamma \leftrightarrow \neg\neg\exists n \in \alpha \exists m \in \beta(k = n \bullet m)))$,
(5) $\neg\neg\exists n \forall y(y \in \alpha \leftrightarrow y = n) \rightarrow \exists n \forall y(y \in \alpha \leftrightarrow y = n)$,
(6) $\forall nm \exists \alpha(n \in \alpha \wedge m \in \alpha)$,
(7) $\forall n \exists \alpha \forall m(m \in \alpha \leftrightarrow m = n)$,
(8) Markov's principle: $\neg\neg\exists y(t = s) \rightarrow \exists y(t = s)$.

We can (and do) think about the $\alpha$'s as some sort of sets of numbers; I shall often refer to the variables $\alpha$ as *set* variables.

In view of the extensionality axiom (3) and axioms (4) and (7) we may pass to a definitional extension of $\mathbf{HA}^\alpha$ and introduce terms $\{n\}$ and partial terms $\alpha \bullet \beta$ with definitions

$$x \in \{n\} \leftrightarrow x = n,$$
$$\alpha \bullet \beta\downarrow \leftrightarrow \forall n \in \alpha \forall m \in \beta(n \bullet m\downarrow),$$
$$\alpha \bullet \beta\downarrow \rightarrow \forall n(n \in \alpha \bullet \beta \leftrightarrow \neg\neg\exists k \in \alpha \exists l \in \beta(n = k \bullet l)).$$

Note that from (6) and (7) we can derive

$$\forall nm \exists \alpha \forall k(k \in \alpha \leftrightarrow k = n \vee k = m).$$

For, given $n$ and $m$ first pick (by 6) a $\beta$ with $n \in \beta \wedge m \in \beta$. If $e$ is such that

$$e \bullet x \simeq \begin{cases} n & \text{if } x = n, \\ m & \text{else,} \end{cases}$$

then for $\alpha = \{e\} \bullet \beta$ we have

$$k \in \alpha \leftrightarrow \neg\neg(k = n \vee k = m) \leftrightarrow k = n \vee k = m.$$

This extends to sequences. I shall therefore also use the notation $\{n, m\}$.

From now on, I call formulas in which no set variables occur (either free or bound) *arithmetical*.

**Proposition 1.8.** (1) $\mathbf{HA}^\alpha$ *is conservative over* **HA**+MP.

(2) *For arithmetical formulas* $A(x)$:

$$\mathbf{HA}^\alpha \vdash \exists\alpha\forall n \in \alpha A(n) \Rightarrow \mathbf{HA}{+}\mathrm{MP} \vdash \exists n A(n).$$

**Proof.** Both results follow directly from a translation of $\mathbf{HA}^\alpha$ into **HA**, which interprets the $\alpha$'s as codes for finite inhabited sets. All axioms of $\mathbf{HA}^\alpha$ are valid under this translation, as well as the axiom $\exists x(x \in \alpha)$; this gives the second statement at once. □

**Definition 1.9** (*Realizability for* $\mathbf{HA}^\alpha$). Define a realizability notion $\alpha\ \mathbf{r}\ A$ for formulas $A$ in the language of $\mathbf{HA}^\alpha$ not containing the variable $\alpha$:

$$\alpha\ \mathbf{r}\ t = s \equiv \alpha = \{t\} \wedge t = s$$

$$\alpha\ \mathbf{r}\ n \in \beta \equiv \alpha = \{n\} \wedge n \in \beta$$

$$\alpha\ \mathbf{r}\ \beta = \gamma \equiv \alpha \subseteq \beta \wedge \beta = \gamma$$

$$\alpha\ \mathbf{r}\ A \wedge B \equiv p_0\alpha\ \mathbf{r}\ A \wedge p_1\alpha\ \mathbf{r}\ B$$

$$\alpha\ \mathbf{r}\ A \to B \equiv \forall\beta(\beta\ \mathbf{r}\ A \to \alpha \bullet \beta\downarrow \wedge\ \alpha \bullet \beta\ \mathbf{r}\ B)$$

$$\alpha\ \mathbf{r}\ \exists n A(n) \equiv \exists n(p_0\alpha = \{n\} \wedge p_1\alpha\ \mathbf{r}\ A(n))$$

$$\alpha\ \mathbf{r}\ \exists\beta A(\beta) \equiv \exists\beta(p_0\alpha \subseteq \beta \wedge p_1\alpha\ \mathbf{r}\ A(\beta))$$

$$\alpha\ \mathbf{r}\ \forall n A(n) \equiv \forall n(\alpha \bullet \{n\}\downarrow \wedge\ \alpha \bullet \{n\}\ \mathbf{r}\ A(n))$$

$$\alpha\ \mathbf{r}\ \forall\beta A(\beta) \equiv \forall\beta(\alpha \bullet \beta\downarrow \wedge\ \alpha \bullet \beta\ \mathbf{r}\ A(\beta)).$$

Here $\alpha \subseteq \beta$ abbreviates $\forall n(n \in \alpha \to n \in \beta)$ and $p_i\alpha = \{\Lambda x.(x)_i\} \bullet \alpha$ for $i = 0, 1$; so

$$x \in p_i\alpha \leftrightarrow \neg\neg\exists y \in \alpha(x = (y)_i).$$

**Lemma 1.10.** *For arithmetical* $A$:

(1) $\mathbf{HA}^\alpha \vdash \forall xy(\neg\neg(x =_A y) \to x =_A y)$,

(2) $\mathbf{HA}^\alpha \vdash \forall\alpha(\neg\neg(\alpha\ \mathbf{r}\ A) \to \alpha\ \mathbf{r}\ A)$.

**Proof.** (1) follows by Markov's principle, since all formulas $x =_A y$ are (equivalent to) almost negative formulas, so by Markov's principle to negated formulas.

(2) is a straightforward induction where one uses the axioms of $\mathbf{HA}^{\alpha}$. One case: $\alpha\,\mathbf{r}\,\exists nA(n)$ is equivalent to

$$\exists n(p_0\alpha = \{n\}) \wedge \forall n(p_0\alpha = \{n\} \rightarrow p_1\alpha\,\mathbf{r}\,A(n))$$

and this is $\neg\neg$-stable by axiom 5). □

**Proposition 1.11.** *For arithmetical $A$,*

$$\mathbf{HA}^{\alpha} \vdash \alpha\,\mathbf{r}\,A \leftrightarrow \forall nm \in \alpha(n =_A m).$$

**Proof.** Induction on $A$; I do two cases, leaving the others to the reader.

- Let $A \equiv B \wedge C$; suppose $\alpha\,\mathbf{r}\,A$ and $n, m \in \alpha$. Then $(n)_i, (m)_i \in p_i\alpha\,(i = 0, 1)$ and since $p_0\alpha\,\mathbf{r}\,B, p_1\alpha\,\mathbf{r}\,C$ we have $(n)_0 =_B (m)_0, (n)_1 =_C (m)_1$ by induction hypothesis, so $n =_A m$.

  Conversely, suppose $\forall nm \in \alpha(n =_A m)$ so $\forall nm \in \alpha((n)_0 =_B (m)_0 \wedge (n)_1 =_C (m)_1)$. Then $\forall nm \in p_0\alpha\neg\neg(n =_B m)$ so $\forall nm \in p_0\alpha(n =_B m)$ by Lemma 1.10, which gives $p_0\alpha\,\mathbf{r}\,B$ by induction hypothesis. Similarly, $p_1\alpha\,\mathbf{r}\,C$ and $\alpha\,\mathbf{r}\,A$.
- Let $A \equiv B \rightarrow C$. Suppose $\alpha\,\mathbf{r}\,B \rightarrow C$ and $n, m \in \alpha$. Given $y, y'$ with $y =_B y'$, by induction hypothesis we have that $\{y, y'\}\,\mathbf{r}\,B$ so $n \bullet y{\downarrow} \wedge n \bullet y'{\downarrow} \wedge \{n \bullet y, n \bullet y'\}\,\mathbf{r}\,C$ so by induction hypothesis it follows that $n\,\mathbf{e}\,B \rightarrow C$. Similarly, $m\,\mathbf{e}\,B \rightarrow C$ and $n =_{B\rightarrow C} m$ follow.

  Conversely, if $\forall nm \in \alpha(n =_{B\rightarrow C} m)$ and $\beta\,\mathbf{r}\,B$ then $\alpha \bullet \beta{\downarrow} \wedge \forall k, l \in \alpha \bullet \beta(\neg\neg k =_C l)$, so by Lemma 1.10 $\forall k, l \in \alpha \bullet \beta(k =_C l)$ so by induction hypothesis $\alpha \bullet \beta\,\mathbf{r}\,C$; so $\alpha\,\mathbf{r}\,B \rightarrow C$. □

**Proposition 1.12.** *For any $A$ in the language of* $\mathbf{HA}^{\alpha}$,

$$\mathbf{HA}^{\alpha} \vdash \alpha\,\mathbf{r}\,A \wedge \beta \subseteq \alpha \rightarrow \beta\,\mathbf{r}\,A.$$

**Proof.** At once. □

**Proposition 1.13** (Soundness for $\alpha\,\mathbf{r}\,A$). *For any $A$ in the language of* $\mathbf{HA}^{\alpha}$,

$$\mathbf{HA}^{\alpha} \vdash A \Rightarrow \mathbf{HA}^{\alpha} \vdash \exists\alpha(\alpha\,\mathbf{r}\,A)$$

**Proof.** In essence, there is nothing new here. To see this, note the following fact: given a set expression $T$ built up from variables $\alpha_1, \ldots, \alpha_n$, the partial application $\bullet$ and $p_0, p_1$; consider the numerical partial term $t = \Lambda x_1 \cdots x_n.T[x_1/\alpha_1, \ldots, x_n/\alpha_n]$ where the $x_i$ are new number variables, the $\bullet$ is now interpreted as Kleene application for numbers, and $p_i$ is replaced by $(\cdot)_i$. Then one proves by induction on $T$:

$$\mathbf{HA}^{\alpha} \vdash \forall\alpha_1 \cdots \alpha_n(T{\downarrow} \rightarrow \{t\} \bullet \alpha_1 \bullet \cdots \bullet \alpha_n{\downarrow} \wedge \subseteq \top).$$

Now one forms the terms realizing formulas about just as in the soundness proof for Kleene realizability, and uses Lemma 1.12.

**Definition 1.14.** $\mathrm{ECT}^\alpha$ is the following axiom scheme:

$$\forall\alpha(\neg A(\alpha) \to \exists\beta B(\alpha,\beta)) \to \exists\gamma\forall\alpha(\neg A(\alpha) \to \gamma\bullet\alpha\downarrow \wedge \exists\beta(\gamma\bullet\alpha \subseteq \beta \wedge B(\alpha,\beta))).$$

**Proposition 1.15.** $\mathrm{ECT}^\alpha$ *is* **r**-*realizable, i.e.*

$$\mathbf{HA}^\alpha \vdash \exists\alpha(\alpha\ \mathbf{r}\ F)$$

*for any instance* $F$ *of* $\mathrm{ECT}^\alpha$.

**Proof.** Of course, the "formulas" $\gamma\bullet\alpha\downarrow$ and $\gamma\bullet\alpha\subseteq\beta$ are read as

$$\forall nm(n\in\gamma \wedge m\in\alpha \to n\bullet m\downarrow)$$

and

$$\forall k(\neg\neg\exists n\in\gamma\exists m\in\alpha(k=n\bullet m) \to k\in\beta),$$

respectively; the reader should convince himself that there are numbers $n$ and $m$ such that

$$\mathbf{HA}^\alpha \vdash \forall\alpha\gamma(\gamma\bullet\alpha\downarrow \leftrightarrow \{n\}\,\mathbf{r}\,(\gamma\bullet\alpha\downarrow)),$$
$$\mathbf{HA}^\alpha \vdash \forall\alpha\beta\gamma(\gamma\bullet\alpha\subseteq\beta \leftrightarrow \{m\}\,\mathbf{r}\,(\gamma\bullet\alpha\subseteq\beta)).$$

Fix these $n$ and $m$ for the rest of the proof. Write

$$f_e = \Lambda x.((e\bullet x)_0)_0,$$
$$g_e = \Lambda xy.\langle n, \langle m, ((e\bullet x)\bullet 0)_1\rangle\rangle.$$

Then I claim that $\{\Lambda e.\langle f_e, g_e\rangle\}$ realizes $\mathrm{ECT}^\alpha$.

A verification of this is left to the reader, who may wish to contemplate the following: $\epsilon\,\mathbf{r}\,\forall\alpha(\neg A(\alpha) \to \exists\beta B(\alpha,\beta))$ is

$$\forall\alpha[\epsilon\bullet\alpha\downarrow \wedge (\neg\exists\delta(\delta\,\mathbf{r}\,A(\alpha)) \to \forall\zeta\{(\epsilon\bullet\alpha)\bullet\zeta\downarrow\ \wedge\ \exists\beta[p_0((\epsilon\bullet\alpha)\bullet\zeta)\subseteq\beta$$
$$\wedge\, p_1((\epsilon\bullet\alpha)\bullet\zeta)\,\mathbf{r}\,B(\alpha,\beta)]\})].$$

On the other hand, writing out

$$\epsilon'\,\mathbf{r}\,\exists\gamma\forall\alpha(\neg A(\alpha) \to \gamma\bullet\alpha\downarrow\ \wedge\ \exists\beta(\gamma\bullet\alpha\subseteq\beta \wedge B(\alpha,\beta)))$$

one gets

$$\exists\gamma[p_0\epsilon'\subseteq\gamma \wedge \forall\alpha(p_1\epsilon'\bullet\alpha\downarrow \wedge (\neg\exists\delta(\delta\,\mathbf{r}\,A(\alpha))$$
$$\to \forall\zeta\{(p_1\epsilon'\bullet\alpha)\bullet\zeta\downarrow\ \wedge\ p_0((p_1\epsilon'\bullet\alpha)\bullet\zeta)\,\mathbf{r}\,\gamma\bullet\alpha\downarrow$$
$$\wedge\,\exists\beta(p_0p_1((p_1\epsilon'\bullet\alpha)\bullet\zeta)\,\mathbf{r}\,\gamma\bullet\alpha\subseteq\beta$$
$$\wedge\, p_1p_1((p_1\epsilon'\bullet\alpha)\bullet\zeta)\,\mathbf{r}\,B(\alpha,\beta))\}))]. \qquad \square$$

**Remark.** The stronger scheme (and perhaps the one some readers expected to turn up):

$$\forall\alpha(\neg A(\alpha) \to \exists\beta B(\alpha,\beta)) \to \exists\gamma\forall\alpha(\neg A(\alpha) \to \gamma\bullet\alpha\downarrow \wedge B(\alpha,\gamma\bullet\alpha))$$

cannot be realizable: think of the interpretation of the $\alpha$'s as finite sets. There can be no finite set $\alpha$ such that for all $\beta$,

$$\beta \bullet \beta\downarrow \Rightarrow \alpha \bullet \beta = \beta \bullet \beta$$

(finite sets with such an application do *not* form a partial combinatory algebra; but see Section 3).

**Proposition 1.16.** *For arithmetical formulas $A$,*
(i) $\mathbf{HA}^\alpha + \mathrm{ECT}^\alpha \vdash A \leftrightarrow \exists\alpha(\alpha\,\mathbf{r}\,A)$,
(ii) $\mathbf{HA} + \mathrm{MP} \vdash \exists x(x\,\mathbf{e}\,A) \Leftrightarrow \mathbf{HA}^\alpha + \mathrm{ECT}^\alpha \vdash A$.

**Proof.** For (i), note that $\mathrm{ECT}^\alpha$, together with Proposition 1.12 and Lemma 1.10, implies for arithmetical $A$ and $B$,

$$\forall\alpha(\alpha\,\mathbf{r}\,A \rightarrow \exists\beta(\beta\,\mathbf{r}\,B)) \rightarrow \exists\gamma(\gamma\,\mathbf{r}\,(A \rightarrow B))$$

which gives the only nontrivial induction step.

For (ii), $\Rightarrow$ follows at once, using (i) and Proposition 1.11; for $\Leftarrow$ suppose $\mathbf{HA}^\alpha + \mathrm{ECT}^\alpha \vdash A$ so $\mathbf{HA}^\alpha \vdash F \rightarrow A$ for a finite conjunction $F$ of instances of $\mathrm{ECT}^\alpha$. Since $F$ is realizable by Proposition 1.15, from Proposition 1.13 we have

$$\mathbf{HA}^\alpha \vdash \exists\alpha(\alpha\,\mathbf{r}\,A).$$

By Proposition 1.11 then

$$\mathbf{HA}^\alpha \vdash \exists\alpha\forall n \in \alpha(n\,\mathbf{e}\,A)$$

so by Proposition 1.8(ii) one gets

$$\mathbf{HA} + \mathrm{MP} \vdash \exists x(x\,\mathbf{e}\,A). \qquad \square$$

This completes the characterization of extensional realizability in a conservative extension of $\mathbf{HA} + \mathrm{MP}$.

**Remarks.** (1) The following weakening of the scheme $\mathrm{ECT}_0$ is, in $\mathbf{HA}^\alpha$, implied by $\mathrm{ECT}^\alpha$:

$$\forall x(\neg A(x) \rightarrow \exists y B(x,y)) \rightarrow \neg\neg\exists z\forall x(\neg A(x) \rightarrow z \bullet x\downarrow \wedge B(x, z \bullet x)).$$

I propose the name $\mathrm{WECT}_0$ (weak extended Church's thesis) for this scheme.

(2) Let us consider (over $\mathbf{HA}^\alpha$ minus axiom 6 of Definition 1.7) the following three axioms in isolation:
(i) Axiom 6: $\forall nm\exists\alpha(n \in \alpha \wedge m \in \alpha)$
(ii) $\mathrm{ECT}^\alpha$
(iii) $\exists x(x \in \alpha)$.
It is not hard to see that these three, taken together, are inconsistent. Our realizability has just (i) and (ii); if one takes (ii) and (iii) one derives $\forall\alpha\exists n(\alpha = \{n\})$, and one has, essentially, Kleene's realizability back.

## 2. Extensional q-realizability and Extensional Church's Rule

**q**-realizability in general is a combination of "realizability and truth". The aim is usually to obtain proof theoretical properties of the formal system one is working in, such as the disjunction property, explicit definability for numbers, or Church's Rule for **HA** [18, 20]. The trick has been played with Kleene realizability, Kleene realizability for second-order arithmetic [8], Lifschitz realizability [23], among others.

Here I give a short treatment for the case of extensional realizability, which results in a derived rule for **HA** (Extensional Church's Rule), a stronger property than the well-known Extended Church's Rule (see [20]).

**Definition 2.1** (*Extensional* **q**-*realizability*). Define simultaneously, by recursion on $A$, formulas $Q_A(x)$ and $x \asymp_A x'$ for $x,x'$ not occurring in $A$, as follows:

$$
\begin{aligned}
Q_{t=s}(x) &\equiv t = s \\
x \asymp_{t=s} x' &\equiv x = x' \wedge t = s \\
Q_{A\wedge B}(x) &\equiv Q_A((x)_0) \wedge Q_B((x)_1) \\
x \asymp_{A\wedge B} x' &\equiv (x)_0 \asymp_A (x')_0 \wedge (x)_1 \asymp_B (x')_1 \\
Q_{A\to B}(x) &\equiv \forall y y'(y \asymp_A y' \to x \bullet y\downarrow \wedge x \bullet y'\downarrow \\
&\qquad \wedge x \bullet y \asymp_B x \bullet y') \wedge A \to B \\
x \asymp_{A\to B} x' &\equiv Q_{A\to B}(x) \wedge Q_{A\to B}(x') \\
&\qquad \wedge \forall y(Q_A(y) \to x \bullet y \asymp_B x' \bullet y) \\
Q_{\forall y A(y)}(x) &\equiv \forall n(x \bullet n\downarrow \wedge Q_{A(n)}(x \bullet n)) \\
x \asymp_{\forall y A(y)} x' &\equiv \forall n(x \bullet n\downarrow \wedge x' \bullet n\downarrow \wedge x \bullet n \asymp_{A(n)} x' \bullet n) \\
Q_{\exists y A(y)}(x) &\equiv Q_{A((x)_0)}((x)_1) \\
x \asymp_{\exists y A(y)} x' &\equiv (x)_0 = (x')_0 \wedge (x)_1 \asymp_{A((x)_0)} (x')_1.
\end{aligned}
$$

Again, $\asymp_A$ is symmetric and transitive, and $Q_A(x)$ is equivalent to $x \asymp_A x$. It also follows by an easy induction that

$$\mathbf{HA} \vdash Q_A(x) \to A \tag{3}$$

for all formulas $A$.

**Proposition 2.2** (Soundness for extensional **q**-realizability).

$$\mathbf{HA} \vdash A \Rightarrow \mathbf{HA} \vdash \exists x Q_A(x).$$

**Proof.** As usual. □

**Proposition 2.3.** *Let* $\psi_A$ *be as in Lemma* 1.3. *Then for almost negative* $A$:

$$\mathbf{HA} \vdash A \leftrightarrow \psi_A\downarrow \wedge\ Q_A(\psi_A).$$

**Proof.** Easy. □

**Proposition 2.4** (Extensional Church's Rule for HA). **HA** *obeys the following rule: if*

$$\mathbf{HA} \vdash \forall e(\forall x \exists y B(e,x,y) \to \exists z C(e,z))$$

*for some almost negative formula B, then there is a number n such that*

$$\begin{aligned}\mathbf{HA} \vdash \forall e(&n \bullet e\downarrow \wedge \\ &\forall f f'(\forall x(f \bullet x\downarrow \wedge f' \bullet x\downarrow \wedge f \bullet x = f' \bullet x \wedge B(e,x,f \bullet x)) \to \\ &(n \bullet e) \bullet f\downarrow \wedge (n \bullet e) \bullet f'\downarrow \wedge (n \bullet e) \bullet f = (n \bullet e) \bullet f' \\ &\wedge C(e,(n \bullet e) \bullet f)))\end{aligned}$$

**Proof.** Let $A$ be the formula $\forall e(\forall x \exists y B(e,x,y) \to \exists z C(e,z)$; suppose $\mathbf{HA} \vdash A$. By Proposition 2.2 and the numerical existence property for **HA**, let $m$ be such that $\mathbf{HA} \vdash Q_A(m)$. Then

$$\mathbf{HA} \vdash \forall e(m \bullet e\downarrow) \wedge \forall f f'(f \asymp_{\forall x \exists y B(e,x,y)} f' \to (m \bullet e) \bullet f \asymp_{\exists z C(e,z)} (m \bullet e) \bullet f').$$

If $\forall x(f \bullet x = f' \bullet x \wedge B(e,f \bullet x))$ then by Proposition 2.3, since $B$ is almost negative,

$$\Lambda x.\langle f \bullet x, \psi_B(e,x,f \bullet x)\rangle \asymp_{\forall x \exists y B(e,x,y)} \Lambda x.\langle f' \bullet x, \psi_B(e,x,f' \bullet x)\rangle.$$

Write this as $a \asymp_{\forall x \exists y B(e,x,y)} a'$. Then for $y = ((m \bullet e) \bullet a)_0 = ((m \bullet e) \bullet a')_0$,

$$Q_{C(e,y)}(((m \bullet e) \bullet a))_1.$$

By (3),

$$C(e,y).$$

Therefore the number $n = \Lambda ef.y$ satisfies the proposition. □

Note that the well-known Extended Church's Rule (see [20]) is a consequence of Extensional Church's Rule by letting $x$ and $y$ be dummy variables.

## 3. Some toposes for extensional realizabilities

In this section I have to assume that the reader is familiar with the elementary concepts of categorical logic (in particular, the notion of validity of a statement in a topos) and some basic topos theory. There is by now a wealth of textbooks in the area, but the reader is sure to find everything that I use in either [12] or [14].

The construction of the toposes below goes via *tripos theory*, a categorical framework treated in [11]. This, and Hyland's paper on the Effective topos [9] will also be used.

The first two subsections define toposes Ext and Ext′, generalizing **e** and $\mathbf{r}_e$-realizability, respectively. Some internal logic is explained. Section 3.1 makes no claim at originality; the material was certainly known to various people [10], but had never been laid down.

Sections 3.3 and 3.4 describe another topos construction, for a topos $\mathscr{A}$ generalizing **e**-realizability. It will be seen that Ext is a sheaf subtopos of $\mathscr{A}$. The construction uses a generalization of the notion of partial combinatory algebra, called $\leqslant$-pca, which I believe may be of independent interest. This is defined in Section 3.3. Section 3.4, finally, shows how the categorical results about $\mathscr{E}\!\mathit{ff}$, obtained in [16] and [4], can be adapted to $\mathscr{A}$.

### *3.1. Pitts' topos Ext*

The topos Ext, defined by A. Pitts in his thesis [13] although he did not give it a name, runs on partial equivalence relations on the natural numbers (pers); first, let us establish some notation for these.

I find it convenient to denote a per by $(A,\sim)$ so $A$(the *domain* of $(A,\sim)$) is a subset of $\mathbb{N}$ and $\sim$ is an equivalence relation on $A$. Now let:

$$(A_1,\sim_1)\times(A_2,\sim_2)\equiv(\{\langle a,a'\rangle \mid a\in A_1, a'\in A_2\},\sim) \text{ with}$$
$$\langle a,a'\rangle\sim\langle b,b'\rangle \text{ iff } a\sim_1 b \text{ and } a'\sim_2 b'$$
$$(A_1,\sim_1)\to(A_2,\sim_2)\equiv(\{c\mid\forall aa'\in A_1(a\sim_1 a'\Rightarrow c\bullet a\sim_2 c\bullet a')\},\sim)$$
$$\text{with } c\sim c' \text{ iff } \forall a\in A_1(c\bullet a\sim_2 c'\bullet a)$$
$$\textstyle\prod_{x\in X}(A_x,\sim_x)\equiv(\bigcap_{x\in X}\{c\mid\forall n\in\mathbb{N}(c\bullet n\in A_x)\},\sim) \text{ with}$$
$$c\sim c' \text{ iff for all } x\in X \text{ and all } n\in\mathbb{N}, c\bullet n\sim_x c'\bullet n$$
$$\textstyle\sum_{x\in X}(A_x,\sim_x)\equiv(\bigcup_{x\in X}A_x,\sim) \text{ with } c\sim c' \text{ the transitive closure}$$
$$\text{of the relation } \exists x\in X(c\sim_x c')$$
$$(A,\delta)\equiv A \text{ with the minimal equivalence relation}$$
$$(A,\top)\equiv A \text{ with the maximal equivalence relation.}$$

Let PER denote the set of pers. There is a tripos $\mathrm{PER}^{(-)}$ on the category of Sets which assigns to each set $X$ the set $\mathrm{PER}^X$ of $X$-indexed families of pers. This is ordered by (writing $\varphi$ and $\psi$ for such families):

$\varphi\vdash\psi$ iff there is a number $n$ such that for all $x\in X$, $n$ is the domain of $\varphi(x)\to\psi(x)$

This ordering is a Heyting prealgebra: the meet $\wedge$ and Heyting implication $\to$ are given respectively by applying the operations $\times$ and $\to$ between pers pointwise.

For any function $f: X \to Y$ the map $\mathrm{PER}^f : \mathrm{PER}^Y \to \mathrm{PER}^X$ is a morphism of Heyting prealgebras (i.e. preserving the propositional structure) that has both adjoints $\exists f$ and $\forall f$ :

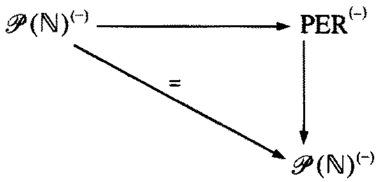
$$(\exists f(\varphi))(y) = \textstyle\sum_{f(x)=y} \varphi(x)$$
$$(\forall f(\varphi))(y) = \textstyle\prod_{f(x)=y} \varphi(x).$$

I call the topos represented by the tripos $\mathrm{PER}^{(-)}$, Ext.

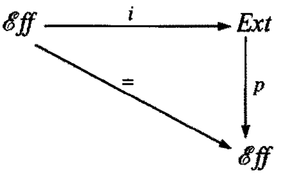
The map $\pi$ : PER $\to \mathscr{P}(\mathbb{N})$ which sends a per to its domain, induces an indexed map of preorders: $\mathrm{PER}^{(-)} \to \mathscr{P}(\mathbb{N})^{(-)}$ ($\mathscr{P}(\mathbb{N})^{(-)}$ denotes the tripos underlying the effective topos $\mathscr{E}\!f\!f$). This has both an indexed left and right adjoint, which are induced respectively by $f_1 f_2 : \mathscr{P}(\mathbb{N}) \to \mathrm{PER}$ given by

$$f_1(A) = (A, \delta), \qquad f_2(A) = (A, \top).$$

Moreover, the indexed left adjoint preserves finite meets; so we have a commutative diagram of geometric morphisms of triposes




inducing geometric morphisms of toposes




I denote the natural numbers object in each topos by $N$; context makes clear in which topos we are.

Since inverse image functors of geometric morphisms preserve natural numbers objects, $N$ in Ext is given (up to isomorphism) by $(\mathbb{N}, =)$ with

$$[\![ n = m ]\!] \text{ is } (\{n\} \cap \{m\}, \sim)$$

($\sim$ being the unique equivalence relation). From this:

**Proposition 3.1.** *First-order arithmetic in Ext is given by* **e**-*realizability.*

**Proof.** Routine. □

In general, computing the direct image functor $i_*$ from the given geometric morphism of triposes is a bit involved (there is a complicated formula for this in [11]). However, we can simplify in the case of *canonically separated* objects of $\mathscr{E}\!f\!f$ (recall that an object $(X,=)$ of $\mathscr{E}\!f\!f$ is canonically separated if $[\![x=y]\!]=\emptyset$ for different $x,y\in X$).

**Proposition 3.2.** *Let $(X,=)$ be canonically separated in $\mathscr{E}\!f\!f$. Then $i_*(X,=)$ is isomorphic to $(X,f_2(=))$.*

**Proof.** First observe that for sets $X,Y$, functions $f:X\to Y$ and $\varphi\in\mathscr{P}(\mathbb{N})^X$, $f_2^Y(\exists f(\varphi))$ is isomorphic to $\exists f(f_2^X(\varphi))$ if for all $x,x'\in X$, $y\in Y$ and $n,m\in\mathbb{N}$:

if $n\in\varphi(x)$, $m\in\varphi(x')$ and $f(x)=f(x')=y$ then there are
$x=x_1,\dots,x_{k+1}=x'$, $n=n_1,\dots,n_k=m$ with $f(x_1)=\cdots=f(x_{k+1})$
and $n_i\in\varphi(x_i)\cap\varphi(x_{i+1})$ for $i=1,\dots,k$.

Clearly, this condition holds if $(X,=)$ is canonically separated, $(Y,=)$ arbitrary, $f$ is a projection $Y\times X\to Y$ and $\varphi\in\mathscr{P}(\mathbb{N})^{Y\times X}$ is a functional relation representing a morphism $(Y,=)\to(X,=)$ in $\mathscr{E}\!f\!f$. So if $\varphi\in\mathscr{P}(\mathbb{N})^{Y\times X}$ represents a morphism into a canonically separated object $(X,=)$, $f_2^{Y\times X}(\varphi)$ represents a morphism in *Ext*: $(Y,f_2(=))\to(X,f_2(=))$.

Now there is a natural isomorphism

$$K:\mathscr{E}\!f\!f((Z,\pi(=)),(X,=))\to Ext((Z,=),(X,f_2(=)))$$

for objects $(Z,=)$ of Ext and canonically separated $(X,=)$ in $\mathscr{E}\!f\!f$ (natural means natural in $(Z,=)$), as follows: given $F:(Z,\pi(=))\to(X,=)$ in $\mathscr{E}\!f\!f$, represented by the functional relation $\varphi$, let $K(F)$ be represented by the functional relation $f_2^{Z\times X}(\varphi)\wedge[\![z=z]\!]$. It has an inverse $L$ defined (again, on representing functional relations) as $L(\psi)=\pi^{Z\times X}(\psi)$:

For, $LK([\varphi])$ is iso to $\pi f_2(\varphi)\wedge\pi(z=z)$ which is iso to $\varphi$ since $\pi f_2$ is the identity and $\varphi$ is strict for the equality $\pi(=)$, and $\varphi\vdash KL(\varphi)$ is easy; and since both are functional relations, they are isomorphic.

This proves that $i_*(X,=)$ must be isomorphic to $(X,f_2(=))$. □

**Proposition 3.3.** *The finite type structure over $N$ in Ext (i.e. the structure built from $N$ and exponentials) is given by*

*The object of type $\sigma$ has as underlying set the hereditarily effective operations of type $\sigma$, and as equality*

$$[\![\alpha=\alpha']\!]=\begin{cases}(\{n\mid n\ \textit{codes}\ \alpha\},\top) & \textit{if}\ \alpha=\alpha',\\ (\emptyset,\emptyset). & \textit{else.}\end{cases}$$

*The type structure of the hereditarily effective operations is defined in* [18].

**Proof.** This follows from the preceding proposition, combining the following ingredients:

1. the description in [9] of the finite type objects in $\mathcal{E}ff$, and the fact that they are canonically separated;

2. the description of $N$ in *Ext* given above, implying $N = i_*(N)$;

3. the fact that $i_*$, being the direct image part of a geometric inclusion, preserves exponents. □

I now discuss briefly some principles that can be expressed in the language of the finite type structure over $N$ (to be precise, the language of the system $\mathbf{HA}^\omega$; again, see [18] for a definition). Some definitions:

- *Church's Thesis* CT is the axiom

$$\forall f : N^N \exists e : N \forall x : N \exists y : N(T(e,x,y) \wedge U(y) = f(x))$$

  expressing in a strong sense that every function from natural numbers to natural numbers is recursive;
- The *axiom of choice* $\mathrm{AC}_{\sigma,\tau}$ for types $\sigma, \tau$, is the axiom scheme

$$\forall x : \sigma \exists y : \tau \varphi(x,y) \rightarrow \exists f : \tau^\sigma \forall x : \sigma \varphi(x, f(x))$$

- The scheme of *weak continuity for numbers* WC-N is:

$$\forall f : N^N \exists x : N \varphi(f,x) \rightarrow \forall f : N^N \exists x, y : N \forall g : N^N(\overline{f}y = \overline{g}y \rightarrow \varphi(g,x))$$

  where $\overline{f}y = \overline{g}y$ abbreviates $\forall z \leqslant y(f(z) = g(z))$
- *Brouwer's principle* BP states that all functions from $N^N$ to $N$ are continuous:

$$\forall \zeta : N^{N^N} \forall f : N^N \exists x : N \forall g : N^N(\overline{g}x = \overline{f}x \rightarrow \zeta(g) = \zeta(f))$$

We also consider two weakenings of these axioms:

- WCT (Weak Church's Thesis) is

$$\forall f : N^N \neg\neg \exists e : N \forall x : N \exists y : N(T(e,x,y) \wedge U(y) = f(x))$$

- WBP (weak Brouwer's principle) is

$$\forall \zeta : N^{N^N} \forall f : N^N \neg\neg \exists x : N \forall g : N^N(\overline{g}x = \overline{f}x \rightarrow \zeta(g) = \zeta(f))$$

**Proposition 3.4.** *The principles $AC_{\sigma,\tau}$, WCT and WBP are valid in* Ext, *but CT, WC-N and BP fail in it.*

**Proof.** Given a realizer for $\forall x : \sigma \exists y : \tau \varphi(x,y)$we find a code for an operation which sends all codes of $x$ to codes of one and the same $y$ (because equivalences must be preserved). Thus, one readily sees that $\mathrm{AC}_{\sigma,\tau}$ must hold.

The validity of WCT is left to the reader. WBP is a consequence of the Kreisel–Lacombe–Shoenfield theorem in recursion theory.

CT fails since by $AC_{1,0}$ it would imply the existence of a $\zeta \in N^{N^N}$ such that $\zeta(f)$ is a code for $f$ as recursive function. But this cannot be true for an effective operation $\zeta$.

$AC_{\sigma,\tau}$ implies that the principles WC-N and BP are equivalent, so it suffices to treat one of them. For any effective operation of type 2, the Kreisel–Lacombe–Shoenfield theorem gives us a modulus of continuity for every function, but this cannot be done extensionally in codes (see [19]). □

### 3.2. A topos for $\mathbf{r}_e$-realizability

I call this topos *Ext′* and the construction is very similar to that of *Ext*. The basic objects are now pairs $(A, \sim)$ where $\sim$ is a *partial* equivalence relation on $A$ (let us call these objects ppers). The basic operations $\times$, $\prod$ and $\sum$ are the same as for pers, and $\rightarrow$ is defined by

$$(A_1, \sim_1) \rightarrow (A_2, \sim_2) \equiv (\{c \mid \forall a \in A_1(c \bullet a \in A_2)\}, \sim) \text{ where } c \sim c' \text{ iff } \forall aa' \in A_1(a \sim_1 a' \Rightarrow c \bullet a \sim_2 c' \bullet a')$$

and the order on $\mathrm{PPER}^X$ (denoting the set of ppers by PPER) is given by

$$\varphi \vdash \psi \text{ iff there is } n \in \mathbb{N} \text{ such that for all } x \in X,\ n \sim n \text{ in } \varphi(x) \rightarrow \psi(x)$$

Analogous to the preceding subsection, there is a tripos $\mathrm{PPER}^{(-)}$. Now consider the following maps:

$u : \mathrm{PER} \rightarrow \mathrm{PPER}$ is the inclusion,
$g_1 : \mathrm{PPER} \rightarrow \mathrm{PER}$ sends $(A, \sim)$ to $(\{a \in A \mid a \sim a\}, \sim)$,
$g_2 : \mathscr{P}(\mathbb{N}) \rightarrow \mathrm{PPER}$ sends $A$ to $(A, \top)$,
$\pi' : \mathrm{PPER} \rightarrow \mathscr{P}(\mathbb{N})$ sends $(A, \sim)$ to $A$.

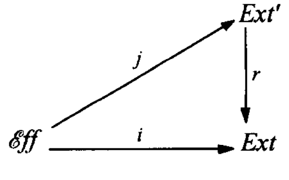
The pair $(g_1, u)$ induces a geometric morphism of triposes:

$$\mathrm{PPER}^{(-)} \rightarrow \mathrm{PER}^{(-)}$$

and $(g_2, \pi')$ gives rise to one:

$$\mathscr{P}(\mathbb{N})^{(-)} \rightarrow \mathrm{PPER}^{(-)}$$

and since $g_1 g_2 = f_2$, we have a commutative diagram of geometric morphisms of toposes:




The proof of the following proposition is left to the reader.

**Proposition 3.5.** *First-order arithmetic in Ext′ corresponds to* $\mathbf{r}_e$*-realizability.*

**Proposition 3.6.** $j:\mathscr{E}\!\mathit{ff}\to Ext'$ *is an open inclusion, i.e. there is a subobject U of 1 in Ext′ such that* $\mathscr{E}\!\mathit{ff}$ *is equivalent to the slice topos Ext′/U, and* $j^*$ *is, modulo this equivalence, the pullback functor* $Ext'\to Ext'/U$.

**Proof.** Let $U$ be the object $(\{*\},=)$ with

$$[\![ * = * ]\!] = (\mathbb{N},\emptyset).$$

Then the slice topos $Ext'/U$ is equivalent to the full subcategory of $Ext'$ on those objects whose equalities all have the empty equivalence relation. But it is clear that this is equivalent to $\mathscr{E}\!\mathit{ff}$, and that pulling back along $U\to 1$ is the same as forgetting the equivalence relation, which is $j^*$. □

**Corollary 3.7.** *Any statement of higher-order arithmetic which holds in* Ext′ *also holds in* $\mathscr{E}\!\mathit{ff}$.

**Proof.** For, pullback functors are logical functors. □

**Corollary 3.8.** *In Ext′, we have* $\neg\neg CT$ *and* $\neg\neg BP$*, but instances of WC-N and AC are false.*

### *3.3. ⩽-partial combinatory algebras*

In the next subsection there will be another topos for **e**-realizability, into which Ext embeds. The construction of this topos can be seen as a generalization of the construction of $\mathscr{E}\!\mathit{ff}$. Because I think this generalization may be of independent interest, I present it separately.

The point is a generalization of the notion of partial combinatory algebra.

**Definition 3.9.** A $\leqslant -pca$ ($\leqslant$ −partial combinatory algebra) is a partially ordered set $A$ together with a partial binary function (written by juxtaposition; $ab\downarrow$ means that the pair $(a,b)$ is in the domain of the function, $ab$ denotes the value), satisfying

1. If $ab\downarrow$, $a'\leqslant a$ and $b'\leqslant b$ then $a'b'\downarrow$ and $a'b'\leqslant ab$.
2. There are elements $k$ and $s$ in $A$ such that
   - for all $a,b\in A$ : $ka\downarrow$ and $kab\downarrow$ and $kab\leqslant a$,
   - for all $a,b,c\in A$: $sa\downarrow$, $sab\downarrow$ and, if $(ac)(bc)\downarrow$, then $sabc\downarrow$ and $sabc\leqslant (ac)(bc)$.

We employ the convention of association to the left: so $abc$ abbreviates $(ab)c$. The partial binary function is often called *application.*

Part of the definition of a $\leqslant -pca$ (the axioms for the combinators $k$ and $s$) already appeared in [1], without any relation to extensional realizability, though.

**Examples.**

1. Given any set $A$ with a partial binary function on it, we may add an element $\bot$, introduce a partial order by just adding $\bot \leqslant a$ for all $a \in A$ and extending the application by

$$\bot a = a\bot = \bot$$

Then $\bot$ serves both as $k$ and as $s$. $\leqslant$ −Pcas like this (with a least element $\bot$ satisfying $\bot a = a\bot = \bot$) will be called *trivial.*

2. Any *pca* is a $\leqslant$ −*pca* with the discrete order; conversely a $\leqslant$ −*pca* for which the order is discrete, is a *pca* iff $sabc\downarrow$ implies $ac(bc)\downarrow$.

3. Given a *pca* $A$, we may define a $\leqslant$ −*pca* structure on the powerset $\mathscr{P}(A)$ as follows: the order is the inclusion order, and $\alpha\beta\downarrow$ if and only if for all $a \in \alpha$ and $b \in \beta$, $ab\downarrow$ in $A$, in which case

$$\alpha\beta = \{ab \mid a \in \alpha, b \in \beta\}.$$

This $\leqslant$ −*pca* is trivial. To make it less trivial, restrict to the nonempty subsets of $A$ (One can also restrict to the nonempty, finite subsets of $A$). This is my motivating example.

4. Suppose $A$ is a *pca* and $(P, \preccurlyeq)$ is a linear order with top element $\top$. Order the set $A \times P$ partially by putting

$$(a, p) \leqslant (b, q) \quad \text{iff} \quad a = b \text{ and } p \preccurlyeq q,$$

and let $(a, p)(b, q)$ be defined iff $ab\downarrow$ in $A$, in which case

$$(a, p)(b, q) = (ab, \min\{p, q\}).$$

5. Given a *pca* $A$, construct a nontrivial total $\leqslant$ −*pca* into which $A$ embeds, as follows: the set of *A-terms* is inductively defined by: every $a \in A$ is an $A$-term, and if $u$ and $v$ are $A$-terms, then so is $(uv)$. Now choose elements $k, s \in A$ satisfying the combinator axioms, and define a *reduction relation* $\rightsquigarrow$ by the clauses:

- $(ab) \rightsquigarrow c$ if, in $A$, $ab$ is defined and equal to $c$.
- If $u \rightsquigarrow u'$ then $(vu) \rightsquigarrow (vu')$ and $(uv) \rightsquigarrow (u'v)$.
- $((ku)v) \rightsquigarrow u$ and $(((su)v)w) \rightsquigarrow ((uw)(vw))$.

The reflexive-transitive closure of $\rightsquigarrow$ gives only a preorder on the set of $A$-terms, so we have to quotient by an equivalence relation: two $A$-terms $u$ and $v$ are equivalent iff there is a sequence

$$u = u_1 \rightsquigarrow u_2 \cdots \rightsquigarrow u_n \rightsquigarrow v = v_1 \rightsquigarrow \cdots \rightsquigarrow v_m = u.$$

Define application by $[u][v] = [(uv)]$; this is well-defined. $A$ is embedded in this in the following sense: the embedding $I$ does not preserve application, but if $ab$ is defined then $I(a)I(b) \leqslant I(ab)$. This seems to me the natural notion of "morphism of $\leqslant$ −*pcas*", but I will not pursue this further here.

6. Let $A$ be a $\wedge$-semilattice (with or without top element). Put $ab = a \wedge b$; any element of $A$ may serve as $k$ and $s$.

7. (Generalizing Example 3) We may relax the definition of a $\leqslant -pca$ by requiring $\leqslant$ to be just a preorder rather than a partial order. Then, given a nontrivial $\leqslant -pca$ $A$, we can preorder the set of nonempty subsets of $A$ by

$$\alpha \leqslant \beta \quad \text{iff} \quad \forall x \in \alpha \exists y \in \beta \, (x \leqslant y).$$

This gives again a nontrivial $\leqslant -pca$ $\mathscr{P}_{\neq\emptyset}(A)$.

From the examples it is immediate that a lot of the beautiful (but also sometimes rather bizarre) theory of *pcas* is lost in this context: we may have $k = s$ without having every possible identity, a $\leqslant -pca$ may be nontotal without there being a nowhere defined element, every *pca* may be embedded in a total $\leqslant -pca$. However, what remains is sufficient for my purposes:

**Proposition 3.10** (Combinatory completeness for $\leqslant -pcas$). *Let $A$ be a $\leqslant -pca$. For every term $t$ composed by elements of $A$, application and the variable $x$, there is an element $[\Lambda x.t]$ in $A$ such that for all $a \in A$ : if $t[a/x]\downarrow$ then $[\Lambda x.t]a\downarrow$, and*

$$[\Lambda x.t]a \leqslant t[a/x].$$

**Proof.** The construction of these terms in the usual proof of combinatory completeness for *pcas* will do. □

In the case of a *pca* $A$ one can form a category $P(A)$ with objects the subsets of $A$, and as morphisms: $\alpha \longrightarrow \beta$ those functions $f : \alpha \longrightarrow \beta$ such that for some $a \in A$ for all $b \in \alpha$: $ab\downarrow$ and $ab = f(b)$.

For $\leqslant -pcas$ $A$ one has to modify this: morphisms are those functions $f$ for which there is $a$ satisfying for all $b \in \alpha$,

$$ab\downarrow \wedge \; ab \leqslant f(b).$$

That this gives a category follows at once by combinatory completeness.

Furthermore:

**Proposition 3.11.** *Given a $\leqslant -pca$ $A$, there is a tripos $I(A)^{(-)}$, where $I(A)$ denotes the set of downwards closed subsets of $A$ ($\alpha \in I(A)$ iff $a \in \alpha$ and $a' \leqslant a$ imply $a' \in \alpha$), and $I(A)^X$ is preordered by: $\varphi \vdash \psi$ iff there is $a \in A$ such that for all $x \in X$ and all $b \in \varphi(x)$, $ab\downarrow$ and $ab \in \psi(x)$.*

**Proof.** The proof for *pcas*, in [11], suffices; use combinatory completeness. The same terms testify all desired entailments. □

Note, that if $A$ is trivial in the sense defined above, the tripos $I(A)^{(-)}$ is equivalent to the tripos $2^{(-)}$ (with the subset order).

In the case of Example 6 ($A$ a $\wedge$-semilattice; for nontriviality let us assume there is no bottom element), the topos represented by $I(A)^{(-)}$ turns out to be a *filter quotient* of the topos $\mathrm{Sets}^{A^{op}}$ by the filter of all non-initial subobjects of 1 (see [14] for this construction). This topos satisfies $\neg\varphi \vee \neg\neg\varphi$, but need not be Boolean as far as I can see.

Still another definition:

**Definition 3.12.** A $\leqslant -pca$ $A$ will be said to have the pasting property iff the underlying partial order has pushouts (i.e. for every $a, b \in A$: if there is $c \leqslant a, c \leqslant b$ in $A$, then the join $a \vee b$ exists in $A$) and application preserves them in each variable separately (i.e. $a(b \vee b') = ab \vee ab'$ and $(a \vee a')b = ab \vee a'b$ whenever this is defined).

**Proposition 3.13.** *Let $A$ be a $\leqslant -pca$ with the pasting property. Denote by $J(A)$ the set of those downwards closed subsets of $A$ which are also closed under pushouts. Preorder $J(A)^X$ in the same way as $I(A)^X$. Then $J(A)^{(-)}$ is a tripos, and the inclusion $J(A) \subseteq I(A)$ induces a geometric inclusion of triposes: $J(A)^{(-)} \to I(A)^{(-)}$*

A geometric inclusion of triposes is a geometric morphism of triposes for which each counit is an isomorphism. It was noted in [13] (and straightforward to check directly), that a geometric inclusion of triposes gives rise to a geometric inclusion of the represented toposes.

**Proof.** Left adjoint to the inclusion is of course the map which takes for every downward closed set, its closure under pushouts. □

### *3.4. Another topos for e-realizability*

The idea for finding a "simpler" topos for $e$-realizability is as follows: instead of looking at partial equivalence relations, look at what "generates" them, in a suitable way (this is familiar practice: e.g. in the theory of locales one often works not with locales but with *presentations* of them).

In [22] the fact is exploited (for an axiomatization of higher order Kleene realizability) that in the effective topos $\mathcal{E}\!f\!f$, there is a surjection $\Delta(\mathscr{P}(\mathbb{N})) \twoheadrightarrow \Omega$ which classifies (viewing $\Delta(\mathscr{P}(\mathbb{N}))$ as the object of $\neg\neg$-closed subsets of $N$) exactly the *inhabited* elements of $\Delta(\mathscr{P}(\mathbb{N}))$.

Somehow this highlighted for me the trivial observation that subsets of $\mathbb{N}$ are generated by singletons under the operation of taking unions. Similarly, partial equivalence relations are generated by nonempty finite sets under the operations of taking unions and closing under pushouts.

So let $\overline{\mathbb{N}}$ be the $\leqslant -pca$ of nonempty subsets of $\mathbb{N}$, as in Example 3 of the preceding subsection. This $\leqslant -pca$ clearly has the pasting property. It is clear that there is a bijection

$$P : \mathrm{PER} \to J(\overline{\mathbb{N}})$$

(where $J(\overline{\mathbb{N}})$ refers to Proposition 3.13), sending $(A,\sim)$ to

$$\{\alpha\in\overline{\mathbb{N}}\mid \alpha\subseteq A\ \wedge\ \forall ab\in\alpha(a\sim b)\}$$

and that, for pers $(A_1,\sim_1)$ and $(A_2,\sim_2)$,

$$P((A_1,\sim_1)\to(A_2,\sim_2))=P(A_1,\sim_1)\to P(A_2,\sim_2),$$

where the $\to$ on the right-hand side refers to the tripos $J(\overline{\mathbb{N}})^{(-)}$.

So $\mathrm{PER}^{(-)}$ is the same as $J(\overline{\mathbb{N}})^{(-)}$ and a subtripos of $I(\overline{\mathbb{N}})^{(-)}$, and it is this last tripos and the topos represented by it, that will be studied a bit in this subsection.

Some logic of the tripos $I(\overline{\mathbb{N}})^{(-)}$: define for $P,Q\in I(\overline{\mathbb{N}})$

$$\begin{aligned}
P\times Q &= \{\alpha\in\overline{\mathbb{N}}\mid p_0\alpha\in P \text{ and } p_1\alpha\in Q\},\\
P\to Q &= \{\alpha\in\overline{\mathbb{N}}\mid \forall\beta\in P(\alpha\bullet\beta\!\downarrow \text{ and } \alpha\bullet\beta\in Q)\},\\
P+Q &= \{\alpha\in\overline{\mathbb{N}}\mid p_0\alpha=\{0\} \text{ and } p_1\alpha\in P\}\\
&\qquad \cup\{\alpha\in\overline{\mathbb{N}}\mid p_0\alpha=\{1\} \text{ and } p_1\alpha\in Q\}\\
\prod_{x\in X}P_x &= \bigcap_{x\in X}(\overline{\mathbb{N}}\to P_x),\\
\sum_{x\in X}P_x &= \bigcup_{x\in X}P_x,
\end{aligned}$$

where $p_i\alpha=\{(a)_i\mid a\in\alpha\}$.

The preorder $I(\overline{\mathbb{N}})^X$ has meets $\varphi\wedge\psi$ given by $\lambda x.\varphi(x)\times\psi(x)$, joins and Heyting implication similarly given by $+$ and $\to$, and top and bottom elements $\top=\lambda x.\overline{\mathbb{N}}$, $\bot=\lambda x.\emptyset$. For maps $f:X\to Y$, left and right adjoint $\exists f$ and $\forall f$ to $I(\overline{\mathbb{N}})^f$ are given by

$$\begin{aligned}
(\exists f(\varphi))(y) &= \sum_{f(x)=y}\varphi(x),\\
(\forall f(\varphi))(y) &= \prod_{f(x)=y}\varphi(x).
\end{aligned}$$

The topos represented by $I(\overline{\mathbb{N}})^{(-)}$ will be called $\mathscr{A}$.

Let us first observe that the geometric morphism of triposes $\mathrm{PER}^{(-)}\to\mathscr{P}(\mathbb{N})^{(-)}$ factors through the inclusion $\mathrm{PER}^{(-)}\to I(\overline{\mathbb{N}})^{(-)}$ by maps $g: I(\overline{\mathbb{N}})\to\mathscr{P}(\mathbb{N})$ and $f:\mathscr{P}(\mathbb{N})\to I(\overline{\mathbb{N}})$ given by

$$g(P)=\bigcup P \quad\text{and}\quad f(A)=\{\alpha\in\overline{\mathbb{N}}\mid\alpha\subseteq A\}.$$

From the description of finite meets in $I(\overline{\mathbb{N}})^{(-)}$ it is immediate that $f^{(-)}$ preserves them. Also, $f\dashv g$.

From this, it follows that the natural number object $N$ of $\mathscr{A}$ can be given as $(\mathbb{N},=)$ with

$$[\![ n=m ]\!]=\begin{cases}\{\{n\}\} & \text{if } n=m,\\ \emptyset & \text{else.}\end{cases}$$

We also need another object of $\mathscr{A}$: the object $\overline{N} = (\overline{\mathbb{N}}, =)$ with

$$[\![ \alpha = \beta ]\!] = \begin{cases} \{\gamma \in \overline{\mathbb{N}} \mid \gamma \subseteq \alpha\} & \text{if } \alpha = \beta, \\ \emptyset & \text{else.} \end{cases}$$

There is an element relation $\varepsilon \to N \times \overline{N}$ represented by

$$[\![ n \,\varepsilon\, \alpha ]\!] = \begin{cases} \{\{n\}\} & \text{if } n \in \alpha, \\ \emptyset & \text{else.} \end{cases}$$

Using this relation $\varepsilon$ we can interpret the language of $\mathbf{HA}^{\alpha}$ (see Definition 1.7) in $\mathscr{A}$, letting $\overline{N}$ be the sort of the set variables $\alpha$ and $N$ be the sort of the natural numbers. We have:

**Proposition 3.14.** *All the axioms of* $\mathbf{HA}^{\alpha}$ *are valid in* $\mathscr{A}$ *under this interpretation and, moreover, truth in* $\mathscr{A}$ *of sentences in this language coincides with the realizability notion of Definition* 1.9.

**Proof.** Again, left to the reader. □

**Corollary 3.15.** *First-order arithmetic in* $\mathscr{A}$ *coincides with e-realizability.*

So, true first-order arithmetic in $\mathscr{A}$ is the same as in *Ext*. I now want to extend this result to the logic of all finite types over $N$.

By a straightforward analogy to [9], there is a geometric inclusion $(\Delta, \Gamma)$: Sets $\to \mathscr{A}$. It is defined in exactly the same way as for $\mathscr{E}\!f\!f$, and Sets is $\neg\neg$-sheaves in $\mathscr{A}$.

Consequently, an object of $\mathscr{A}$ is *separated* iff it is isomorphic to an object $(X, =)$ which has the property that $[\![ x = y ]\!] = \emptyset$ for different $x, y \in X$. Such objects are called *canonically separated*.

From now on I identify *Ext* with the topos represented by the tripos $J(\overline{\mathbb{N}})^{(-)}$; the sheafification $\mathscr{A} \to Ext$ is induced by the map $J : I(\overline{\mathbb{N}}) \to J(\overline{\mathbb{N}})$ which takes every downwards closed subset of $\overline{\mathbb{N}}$ to its closure under pushouts. The internal topology in $\mathscr{A}$ to which this gives rise, is denoted by $j$.

**Proposition 3.16.** *Suppose* $(X, =)$ *is a canonically separated object of* $\mathscr{A}$ *such that* $[\![ x = x ]\!] \in J(\overline{\mathbb{N}})$ *for all* $x \in X$. *Then* $(X, =)$ *is a* $j$*-sheaf.*

**Proof.** The heart of the matter is that if $F : Y \times X \to J(\overline{\mathbb{N}})$ represents a morphism $(Y, J(=)) \to (X, J(=))$ in *Ext* and $(X, =)$ is canonically separated in $\mathscr{A}$, then $F$ is a total relation (for the equalities $J(=)$), i.e. $J(y = y) \to \exists x F(y, x)$ is valid in the topos $J(\overline{\mathbb{N}})^{(-)}$. Now

$$[\![ \exists x F(y, x) ]\!] = J\left( \bigcup_{x \in X} F(y, x) \right).$$

Since $F$ is single-valued and $(X,=)$ canonically separated, this is equal to $\bigcup_{x\in X} J(F(y,x))$, which is

$$\bigcup_{x\in X} F(y,x)$$

because $F$ maps into $J(\overline{\mathbb{N}})$. So if we define, for such $F$, a map $\tilde{F} : Y \times X \to I(\overline{\mathbb{N}})$ by

$$\tilde{F}(y,x) = \bigcup_{y'\in Y} ([\![ y' = y ]\!] \wedge F(y',x')),$$

then $\tilde{F}$ represents a map $(Y,=) \to (X,=)$ in $\mathscr{A}$ (check that it is single-valued!). So there is a natural 1–1 correspondence between maps: $(Y,J(=)) \to (X,J(=))$ in *Ext*, and maps $(Y,=) \to (X,=)$ in $\mathscr{A}$; which by the Yoneda Lemma proves that $(X,=)$ is a $j$-sheaf.

So $N$ is a $j$-sheaf, and since for any topology the sheaves form an exponential ideal, the finite type structure over $N$ consists of $j$-sheaves. The computation of this structure is easy: if $(Y,=)$ is separated there is an expression for $(Y,=)^{(X,=)}$ completely similar to the one in [9], and we see:

**Proposition 3.17.** *The finite type structure over $N$ in $\mathscr{A}$ is the following: the object of type $\sigma$ has as underlying set the effective operations of type $\sigma$, and equality: $[\![ x = x ]\!]$ is the set of those $\alpha$ which consist of codes for $x$, whereas $[\![ x = y ]\!]$ is empty for $x \neq y$.*

**Proposition 3.18.** *The logic of the finite type structure over $N$ in $\mathscr{A}$ is the same as that in Ext.*

**Proof.** The finite type objects in $\mathscr{A}$ as defined in the preceding proposition, have the following properties:
- They are *modest*, i.e. canonically separated and such that for different $x, y$, the set $[\![ x = x ]\!] \cap [\![ y = y ]\!]$ is empty.
- The equalities are all closed under pushouts, as well as the relations representing the evaluation maps.

Now if $\varphi(x)$ is a strict predicate for $x$ of type a modest object, and $\varphi(x)$ is closed under pushouts, then $\exists x\varphi(x)$ is also closed under pushouts. It is trivial that the property of being closed under pushouts (for predicates) is preserved under the logical operations $\to, \wedge, \vee$ and $\forall$, so there you are. □

### 3.5. $\mathscr{A}$ and Ext as exact completions

The categorical and logical analysis of $\mathscr{A}$ can be pushed a lot further, exploiting the analogy with $\mathscr{E}\!f\!f$ and what is known for that topos. For example, there is a surjection: $\Delta(I(\overline{\mathbb{N}})) \twoheadrightarrow \Omega$ in $\mathscr{A}$, which classifies, viewing $\Delta(I(\overline{\mathbb{N}}))$ as the object of $\neg\neg$-closed, downwards closed subsets of $\overline{N}$ in $\mathscr{A}$, exactly the inhabited ones; this should be the starting point for the definition of an "internal realizability" as in [22], and an axiomatization (over a suitable expansion of higher-order arithmetic) of an extension of the realizability of Definition 1.9. In the expansion of arithmetic one will need sorts for $\overline{N}$ and its powers; it would be nice if these could be eliminated, i.e. if $\overline{N}$ would be, in $\mathscr{A}$, definable from higher-order arithmetic. I doubt this, but do not know.

Here I just present an analogy of a characterization of $\mathscr{E}\!f\!f$ as *exact completion of its category of projectives*, obtained in [16] and also explained in [4]. Both papers start from the basic result in [5], which is the construction of the exact completion $E_{\mathrm{ex/lex}}$ of a left exact category $E$. Let me first explain what it means. A left exact category is said to be *exact* if

(a) For every map $f : A \to B$ the coequalizer of the kernel pair of $f$ (i.e. the two projections $A \times_B A \to A$) exists.

(b) Regular epimorphisms (i.e. those which are coequalizers) are stable under pullback.

(c) Equivalence relations are effective, that is: kernel pairs.

A left exact functor between exact categories is called exact if it preserves regular epimorphisms.

If *EX* denotes the (2-)category of exact categories and exact functors, and *LEX* is the category of left exact categories and left exact functors, then the exact completion $E_{\mathrm{ex/lex}}$ of a left exact category $E$ is its image under the reflection of *LEX* to *EX* (the left adjoint to the inclusion of *EX* into *LEX*). It is important to notice that the inclusion of *EX* into *LEX* is *not* full and faithful, so an exact category is not automatically equivalent to the exact completion of something. For this to be the case, we need to look at the *projective* objects of the category: an object $A$ is projective (One should say: regular projective, but never mind) iff every regular epimorphism to $A$ has a section.

It turns out that an exact category $E$ is an exact completion if and only if the following two conditions hold:

(i) *$E$ has enough projectives*, which means that for every object $A$ of $E$ there is a projective object $B$ and a regular epimorphism $B \twoheadrightarrow A$.

(ii) The full subcategory of $E$ on the projective objects is left exact.

([4]) If these conditions are satisfied, $E$ is the exact completion of its category of projectives.

The authors of [16] were able to identify the projectives of $\mathscr{E}\!f\!f$ and show that $\mathscr{E}\!f\!f$ is the exact completion of its category of projectives. This category looks as follows: objects are surjective functions $X \twoheadrightarrow I$ where $X$ is a set and $I \subseteq \mathbb{N}$; morphisms are commutative diagrams

$$\begin{array}{ccc} X & \twoheadrightarrow & I \\ \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle \psi} \\ Y & \twoheadrightarrow & J \end{array}$$

where $\psi : I \to J$ is the restriction to $I$ of a partial recursive function ($\psi$ is uniquely determined by $f$ since the horizontal maps are surjective). In other words, this category is the full subcategory of the comma category $(Sets \downarrow P(\mathbb{N}))$ on the surjections ($P(\mathbb{N})$ is the category of subsets of $\mathbb{N}$ and partial recursive functions).

Since $\overline{\mathbb{N}}$ is only a $\leqslant -pca$, the category $P(\overline{\mathbb{N}})$ (as defined in Section 3.3) has as maps $f : A \to B$, those functions $f$ such that for some partial recursive function $\psi$, for all $\alpha \in A$ and for all $n \in \alpha$, $\psi(n)$ is defined, and

$$\psi[\alpha] \subseteq f(\alpha)$$

So the full subcategory of $(Sets \downarrow P(\overline{\mathbb{N}}))$ on the surjective functions looks as follows:

- Objects are surjective functions $f : X \longrightarrow\!\!\triangleright A$ from a set $X$ to a subset $A$ of $\overline{\mathbb{N}}$.
- Morphisms from $f$ to $g : Y \longrightarrow\!\!\triangleright B$ are functions $h : X \to Y$ such that for some partial recursive function $\psi$ there is a diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & A \\ {\scriptstyle h}\downarrow & \supseteq & \downarrow{\scriptstyle \psi} \\ Y & \xrightarrow{g} & B \end{array}$$

  where it is meant that $\psi$ acts on $\alpha$ to give $\psi[\alpha] = \{\psi(n) \mid n \in \alpha\}$, and the diagram commutes "up to inclusion".

The claim is now that this category is the category of projectives in $\mathscr{A}$, that it is left exact and that $\mathscr{A}$ has enough projectives, so that $\mathscr{A}$ is the exact completion of this category.

The result is a direct adaptation of the method of [16]. First a lemma:

**Lemma 3.19.** *Every object of $\mathscr{A}$ is covered by a separated object.*

**Proof.** Given $(X, =)$, define the object $(Q, =)$ by

$$Q = \{(x, \alpha) \mid x \in X,\ \alpha \in [\![ x = x ]\!]\},$$

$$[\![ (x, \alpha) = (y, \beta) ]\!] = \begin{cases} \{\gamma \in \overline{\mathbb{N}} \mid \gamma \subseteq \alpha\} & \text{if } x = y \text{ and } \alpha = \beta, \\ \emptyset & \text{otherwise.} \end{cases}$$

The reader can check that the function $Ku : Q \times X \to I(\overline{\mathbb{N}})$ given by

$$Ku((x, \alpha), y) = \{\gamma \mid \gamma \subseteq \alpha\} \times [\![ x = y ]\!]$$

represents a surjection $(Q, =) \longrightarrow\!\!\triangleright (X, =)$. $\square$

**Proposition 3.20.** *Call an object* $(X,=)$ *of* $\mathscr{A}$ canonically projective *if it is canonically separated, and for all* $x \in X$ *there is* $\alpha \in \overline{\mathbb{N}}$ *such that*

$$[\![ x = x ]\!] = \{\gamma \mid \gamma \subseteq \alpha\}.$$

*Then an object of* $\mathscr{A}$ *is projective if and only if it is isomorphic to a canonically projective object.*

It is easily seen that the category of canonically projectives is the same as the full subcategory of $(Sets\downarrow P(\overline{\mathbb{N}}))$ on the surjections, as given above. Also, from this description it is obvious that this is a left exact category: it has products because if $[\![ x = x ]\!] = \{\gamma \mid \gamma \subseteq \alpha\}$ and $[\![ y = y ]\!] = \{\gamma \mid \gamma \subseteq \beta\}$ then for $(x, y)$ (in the product) we have

$$[\![ (x,y) = (x,y) ]\!] = \{\gamma \mid \gamma \subseteq \alpha \times \beta\}$$

with $\alpha \times \beta = \{\langle n,m\rangle \mid n \in \alpha \wedge m \in \beta\}$; it has equalizers because $\neg\neg$-closed subobjects of canonically projective objects are canonically projective: in complete analogy to the situation for $\mathscr{E}\!f\!f$ (see [9]), a subobject of $(X,=)$ is $\neg\neg$-closed iff isomorphic to one of form $(A,=)$ with $A \subseteq X$ and $=$ the restriction to $A$ of the equality $=$ on $X$.

**Proof.** Suppose $X$ is projective; let $Q \overset{m}{\twoheadrightarrow} X$ be the cover in the proof of Lemma 3.19. Then $Q$ is canonically projective, $m$ has a section $i$, and $X$ is isomorphic to the equalizer of $im$ and $id : Q \to Q$. Since the canonically projective objects are closed under equalizers, $X$ is isomorphic to a canonically projective object.

To show the converse, suppose $(P,=)$ canonically projective and

$$(X,=) \twoheadrightarrow (P,=)$$

a surjection, represented by $G$. By surjectivity, pick a $\beta$ with

$$\beta \in \bigcap_{p \in P} ([\![ p = p ]\!] \to [\![ \exists x G(x,p) ]\!]).$$

Let $[\![ p = p ]\!] = \{\gamma \mid \gamma \subseteq \alpha_p\}$. Pick for each $p \in P$ a $x_p$ with $\beta \bullet \alpha_p \in G(x_p, p)$; then the relation

$$H(p,y) = [\![ p = p ]\!] \times [\![ y = x_p ]\!]$$

is easily seen to represent a section for the given surjection. □

Since the category of canonically projective objects of $\mathscr{A}$ is clearly isomorphic to the full subcategory of $(Sets\downarrow P(\overline{\mathbb{N}})$ on the surjective functions, described just before Lemma 3.19, and this category is left exact, we have that $\mathscr{A}$ is the exact completion of its category of projectives.

Of course, the real content of [16] is in their argument that the category of projectives of $\mathscr{E}\!f\!f$ is itself a completion: it is the category which results from freely adding nonempty, recursively indexed coproducts to the category of sets.

We can relate the categories of projectives in $\mathscr{E}\!f\!f$ and $\mathscr{A}$ as follows. There is, analogously to the exact completion, the notion of a *regular completion* of a left exact category, denoted $(-)_{\text{reg/lex}}$ (a regular category is a category satisfying the axioms for an exact one minus the requirement that equivalence relations be effective). Carboni has shown in [4], that the category of $\neg\neg$-separated objects in $\mathscr{E}\!f\!f$ is the regular completion of the category of projectives in $\mathscr{E}\!f\!f$. Now it does not take much inspiration to see, that the separated objects in $\mathscr{E}\!f\!f$ are equivalent to the projectives of $\mathscr{A}$. Summing up:

**Theorem 3.21.** *Let Proj and Sep denote, respectively, the categories of projective and $\neg\neg$-separated objects in the effective topos.*
*Then $\mathscr{A}$ is $(Sep)_{\text{ex/lex}}$, equivalently $((Proj)_{\text{reg/lex}})_{\text{ex/lex}}$.*

What about *Ext*? Does it have enough projectives? Is it an exact completion? The first question would be easy to answer if the inclusion $Ext \to \mathscr{A}$ would preserve epimorphisms. However:

**Proposition 3.22.** *The inclusion $Ext \to \mathscr{A}$ does not preserve epis.*

**Proof.** Given the inclusion: $Ext \to \mathscr{A}$ and the fact that $\mathscr{A}$ has enough projectives, this is equivalent to the statement that the inverse image functor of the inclusion does not preserve projectives. So I give a counterexample to this.

Let $(X,=)$ and $(Y,=)$ be the canonically separated objects of $\mathscr{A}$ given by

$$X = \{x_1,x_2,x_3,x_4\} \quad \text{and} \quad \begin{cases} [\![ x_1 = x_1 ]\!] = [\![ x_2 = x_2 ]\!] = \downarrow\{0,1\} \\ [\![ x_3 = x_3 ]\!] = [\![ x_4 = x_4 ]\!] = \downarrow\{0,2\} \end{cases}$$

$$Y = \{y_1,y_2,y_3\} \quad \text{and} \quad \begin{cases} [\![ y_1 = y_1 ]\!] = \downarrow\{0,1\} \\ [\![ y_2 = y_2 ]\!] = \downarrow\{0,1,2\} \\ [\![ y_3 = y_3 ]\!] = \downarrow\{0,2\}. \end{cases}$$

I have started writing $\downarrow\alpha$ for $\{\gamma \mid \gamma \subseteq \alpha\}$.

Let $f : X \to Y$ be the function: $f(x_1) = y_1$, $f(x_2) = f(x_3) = y_2$, $f(x_4) = y_3$. Then

$$\{\Lambda x.x\} \in \bigcap_{x \in X} ([\![ x = x ]\!] \to [\![ f(x) = f(x) ]\!])$$

so the predicate $F$ defined by $F(x,y) = [\![ x = x ]\!] \wedge [\![ f(x) = y ]\!]$ represents a morphism $[F]$ in $\mathscr{A}$. Now the objects $(X,=)$, $(Y,=)$ and the morphism $[F]$ also live in *Ext*; and by Proposition 3.20 both are projective in $\mathscr{A}$. But $(Y,=)$, taken as object of *Ext* (which is the inverse image of itself as object of $\mathscr{A}$) is not projective: the map $[F]$ is surjective in *Ext* (not in $\mathscr{A}$!) since

$$[\![ \exists x F(x,y_2) ]\!]_{Ext} = J(F(x_2,y_2) \cup F(x_3,y_2))$$

which is equal to

$$\downarrow\{0,1,2\} \times \downarrow\{0,1,2\}.$$

So $\{\Lambda x.\langle x,x\rangle\} \in \bigcap_{y\in Y}[\![ y = y ]\!] \to [\![ \exists x F(x,y) ]\!]$. But of course, $[F]$ cannot have a section in *Ext*. □

The above proof clearly indicates what should be the projectives in *Ext*, since obviously, in the example, the element $y_2$ is the problematic guy. Its existence, the downset of a three-element set, can be glued together from two downsets of two-element sets by pushout. This suggests the following definition, which embodies the deep mathematical intuition that an equivalence relation is generated by sets of "equivalent pairs".

**Definition 3.23.** Call an object $(X,=)$ of *Ext canonically projective* if it is canonically separated, and for all $x \in X$ there is an $\alpha$ with at most two elements such that

$$[\![ x = x ]\!] = \downarrow\alpha = \{\gamma \mid \gamma \subseteq \alpha\}.$$

**Lemma 3.24.** *Every object of Ext is covered by a canonically projective object.*

**Proof.** Given $(X,=)$, let $(Q,=)$ be defined by

$$Q = \{(x,\alpha) \mid \alpha \in [\![ x = x ]\!] \text{ and } \sharp\alpha \leqslant 2\},$$
$$[\![ (x,\alpha) = (y,\beta) ]\!] = \begin{cases} \downarrow\alpha & \text{if } x = y \text{ and } \alpha = \beta, \\ \emptyset & \text{else.} \end{cases}$$

The rest is left to the reader, who just has to keep in mind how an existential quantifier is interpreted in the tripos $J(\overline{\mathbb{N}})^{(-)}$. □

**Proposition 3.25.** *An object of Ext is projective if and only if it is isomorphic to a canonically projective object.*

**Proof.** This is completely analogous to the proof of Proposition 3.20; one uses the canonically projective cover and one realizes that, if

$$\bigcap_{p\in P}\left([\![ p = p ]\!] \to J\left(\bigcup_{x\in X} G(x,p)\right)\right)$$

is nonempty and $[\![ p = p ]\!]$ is of form $\downarrow\alpha$ with $\sharp\alpha \leqslant 2$, then so is

$$\bigcap_{p\in P}\left([\![ p = p ]\!] \to \bigcup_{x\in X} G(x,p)\right)$$

nonempty. □

So *Ext* does have enough projectives; from the definition of canonically projective however, it is clear that this is not closed under products. In fact, if $A$ is the canonically separated object $(\{a,b\},=)$ with $[\![ a = a ]\!] = \downarrow\{0\}$ and $[\![ b = b ]\!] = \downarrow\{0,1\}$ then one can check that $A \times A$ is not projective in *Ext*.

It follows, that in *Ext*, the notions of projective and *internally projective* (in a topos, an object $Q$ is called internally projective if the functor $(-)^Q$ preserves surjections) do *not* coincide, as they do in $\mathscr{E}\!f\!f$.

Since the projectives in *Ext* do not form a left exact category, *Ext* can not be an exact completion of a left exact category. Of course, the question then naturally arises whether there are exact completions with respect to weaker structures than finite limits. This question has been solved by Carboni and Vitale in [6] and [21] in the following way. A category $E$ is said to have *weak finite limits* if for any finite diagram in $E$ there is a cone $C$ in $E$ such that every other cone for the diagram factors through $C$ (not necessarily uniquely).

For every category $E$ with weak finite limits then, there is an exact completion $E_{\mathrm{ex/wlex}}$ with the following universal property: there is, for every exact category $\mathscr{A}$, an equivalence of categories between exact functors: $E_{\mathrm{ex/wlex}} \to \mathscr{A}$ and *left covering functors* $E \to \mathscr{A}$, where a functor $F : E \to \mathscr{A}$ from a category $E$ with weak finite limits to an exact category $\mathscr{A}$ is left covering if the factorization in $\mathscr{A}$ of the $F$-image of a weak limiting cone through the limiting cone, is a regular epimorphism.

An exact category $E$ will be an exact completion (over weak limits) of its category of projectives if it has enough projectives: it is easy to show that in this case, the full subcategory of $E$ on the projectives has weak finite limits.

So *Ext* is the exact completion over weak limits of its category of projectives.

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