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Locally convex quasi C^*-normed algebras ☆

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ABSTRACT

If $\mathcal{A}_0[\|\cdot\|_0]$ is a C^*-normed algebra and τ a locally convex topology on $\mathcal{A}_0$ making its multiplication separately continuous, then $\widetilde{\mathcal{A}}_0[\tau]$ (completion of $\mathcal{A}_0[\tau]$) is a locally convex quasi $*$-algebra over $\mathcal{A}_0$, but it is not necessarily a locally convex quasi $*$-algebra over the C^*-algebra $\widetilde{\mathcal{A}}_0[\|\cdot\|_0]$ (completion of $\mathcal{A}_0[\|\cdot\|_0]$). In this article, stimulated by physical examples, we introduce the notion of a locally convex quasi C^*-normed algebra, aiming at the investigation of $\widetilde{\mathcal{A}}_0[\tau]$; in particular, we study its structure, $*$-representation theory and functional calculus.

1. Introduction

In the present paper we continue the study introduced in [7] and carried over in [13] and [8]. At this stage, it concerns the investigation of the structure of the completion of a C^*-normed algebra $\mathcal{A}_0[\|\cdot\|_0]$, under a locally convex topology τ "compatible" to $\|\cdot\|_0$, that makes the multiplication of $\mathcal{A}_0$ separately continuous. The case when $\mathcal{A}_0[\|\cdot\|_0]$ is a C^*-algebra and τ makes the multiplication jointly continuous was considered in [7,13], while the analogue case corresponding to separately continuous multiplication was discussed in [8], where the so-called *locally convex quasi C^*-algebras* were introduced. In this work, prompted by examples that one meets in physics, we introduce the notion of *locally convex quasi C^*-normed algebras*, which is wider than that of locally convex quasi C^*-algebras, starting with a C^*-normed algebra $\mathcal{A}_0[\|\cdot\|_0]$ and a locally convex topology τ, compatible with $\|\cdot\|_0$, making the multiplication of $\mathcal{A}_0$ separately continuous. For example, let $\mathcal{M}_0$ be a C^*-normed algebra of operators on a Hilbert space $\mathcal{H}$, endowed with the operator norm $\|\cdot\|_0$, $\mathcal{D}$ a dense subspace of $\mathcal{H}$ such that $\mathcal{M}_0\mathcal{D} \subset \mathcal{D}$ and τ_{s^*} the strong*-topology on $\mathcal{M}_0$ defined by $\mathcal{D}$. Then, the C^*-algebra $\widetilde{\mathcal{M}}_0[\|\cdot\|_0]$ does not leave $\mathcal{D}$ invariant, in general, and so the multiplication ax of $a \in \widetilde{\mathcal{M}}_0[\tau_{s^*}]$ and $x \in \widetilde{\mathcal{M}}_0[\|\cdot\|_0]$ is not necessarily well defined, therefore $\widetilde{\mathcal{M}}_0[\tau_{s^*}]$ is not a locally convex quasi C^*-algebra over the C^*-algebra $\widetilde{\mathcal{M}}_0[\|\cdot\|_0]$. Hence, it is meaningful to study not only locally convex quasi C^*-algebras, but also locally convex quasi C^*-normed algebras.

For locally convex quasi "C^*-normed algebras" we obtain analogous results to those in [8] for locally convex quasi "C^*-algebras" despite of the lack of completion and of weakening the condition (T_3) of [8].

In Section 3 we consider a C^*-algebra $\mathcal{A}_0[\|\cdot\|_0]$ with a "regular" locally convex topology τ and show that every unital pseudo-complete symmetric locally convex $*$-algebra $\mathcal{A}[\tau]$ such that $\mathcal{A}_0[\|\cdot\|_0] \subset \mathcal{A}[\tau] \subset \widetilde{\mathcal{A}}_0[\tau]$ is a GB^*-algebra over the

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unit ball $\mathcal{U}(\mathcal{A}_0)$ of $\mathcal{A}_0[\|\cdot\|_0]$. The latter algebras have been defined by G.R. Allan [2] and P.G. Dixon [12] and play an essential role in the unbounded $*$-representation theory. In Section 4 we define the notion of locally convex quasi C^*-normed algebras and study their general theory, while in Section 5 we investigate the structure of commutative locally convex quasi C^*-normed algebras. In the final Section 6 we present locally convex quasi C^*-normed algebras of operators and then we study questions on the $*$-representation theory of locally convex quasi C^*-normed algebras and functional calculus for the "commutatively quasi-positive" elements of $\widetilde{\mathcal{A}_0}[\tau]$.

Topological quasi $*$-algebras were introduced in 1981 by G. Lassner [15,16], for facing solutions of certain problems in quantum statistics and quantum dynamics. But only later (see [17, p. 90]) the initial definition was reformulated in the right way, having thus included many more interesting examples. Quasi $*$-algebras came in light in 1988 (see [19], as well as [20,9,10]), serving as important examples of partial $*$-algebras initiated by J.-P. Antoine and W. Karwowski in [4,5]. A lot of works have been done on this topic, which can be found in the treatise [3], where the reader will also find a relevant rich literature. Partial $*$-algebras and quasi $*$-algebras keep a very prominent place in the study of unbounded operators, where the latter are the foundation stones for mathematical physics and quantum field theory (see, for instance, [3,14,6,20]).

Our motivation for such studies comes, on the one hand, from the preceding discussion and the promising contribution of the powerful tool that the C^*-property offers to such studies and, on the other hand, from the physical examples of locally convex quasi C^*-normed algebras in "dynamics of the BCS–Bogolubov model" [16] that will be shortly discussed in Section 7.

2. Preliminaries

Throughout the whole paper we consider complex algebras and we suppose that all topological spaces are Hausdorff. If an algebra $\mathcal{A}$ has an identity element, this will be denoted by *1*, and an algebra $\mathcal{A}$ with identity *1* will be called *unital*.

Let $\mathcal{A}_0[\|\cdot\|_0]$ be a C^*-normed algebra. The symbol $\|\cdot\|_0$ of the C^*-norm will also denote the corresponding topology. Let τ be a topology on $\mathcal{A}_0$ such that $\mathcal{A}_0[\tau]$ is a locally convex $*$-algebra. The topologies τ, $\|\cdot\|_0$ on $\mathcal{A}_0$ are called *compatible*, whenever for any Cauchy net $\{x_\alpha\}$ in $\mathcal{A}_0[\|\cdot\|_0]$ such that $x_\alpha \to 0$ in τ, $x_\alpha \to 0$ in $\|\cdot\|_0$ [8]. The completion of $\mathcal{A}_0$ with respect to τ will be denoted by $\widetilde{\mathcal{A}_0}[\tau]$. In the sequel, we shall call a directed family of seminorms that defines a locally convex topology τ, *a defining family of seminorms*.

A *partial $*$-algebra* is a vector space $\mathcal{A}$ equipped with a vector space involution $* : \mathcal{A} \to \mathcal{A} : x \mapsto x^*$ and a partial multiplication defined on a set $\Gamma \subset \mathcal{A} \times \mathcal{A}$ such that:

(i) $(x, y) \in \Gamma$ implies $(y^*, x^*) \in \Gamma$;
(ii) $(x, y_1), (x, y_2) \in \Gamma$ and $\lambda, \mu \in \mathbb{C}$ imply $(x, \lambda y_1 + \mu y_2) \in \Gamma$;
(iii) for every $(x, y) \in \Gamma$, a product $xy \in \mathcal{A}$ is defined, such that xy depends linearly on x and y and satisfies the equality $(xy)^* = y^* x^*$.

Given a pair $(x, y) \in \Gamma$, we say that x is a *left multiplier* of y and y is a *right multiplier* of x.

Quasi $*$-algebras are essential examples of partial $*$-algebras. If $\mathcal{A}$ is a vector space and $\mathcal{A}_0$ a subspace of $\mathcal{A}$, which is also a $*$-algebra, then $\mathcal{A}$ is said to be a *quasi $*$-algebra over $\mathcal{A}_0$* whenever:

(i)′ The multiplication of $\mathcal{A}_0$ is extended on $\mathcal{A}$ as follows: The correspondences

$$\mathcal{A} \times \mathcal{A}_0 \to \mathcal{A} : (a, x) \mapsto ax \quad (\text{left multiplication of } x \text{ by } a) \quad \text{and}$$

$$\mathcal{A}_0 \times \mathcal{A} \to \mathcal{A} : (x, a) \mapsto xa \quad (\text{right multiplication of } x \text{ by } a)$$

are always defined and are bilinear;
(ii)′ $x_1(x_2 a) = (x_1 x_2)a$, $(a x_1)x_2 = a(x_1 x_2)$ and $x_1(a x_2) = (x_1 a)x_2$, for all $x_1, x_2 \in \mathcal{A}_0$ and $a \in \mathcal{A}$;
(iii)′ the involution $*$ of $\mathcal{A}_0$ is extended on $\mathcal{A}$, denoted also by $*$, such that $(ax)^* = x^* a^*$ and $(xa)^* = a^* x^*$, for all $x \in \mathcal{A}_0$ and $a \in \mathcal{A}$.

For further information cf. [3]. If $\mathcal{A}_0[\tau]$ is a locally convex $*$-algebra, with separately continuous multiplication, its completion $\widetilde{\mathcal{A}_0}[\tau]$ is a quasi $*$-algebra over $\mathcal{A}_0$ with respect to the operations:

- $ax := \lim_\alpha x_\alpha x$ (left multiplication), $x \in \mathcal{A}_0$, $a \in \widetilde{\mathcal{A}_0}[\tau]$,
- $xa := \lim_\alpha x x_\alpha$ (right multiplication), $x \in \mathcal{A}_0$, $a \in \widetilde{\mathcal{A}_0}[\tau]$, where $\{x_\alpha\}_{\alpha \in \Sigma}$ is a net in $\mathcal{A}_0$ such that $a = \tau\text{-}\lim_\alpha x_\alpha$.
- An involution on $\widetilde{\mathcal{A}_0}[\tau]$ like in (iii)′ is the continuous extension of the involution on $\mathcal{A}_0$.

A $*$-invariant subspace $\mathcal{A}$ of $\widetilde{\mathcal{A}_0}[\tau]$ containing $\mathcal{A}_0$ is called a *quasi $*$-subalgebra* of $\widetilde{\mathcal{A}_0}[\tau]$ if ax, xa belong to $\mathcal{A}$ for any $x \in \mathcal{A}_0$, $a \in \mathcal{A}$. One easily shows that $\mathcal{A}$ is a quasi $*$-algebra over $\mathcal{A}_0$. Moreover, $\mathcal{A}[\tau]$ is a locally convex space that contains $\mathcal{A}_0$ as a dense subspace and for every fixed $x \in \mathcal{A}_0$, the maps $\mathcal{A}[\tau] \to \mathcal{A}[\tau]$ with $a \mapsto ax$ and $a \mapsto xa$ are continuous. An algebra of this kind is called *locally convex quasi $*$-algebra over $\mathcal{A}_0$*.

We denote by $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ the set of all (closable) linear operators X such that $D(X)=\mathcal{D}$, $D(X^*)\supseteq\mathcal{D}$. The set $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ is a partial $*$-algebra with respect to the following operations: the usual sum X_1+X_2, the scalar multiplication λX, the involution $X\mapsto X^\dagger=X^*\restriction\mathcal{D}$ and the (*weak*) partial multiplication $X_1\,\Box\,X_2={X_1}^{\dagger *}X_2$, defined whenever X_2 is a weak right multiplier of X_1 (we shall write $X_2\in R^{\mathrm{w}}(X_1)$ or $X_1\in L^{\mathrm{w}}(X_2)$), that is, iff $X_2\mathcal{D}\subset D({X_1}^{\dagger *})$ and ${X_1}^*\mathcal{D}\subset D({X_2}^*)$. $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ is neither associative nor semiassociative.

Definition 2.1. Let $\mathcal{D}$ be a dense subspace of a Hilbert space $\mathcal{H}$. A *$*$-representation* π of $\mathcal{A}[\tau]$ is a linear map from $\mathcal{A}$ into $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ (see beginning of Section 4) with the following properties:

(i) π is a $*$-representation of $\mathcal{A}_0$;
(ii) $\pi(a)^\dagger=\pi(a^*)$, $\forall a\in\mathcal{A}$;
(iii) $\pi(ax)=\pi(a)\,\Box\,\pi(x)$ and $\pi(xa)=\pi(x)\,\Box\,\pi(a)$, $\forall a\in\mathcal{A}$ and $x\in\mathcal{A}_0$, where $\Box$ is the (weak) partial multiplication of $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ (ibid.) Having a $*$-representation π as before, we write $\mathcal{D}(\pi)$ in the place of $\mathcal{D}$ and $\mathcal{H}_\pi$ in the place of $\mathcal{H}$. By a *(τ,τ_{s^*})-continuous $*$-representation* π of $\mathcal{A}[\tau]$, we clearly mean continuity of π, when $\mathcal{L}^\dagger(\mathcal{D}(\pi),\mathcal{H}_\pi)$ carries the locally convex topology τ_{s^*} (see Section 4).

In what follows, we shall need the concept of a *GB**-algebra introduced by G.R. Allan [2] (see also [12]), which we remind here. Let $\mathcal{A}[\tau]$ be a locally convex $*$-algebra with identity 1 and let $\mathcal{B}^*$ denote the collection of all closed, bounded, absolutely convex subsets B of $\mathcal{A}[\tau]$ with the properties: $1\in B$, $B^*=B$ and $B^2\subset B$. For each $B\in\mathcal{B}^*$, the linear span $A[B]$ of B is a normed $*$-algebra under the Minkowski functional $\|\cdot\|_B$ of B. When $A[B]$ is complete for each $B\in\mathcal{B}^*$, then $\mathcal{A}[\tau]$ is called *pseudo-complete*. Every unital sequentially complete locally convex $*$-algebra is pseudo-complete [1, Proposition (2.6)]. A unital locally convex $*$-algebra $\mathcal{A}[\tau]$ is called *symmetric* (resp. algebraically symmetric) if for every $x\in\mathcal{A}$ the element $1+x^*x$ has an Allan-bounded inverse in $\mathcal{A}$ [2, pp. 91, 93] (resp. if $1+x^*x$ has an inverse in $\mathcal{A}$). A unital symmetric pseudo-complete locally convex $*$-algebra $\mathcal{A}[\tau]$, such that $\mathcal{B}^*$ has a greatest member, say B_0, is said to be a *GB***-algebra over* B_0. In this case, $A[B_0]$ is a C^*-algebra.

3. C^*-normed algebras with regular locally convex topology

Let $\mathcal{A}_0[\|\cdot\|_0]$ be a C^*-normed algebra and $\widetilde{\mathcal{A}_0}[\|\cdot\|_0]$ the C^*-algebra completion of $\mathcal{A}_0[\|\cdot\|_0]$. Consider a locally convex topology τ on $\mathcal{A}_0$ with the following properties:

(T_1) $\mathcal{A}_0[\tau]$ is a locally convex $*$-algebra with separately continuous multiplication.
(T_2) $\tau\preccurlyeq\|\cdot\|_0$, with τ and $\|\cdot\|_0$ being compatible.

Then, compatibility of τ, $\|\cdot\|_0$ implies that:

- $\mathcal{A}_0[\|\cdot\|_0]\hookrightarrow\widetilde{\mathcal{A}_0}[\|\cdot\|_0]\hookrightarrow\widetilde{\mathcal{A}_0}[\tau]$;
- $\widetilde{\mathcal{A}_0}[\tau]$ is a locally convex quasi $*$-algebra over the C^*-normed algebra $\mathcal{A}_0[\|\cdot\|_0]$, but it is not necessarily a locally convex quasi $*$-algebra over the C^*-algebra $\widetilde{\mathcal{A}_0}[\|\cdot\|_0]$, since $\widetilde{\mathcal{A}_0}[\|\cdot\|_0]$ is not a locally convex $*$-algebra under the topology τ.

Question. Under which conditions one could have a well defined multiplication of elements in $\widetilde{\mathcal{A}_0}[\tau]$ with elements in $\widetilde{\mathcal{A}_0}[\|\cdot\|_0]$?

We consider the case that the locally convex topology τ defined by a directed family of seminorms, say $(p_\lambda)_{\lambda\in\Lambda}$, satisfies in addition the conditions (T_1) and (T_2), an extra "good" condition for the C^*-norm $\|\cdot\|_0$, called *regularity condition*, denoted by (R). That is,

$$(\mathrm{R})\quad \forall\lambda\in\Lambda,\ \exists\lambda'\in\Lambda \text{ and } \gamma_\lambda>0:\quad p_\lambda(xy)\leqq\gamma_\lambda\|x\|_0\,p_{\lambda'}(y),\quad \forall x,y\in\mathcal{A}_0\big[\|\cdot\|_0\big].$$

In this regard, we have the following

Lemma 3.1. *Suppose $\mathcal{A}_0[\|\cdot\|_0]$ is a C^*-normed algebra and τ a locally convex topology on $\mathcal{A}_0$ satisfying the conditions* (T_1), (T_2) *and the regularity condition* (R) *for $\|\cdot\|_0$. Let a be an arbitrary element in $\widetilde{\mathcal{A}_0}[\tau]$ and y an arbitrary element in $\widetilde{\mathcal{A}_0}[\|\cdot\|_0]$. Then, the left resp. right multiplication of a with y is defined by*

$$a\cdot y=\tau\text{-}\lim_{\alpha,n}x_\alpha y_n\quad\textit{resp.}\quad y\cdot a=\tau\text{-}\lim_{\alpha,n}y_n x_\alpha,$$

where $\{x_\alpha\}_{\alpha\in\Sigma}$ is a net in $\mathcal{A}_0[\tau]$ converging to a, $\{y_n\}_{n\in\mathbb{N}}$ is a sequence in $\mathcal{A}_0[\|\cdot\|_0]$ converging to y and $\forall\lambda\in\Lambda$, $\exists\lambda'\in\Lambda$ and $\gamma_\lambda>0$:

$$p_\lambda(a\cdot y)\leqq\gamma_\lambda\|y\|_0\,p_{\lambda'}(a),\qquad p_\lambda(y.a)\leqq\gamma_\lambda\|y\|_0\,p_{\lambda'}(a).$$

Under this multiplication $\widetilde{\mathcal{A}_0}[\tau]$ is a locally convex quasi $*$-algebra over the C^*-algebra $\widetilde{\mathcal{A}_0}[\|\cdot\|_0]$.

The proof of Lemma 3.1 follows directly from the regularity condition (R). If $\mathcal{A}_0[\tau]$ is a locally convex $*$-algebra with jointly continuous multiplication and $\tau \preccurlyeq \|\cdot\|_0$, then it satisfies the regular condition (R) for $\|\cdot\|_0$.

Lemma 3.2. *Let $\mathcal{A}_0[\|\cdot\|_0]$ be a C^*-normed algebra and $\mathcal{A}_0[\tau]$ an m^*-convex algebra satisfying conditions* (T$_2$) *and* (R). *If $(p_\lambda)_{\lambda\in\Lambda}$ is a defining family of m^*-seminorms for τ* (i.e., submultiplicative $*$-preserving seminorms) *and there is $\lambda_0 \in \Lambda$ such that p_{λ_0} is a norm, then $\tau \sim \|\cdot\|_0$, where $\sim$ means equivalence of the respective topologies. In particular, if $\mathcal{A}_0[\|\cdot\|]$ is a normed $*$-algebra such that $\|\cdot\| \leqslant \|\cdot\|_0$ and $\|\cdot\|$, $\|\cdot\|_0$ are compatible, then $\|\cdot\| \sim \|\cdot\|_0$.*

Proof. By (T$_2$) and (R) we have $\widetilde{\mathcal{A}_0}[\|\cdot\|_0] \hookrightarrow \widetilde{\mathcal{A}_0}[\tau] \hookrightarrow \widetilde{\mathcal{A}_0}[p_{\lambda_0}]$, which by the basic theory of C^*-algebras (see e.g., [18, Proposition 5.3]) implies that $\|x\|_0 \leqslant p_{\lambda_0}(x)$, for all $x \in \mathcal{A}_0$. Hence, $\tau \sim \|\cdot\|_0$. □

By Lemma 3.2 there does not exist any normed $*$-algebra containing the C^*-algebra $\widetilde{\mathcal{A}_0}[\|\cdot\|_0]$ properly and densely.

We now consider whether a GB^*-algebra over the unit ball $\mathcal{U}(\widetilde{\mathcal{A}_0}[\|\cdot\|_0])$ exists in $\widetilde{\mathcal{A}_0}[\tau]$. If $\widetilde{\mathcal{A}_0}[\tau]$ has jointly continuous multiplication and $\mathcal{U}(\widetilde{\mathcal{A}_0}[\|\cdot\|_0])$ is τ-closed in $\widetilde{\mathcal{A}_0}[\tau]$, then $\widetilde{\mathcal{A}_0}[\tau]$ is a GB^*-algebra over $\mathcal{U}(\widetilde{\mathcal{A}_0}[\|\cdot\|_0])$ (cf. [13, Theorem 2.1]).

Theorem 3.3. *Let $\mathcal{A}_0[\|\cdot\|_0]$ be a unital C^*-normed algebra and $\mathcal{A}_0[\tau]$ a locally convex $*$-algebra such that τ satisfies the conditions* (T$_1$), (T$_2$), *the regularity condition* (R) *for $\|\cdot\|_0$ and makes the unit ball $\mathcal{U}(\widetilde{\mathcal{A}_0}[\|\cdot\|_0])$ τ-closed in $\widetilde{\mathcal{A}_0}[\tau]$. Then every algebraically symmetric locally convex $*$-algebra $\mathcal{A}[\tau]$ such that $\widetilde{\mathcal{A}_0}[\|\cdot\|_0] \subset \mathcal{A}[\tau] \subset \widetilde{\mathcal{A}_0}[\tau]$ is a GB^*-algebra over $\mathcal{U}(\widetilde{\mathcal{A}_0}[\|\cdot\|_0])$.*

Proof. The proof can be done in a similar way to that of [7, Theorem 2.2]. Here we give a simpler proof. Without loss of generality we may assume that $\mathcal{A}_0[\|\cdot\|_0]$ is a C^*-algebra. Then we have (see, e.g., proof of [7, Lemma 2.1]):

(**1**) $(1 + a^*a)^{-1} \in \mathcal{U}(\mathcal{A}_0)$, $\forall a \in \mathcal{A}$.

Moreover, we show that

(**2**) $\mathcal{U}(\mathcal{A}_0)$ is the largest member in $\mathcal{B}^*(\mathcal{A})$.

It is clear that $\mathcal{U}(\mathcal{A}_0) \in \mathcal{B}^*(\mathcal{A})$. Suppose now that B is an arbitrary element in $\mathcal{B}^*(\mathcal{A})$ and take $a = a^*$ in B. Let $\mathcal{C}(a)$ be the maximal commutative $*$-subalgebra of $\mathcal{A}$ containing a and

$$\mathcal{C}_1 \equiv \big(\mathcal{U}(\mathcal{A}_0) \cap \mathcal{C}(a)\big) \cdot \big(B \cap \mathcal{C}(a)\big).$$

Then, clearly $\mathcal{C}_1^* = \mathcal{C}_1$; by the regular condition (R) $\mathcal{C}_1$ is τ-bounded in $\mathcal{C}(a)$, while by the commutativity of $\mathcal{U}(\mathcal{A}_0) \cap \mathcal{C}(a)$ and $B \cap \mathcal{C}(a)$ one has that $\mathcal{C}_1^2 \subset \mathcal{C}_1$. It is now easily seen that $\overline{\mathcal{C}_1}^\tau \in \mathcal{B}^*(\mathcal{C}(a))$, where $\mathcal{B}^*(\mathcal{C}(a)) = \{B \cap \mathcal{C}(a)\colon\ B \in \mathcal{B}^*(\mathcal{A})\}$. Thus, there is $B_1 \in \mathcal{B}^*(\mathcal{A})$ such that $\overline{\mathcal{C}_1}^\tau = B_1 \cap \mathcal{C}(a)$.

Since $\mathcal{C}(a)$ is commutative and pseudo-complete, $\mathcal{B}^*(\mathcal{C}(a))$ is directed [1, Theorem (2.10)]. So for each $B \in \mathcal{B}^*(\mathcal{A})$ there is $B_1 \in \mathcal{B}^*(\mathcal{A})$ such that

$$\big(B \cup \mathcal{U}(\mathcal{A}_0)\big) \cap \mathcal{C}(a) \subset B_1 \cap \mathcal{C}(a).$$

Hence

$$\mathcal{A}_0 \cap \mathcal{C}(a) \subset A[B_1] \cap \mathcal{C}(a),$$

where $\mathcal{A}_0 \cap \mathcal{C}(a)$ is a C^*-algebra and $A[B_1] \cap \mathcal{C}(a)$ a normed $*$-algebra. An application of Lemma 3.2 gives

$$\|x\|_0 = \|x\|_{B_1}, \quad \forall x \in \mathcal{A}_0 \cap \mathcal{C}(a). \tag{3.1}$$

Furthermore, it follows from (**1**) that $x(1 + \frac{1}{n}x^*x)^{-1} \in \mathcal{A}_0$. Thus,

$$\left\| x\left(1 + \frac{1}{n}x^*x\right)^{-1} - x \right\|_{B_1} \leqslant \frac{1}{n}\left\|xx^*x\right\|_{B_1}, \quad \forall x \in A[B_1] \cap \mathcal{C}(a),\ n \in \mathbb{N},$$

which implies that $\mathcal{A}_0 \cap \mathcal{C}(a)$ is $\|\cdot\|_{B_1}$-dense in $A[B_1] \cap \mathcal{C}(a)$. Therefore, from (3.1) and the fact that $\mathcal{A}_0 \cap \mathcal{C}(a)$ is a C^*-algebra we get $\mathcal{A}_0 \cap \mathcal{C}(a) = A[B_1] \cap \mathcal{C}(a)$. It follows that $B \cap \mathcal{C}(a) \subset B_1 \cap \mathcal{C}(a) = \mathcal{U}(\mathcal{A}_0) \cap \mathcal{C}(a)$, from which we conclude

$$a \in \mathcal{U}(\mathcal{A}_0), \quad \forall a \in B,\ \text{with } a^* = a. \tag{3.2}$$

Now taking an arbitrary $a \in B$ we clearly have $a^*a \in B$, hence from (3.2) $a^*a \in \mathcal{U}(\mathcal{A}_0)$, which gives $a \in \mathcal{U}(\mathcal{A}_0)$. So, $B \subset \mathcal{U}(\mathcal{A}_0)$ and the proof of (**2**) is complete. Now, since $\mathcal{U}(\mathcal{A}_0)$ is the greatest member in $\mathcal{B}^*(\mathcal{A})$, we have that $A[\mathcal{U}(\mathcal{A}_0)]$ coincides with the C^*-algebra $\mathcal{A}_0$, therefore it is complete. So [1, Proposition 2.7] implies that $\mathcal{A}[\tau]$ is pseudo-complete, hence a GB^*-algebra over $\mathcal{U}(\mathcal{A}_0)$. □

4. Locally convex quasi C^*-normed algebras

Let $\mathcal{A}_0[\|\cdot\|_0]$ be a C^*-normed algebra and τ a locally convex topology on $\mathcal{A}_0$ with $\{p_\lambda\}_{\lambda\in\Lambda}$ a defining family of seminorms. Suppose that τ satisfies the properties (T_1), (T_2). The *regularity condition* (R), considered in the previous Section 2, for $\|\cdot\|_0$, is too strong (see Section 6). So in the present section we weaken this condition, and we use it together with the conditions (T_1), (T_2), in order to investigate the locally convex quasi $*$-algebra $\widetilde{\mathcal{A}_0}[\tau]$. *The weakened condition* (R) *will be denoted by* (T_3) *and it will read as follows*:

(T_3) $\forall\lambda\in\Lambda$, $\exists\lambda'\in\Lambda$ and $\gamma_\lambda>0$: $p_\lambda(xy)\leqq\gamma_\lambda\|x\|_0 p_{\lambda'}(y)$, for all $x,y\in\mathcal{A}_0$ with $xy=yx$.

Then, we first consider the question stated in Section 3, just before Lemma 3.1, concerning a well defined multiplication between elements of $\widetilde{\mathcal{A}_0}[\tau]$ and $\widetilde{\mathcal{A}_0}[\|\cdot\|_0]$.

If $\mathcal{A}_0[\|\cdot\|_0]$ is commutative and τ satisfies the conditions (T_1)–(T_3), then τ fulfills clearly the regularity condition (R) for $\|\cdot\|_0$, and so by Lemma 3.1, for arbitrary $a\in\widetilde{\mathcal{A}_0}[\tau]$ and $y\in\widetilde{\mathcal{A}_0}[\|\cdot\|_0]$ the left and right multiplications $a\cdot y$ and $y\cdot a$ are defined, respectively, and $\widetilde{\mathcal{A}_0}[\tau]$ is a locally convex quasi $*$-algebra over the C^*-algebra $\widetilde{\mathcal{A}_0}[\|\cdot\|_0]$.

We consider now the afore-mentioned question in the noncommutative case; for this we set the following

Definition 4.1. Let $a\in\widetilde{\mathcal{A}_0}[\tau]$ and $y\in\widetilde{\mathcal{A}_0}[\|\cdot\|_0]$. We shall say that y *commutes strongly* with a if there is a net $\{x_\alpha\}_{\alpha\in\Sigma}$ in $\widetilde{\mathcal{A}_0}[\|\cdot\|_0]$ such that $x_a \underset{\tau}{\longrightarrow} a$ and $x_\alpha y=yx_\alpha$, for every $\alpha\in\Sigma$.

• In the rest of the paper, $\widetilde{\mathcal{A}_0}[\|\cdot\|_0]^{\sim}[\tau]$, denotes the completion of the C^*-algebra $\widetilde{\mathcal{A}_0}[\|\cdot\|_0]$ with respect to the locally convex topology τ. As a set it clearly coincides with $\widetilde{\mathcal{A}_0}[\tau]$, but there are cases that we need to distinguish them (see Remark 4.6).

Remark 4.2. Let $a\in\widetilde{\mathcal{A}_0}[\tau]$ and $y\in\widetilde{\mathcal{A}_0}[\|\cdot\|_0]$. Whenever $y\in\mathcal{A}_0$, the multiplications ay and ya are always defined by

$$ay=\lim_\alpha x_\alpha y \quad\text{and}\quad ya=\lim_\alpha yx_\alpha,$$

where $\{x_\alpha\}_{\alpha\in\Sigma}$ is a net in $\mathcal{A}_0$ converging to a with respect to τ. Hence, we may define the notion y *commutes with* a, as usually, i.e., when $ay=ya$. But, even if y *commutes with* a, one has, in general, that y *does not commute strongly with* a. Thus, the notion of strong commutativity is clearly stronger than that of commutativity.

Lemma 4.3. *Let $\mathcal{A}_0[\|\cdot\|_0]$ be a C^*-normed algebra and τ a locally convex topology on $\mathcal{A}_0$ that satisfies the properties* (T_1)–(T_3). *Let $a\in\widetilde{\mathcal{A}_0}[\tau]$ and $y\in\widetilde{\mathcal{A}_0}[\|\cdot\|_0]$ be strongly commuting. Then the multiplications $a\cdot y$ resp. $y\cdot a$ are defined by*

$$a\cdot y=\tau\text{-}\lim_\alpha x_\alpha y \quad\textit{resp.}\quad y\cdot a=\tau\text{-}\lim_\alpha yx_\alpha \quad\textit{and}\quad a\cdot y=y\cdot a,$$

where $\{x_\alpha\}_{\alpha\in\Sigma}$ is a net in $\widetilde{\mathcal{A}_0}[\|\cdot\|_0]$, τ-converging to a and commutating with y. The preceding multiplications provide an extension of the multiplication of $\mathcal{A}_0$. Moreover, an analogous condition to (T_3) *holds for the elements a, y, i.e.,*

(T_3') $\forall\lambda\in\Lambda$, $\exists\lambda'\in\Lambda$ *and* $\gamma_\lambda>0$: $p_\lambda(a\cdot y)\leqq\gamma_\lambda\|y\|_0 p_{\lambda'}(a)$.

Proof. Existence of the $\tau\text{-}\lim_\alpha x_\alpha y$ in $\widetilde{\mathcal{A}_0}[\tau]$:

Note that $\{x_\alpha y\}_{\alpha\in\Sigma}$ is a τ-Cauchy net in $\widetilde{\mathcal{A}_0}[\|\cdot\|_0]$. Indeed, from (T_3), for every $\lambda\in\Lambda$, there are $\lambda'\in\Lambda$ and $\gamma_\lambda>0$ such that

$$p_\lambda(x_\alpha y-x_{\alpha'}y)=p_\lambda\big((x_\alpha-x_{\alpha'})y\big)\leqq\gamma_\lambda\|y\|_0 p_{\lambda'}(x_\alpha-x_{\alpha'}) \underset{\alpha,\alpha'}{\longrightarrow} 0.$$

Hence, $\tau\text{-}\lim_\alpha x_\alpha y$ exists in $\widetilde{\mathcal{A}_0}[\|\cdot\|_0]^{\sim}[\tau]$, which, as already noticed, as a set clearly coincides with $\widetilde{\mathcal{A}_0}[\tau]$.

The existence of the $\tau\text{-}\lim_\alpha yx_\alpha$ in $\widetilde{\mathcal{A}_0}[\|\cdot\|_0]^{\sim}[\tau]$ is similarly shown and clearly $\tau\text{-}\lim_\alpha yx_\alpha=\tau\text{-}\lim_\alpha x_\alpha y$.

Independence of $\tau\text{-}\lim_\alpha x_\alpha y$ of the choice of the net $\{x_\alpha\}_{\alpha\in\Sigma}$:

Let $\{x'_\beta\}_{\beta\in\Sigma'}$ be another net in $\mathcal{A}_0$ such that $x'_\beta \xrightarrow{\tau} a$ and $x'_\beta y=yx'_\beta$, for all $\beta\in\Sigma'$. Then,

$$x_\alpha-x'_\beta \xrightarrow{\tau} 0 \quad\text{with } \big(x_\alpha-x'_\beta\big)y=y\big(x_\alpha-x'_\beta\big),\ \forall(\alpha,\beta)\in\Sigma\times\Sigma'.$$

Moreover, by (T_3), for every $\lambda\in\Lambda$, there exist $\lambda'\in\Lambda$ and $\gamma_\lambda>0$ such that

$$p_\lambda\big(\big(x_\alpha-x'_\beta\big)y\big)\leqq\gamma_\lambda\|y\|_0 p_{\lambda'}\big(x_\alpha-x'_\beta\big) \underset{\alpha,\beta}{\longrightarrow} 0;$$

this completes the proof of our claim. Thus, we set

$$a\cdot y:=\tau\text{-}\lim x_\alpha y,\quad\text{resp.}\quad y\cdot a:=\tau\text{-}\lim yx_\alpha;$$

this clearly implies $a \cdot y = y \cdot a$. Furthermore, using again (T_3) we conclude that

$$\forall \lambda \in \Lambda,\ \exists \lambda' \in \Lambda \text{ and } \gamma_\lambda > 0\colon \quad p_\lambda(a \cdot y) \leqq \gamma_\lambda \|y\|_0 p_{\lambda'}(a), \quad \forall a \in \widetilde{\mathcal{A}_0}[\tau] \text{ and } y \in \widetilde{\mathcal{A}_0}\big[\| \cdot \|_0\big],$$

and this proves (T'_3). □

Now, following [8] we define notions of positivity for the elements of $\widetilde{\mathcal{A}_0}[\tau]$.

Definition 4.4. Let $a \in \widetilde{\mathcal{A}_0}[\tau]$. Consider the set

$$(\mathcal{A}_0)_+ := \big\{x \in \mathcal{A}_0\colon x^* = x \text{ and } sp_{\mathcal{A}_0}(x) \subseteq [0, \infty)\big\},$$

where $sp_{\mathcal{A}_0}(x)$ means spectrum of x in $\mathcal{A}_0$. Clearly $(\mathcal{A}_0)_+$ is contained in the positive cone of the C^*-algebra $\widetilde{\mathcal{A}_0}[\| \cdot \|_0]$. The element a is called *quasi-positive* if there is a net $\{x_\alpha\}_{\alpha \in \Sigma}$ in $(\mathcal{A}_0)_+$ such that $x_\alpha \underset{\tau}{\longrightarrow} a$. In particular, a is called *commutatively quasi-positive* if there is a commuting net $\{x_\alpha\}_{\alpha \in \Sigma}$ in $(\mathcal{A}_0)_+$ such that $x_\alpha \underset{\tau}{\longrightarrow} a$.

Denote by $\widetilde{\mathcal{A}_0}[\tau]_{q+}$ *the set of all quasi-positive elements of* $\widetilde{\mathcal{A}_0}[\tau]$ and by $\widetilde{\mathcal{A}_0}[\tau]_{cq+}$ *the set of all commutatively quasi-positive elements of* $\widetilde{\mathcal{A}_0}[\tau]$.

An easy consequence of Definition 4.4 is the following

Lemma 4.5.

(1)
$$\begin{array}{ccc} (\mathcal{A}_0)_+ & \subset & \widetilde{\mathcal{A}_0}[\tau]_{cq+} \\ \cap & & \cap \\ \overline{(\mathcal{A}_0)_+}^{\|\cdot\|_0} = \widetilde{\mathcal{A}_0}\big[\| \cdot \|_0\big]_+ & \subset & \widetilde{\mathcal{A}_0}[\tau]_{q+}. \end{array}$$

(2) $\widetilde{\mathcal{A}_0}[\tau]_{q+}$ *is a positive wedge, but it is not necessarily a positive cone.* $\widetilde{\mathcal{A}_0}[\tau]_{cq+}$ *is not even a positive wedge, in general.*

Remark 4.6. As we have mentioned before, the equality $\widetilde{\mathcal{A}_0}[\| \cdot \|_0]^\sim[\tau] = \widetilde{\mathcal{A}_0}[\tau]$ holds set-theoretically. We consider the following notation:

$$\widetilde{\mathcal{A}_0}\big[\| \cdot \|_0\big]^\sim[\tau]_{q+} \equiv \big\{a \in \widetilde{\mathcal{A}_0}[\tau]\colon \exists \text{ a net } \{x_\alpha\}_{\alpha \in \Sigma} \text{ in } \widetilde{\mathcal{A}_0}\big[\| \cdot \|_0\big]_+ : x_\alpha \underset{\tau}{\longrightarrow} a\big\},$$
$$\widetilde{\mathcal{A}_0}\big[\| \cdot \|_0\big]^\sim[\tau]_{cq+} \equiv \big\{a \in \widetilde{\mathcal{A}_0}[\tau]\colon \exists \text{ a commuting net } \{x_\alpha\}_{\alpha \in \Sigma} \text{ in } \widetilde{\mathcal{A}_0}\big[\| \cdot \|_0\big]_+ : x_\alpha \underset{\tau}{\longrightarrow} a\big\}.$$

Then,

$$\widetilde{\mathcal{A}_0}\big[\| \cdot \|_0\big]^\sim[\tau]_{q+} = \widetilde{\mathcal{A}_0}[\tau]_{q+}, \quad \text{but} \quad \widetilde{\mathcal{A}_0}\big[\| \cdot \|_0\big]^\sim[\tau]_{cq+} \supsetneqq \widetilde{\mathcal{A}_0}[\tau]_{cq+}, \quad \text{in general.} \tag{4.1}$$

If $\mathcal{A}_0$ is commutative, then

$$\widetilde{\mathcal{A}_0}[\tau]_{cq+} = \widetilde{\mathcal{A}_0}\big[\| \cdot \|_0\big]^\sim[\tau]_{cq+} = \widetilde{\mathcal{A}_0}\big[\| \cdot \|_0\big]^\sim[\tau]_{q+} = \widetilde{\mathcal{A}_0}[\tau]_{q+}.$$

The following Proposition 4.7 plays an important role in the present paper. It is a generalization of Proposition 3.2 in [8], stated for locally convex quasi C^*-algebras, to the case of locally convex quasi C^*-normed algebras.

Proposition 4.7. *Let $\mathcal{A}_0[\| \cdot \|_0]$ be a unital C^*-normed algebra and τ a locally convex topology on $\mathcal{A}_0$ that fulfills the conditions* (T_1)–(T_3). *Suppose that the next condition* (T_4) *holds*:

(T_4) *The set $\mathcal{U}(\widetilde{\mathcal{A}_0}[\| \cdot \|_0])_+ \equiv \{x \in \widetilde{\mathcal{A}_0}[\| \cdot \|_0]_+\colon \|x\|_0 \leqq 1\}$ is τ-closed in $\widetilde{\mathcal{A}_0}[\tau]$ (or, equivalently, it is τ-complete).*

Then, $\widetilde{\mathcal{A}_0}[\tau]$ is a locally convex quasi $*$-algebra over $\mathcal{A}_0$ with the properties:

(1) *$a \in \widetilde{\mathcal{A}_0}[\tau]_{cq+}$ implies that $1 + a$ is invertible with $(1 + a)^{-1}$ in $\mathcal{U}(\widetilde{\mathcal{A}_0}[\| \cdot \|_0])_+$.*
(2) *For $a \in \widetilde{\mathcal{A}_0}[\tau]_{cq+}$ and $\varepsilon > 0$, the element $a_\varepsilon := a \cdot (1 + \varepsilon a)^{-1}$ is well defined, $a - a_\varepsilon \in \widetilde{\mathcal{A}_0}[\| \cdot \|_0]^\sim[\tau]_{cq+}$ and $a = \tau\text{-}\lim_{\varepsilon \downarrow 0} a_\varepsilon$.*
(3) *$\widetilde{\mathcal{A}_0}[\tau]_{cq+} \cap (-\widetilde{\mathcal{A}_0}[\tau]_{cq+}) = \{0\}$.*
(4) *Furthermore, suppose that the following condition*
(T_5) *$\widetilde{\mathcal{A}_0}[\tau]_{q+} \cap \widetilde{\mathcal{A}_0}[\| \cdot \|_0] = \widetilde{\mathcal{A}_0}[\| \cdot \|_0]_+$*
is satisfied. Then, if $a \in \widetilde{\mathcal{A}_0}[\tau]_{cq+}$ and $y \in \widetilde{\mathcal{A}_0}[\| \cdot \|_0]_+$ with $y - a \in \widetilde{\mathcal{A}_0}[\tau]_{q+}$, one has that $a \in \widetilde{\mathcal{A}_0}[\| \cdot \|_0]_+$.

Proof. (1) There exists a commuting net $\{x_\alpha\}_{\alpha\in\Sigma}$ in $(\mathcal{A}_0)_+$ with $x_\alpha \xrightarrow[\tau]{} a$ and $x_\alpha x_{\alpha'} = x_{\alpha'} x_\alpha$, for all $\alpha, \alpha' \in \Sigma$. Using properties of the positive elements in a C^*-algebra, and condition (T_3) we get that for every $\lambda \in \Lambda$ there are $\lambda' \in \Lambda$ and $\gamma_\lambda > 0$ such that:

$$p_\lambda\big((1+x_\alpha)^{-1} - (1+x_{\alpha'})^{-1}\big) = p_\lambda\big((1+x_\alpha)^{-1}(x_{\alpha'} - x_\alpha)(1+x_{\alpha'})^{-1}\big)$$
$$\leqq \gamma_\lambda \big\|(1+x_\alpha)^{-1}\big\|_0 \big\|(1+x_{\alpha'})^{-1}\big\|_0 p_{\lambda'}(x_{\alpha'} - x_\alpha) \leqq \gamma_\lambda p_{\lambda'}(x_{\alpha'} - x_\alpha) \xrightarrow[\alpha,\alpha']{} 0.$$

Hence $\{(1+x_\alpha)^{-1}\}_{\alpha\in\Sigma}$ is a Cauchy net in $\widetilde{\mathcal{A}}_0[\tau]$ consisting of elements of $\mathcal{U}(\widetilde{\mathcal{A}}_0[\|\cdot\|_0])_+$, the latter set being τ-closed by (T_4). Hence, there exists $y \in \mathcal{U}(\widetilde{\mathcal{A}}_0[\|\cdot\|_0])_+$ such that

$$(1+x_\alpha)^{-1} \xrightarrow[\tau]{} y. \tag{4.2}$$

We shall show that $(1+a)^{-1}$ exists in $\mathcal{U}(\widetilde{\mathcal{A}}_0[\|\cdot\|_0])_+$ and coincides with y. It is easily seen that, for each index $\alpha \in \Sigma$, $(1+x_\alpha)^{-1}$ commutes strongly with $(1+a)$, so that $(1+a)\cdot(1+x_\alpha)^{-1}$ is well defined (Lemma 4.3). Similarly, $(x_\alpha - a)\cdot(1+x_\alpha)^{-1} = 1 - (1+a)\cdot(1+x_\alpha)^{-1}$ is well defined, therefore using (T_3') of Lemma 4.3, we have that for all $\lambda \in \Lambda$ there are $\lambda' \in \Lambda$ and $\gamma_\lambda > 0$ with

$$p_\lambda\big(1 - (1+a)\cdot(1+x_\alpha)^{-1}\big) = p_\lambda\big((x_\alpha - a)\cdot(1+x_\alpha)^{-1}\big) \leqq \gamma_\lambda p_{\lambda'}(x_\alpha - a) \xrightarrow[\alpha]{} 0.$$

Thus, $(1+a)\cdot(1+x_\alpha)^{-1} \xrightarrow[\tau]{} 1$. By the above,

$$1 + x_\alpha \xrightarrow[\tau]{} 1 + a \quad \text{and} \quad (1+x_\alpha)y = y(1+x_\alpha), \quad \forall \alpha \in \Sigma.$$

Hence, y commutes strongly with $1+a$, therefore $(1+a)\cdot y$ is well defined by Lemma 4.3. Now, since $x_\alpha \xrightarrow[\tau]{} a$, we have that

$$\forall \lambda \in \Lambda \text{ and } \forall \varepsilon > 0,\ \exists \alpha_0 \in \Sigma\colon \quad p_\lambda(x_{\alpha'} - a) < \varepsilon, \quad \forall \alpha' \geqq \alpha_0. \tag{4.3}$$

Using (T_3), (T_3') of Lemma 4.3, and relations (4.3), (4.2) we obtain

$$p_\lambda\big((1+a)\cdot(1+x_\alpha)^{-1} - (1+a)\cdot y\big) \leqq p_\lambda\big((1+a)\cdot(1+x_\alpha)^{-1} - (1+x_{\alpha_0})(1+x_\alpha)^{-1}\big)$$
$$+ p_\lambda\big((1+x_{\alpha_0})(1+x_\alpha)^{-1} - (1+x_{\alpha_0})y\big) + p_\lambda\big((1+x_{\alpha_0})y - (1+a)y\big)$$
$$\leqq \gamma_\lambda p_{\lambda'}(a - x_{\alpha_0}) + \gamma_\lambda \|1 + x_{\alpha_0}\|_0 p_{\lambda'}\big((1+x_\alpha)^{-1} - y\big) + \gamma_\lambda p_{\lambda'}(x_{\alpha_0} - a)$$
$$< 2\varepsilon + \gamma_\lambda \|1 + x_{\alpha_0}\|_0 p_{\lambda'}\big((1+x_\alpha)^{-1} - y\big), \quad \forall \varepsilon > 0.$$

Hence,

$$0 \leqq \lim_\alpha p_\lambda\big((1+a)\cdot(1+x_\alpha)^{-1} - (1+a)\cdot y\big) \leqq 2\varepsilon, \quad \forall \varepsilon > 0,$$

which implies

$$\lim_\alpha p_\lambda\big((1+a)\cdot(1+x_\alpha)^{-1} - (1+a)\cdot y\big) = 0.$$

Consequently,

$$(1+a)\cdot(1+x_\alpha)^{-1} \xrightarrow[\tau]{} (1+a)\cdot y. \tag{4.4}$$

Similarly, $(1+x_\alpha)^{-1}\cdot(1+a) \xrightarrow[\tau]{} y\cdot(1+a)$. So from (4.3) and (4.4) we conclude that $(1+a)\cdot y = y\cdot(1+a) = 1$, therefore $y = (1+a)^{-1}$.

(2) By (1), for every $\varepsilon > 0$, the element $(1+\varepsilon a)^{-1}$ exists in $\mathcal{U}(\widetilde{\mathcal{A}}_0[\|\cdot\|_0])_+$, and commutes strongly with a. Hence (see Lemma 4.3), $a_\varepsilon := a\cdot(1+\varepsilon a)^{-1}$ is well defined. Moreover, applying (T_3') of Lemma 4.3, we have that for all $\lambda \in \Lambda$, there exist $\lambda' \in \Lambda$ and $\gamma_\lambda > 0$ such that

$$p_\lambda\big(1 - (1+\varepsilon a)^{-1}\big) = \varepsilon p_\lambda\big(a\cdot(1+\varepsilon a)^{-1}\big) \leqq \varepsilon\gamma_\lambda \big\|(1+\varepsilon a)^{-1}\big\|_0 p_{\lambda'}(a) \leqq \varepsilon\gamma_\lambda p_{\lambda'}(a).$$

Therefore,

$$\tau\text{-}\lim_{\varepsilon\downarrow 0}(1+\varepsilon a)^{-1} = 1. \tag{4.5}$$

On the other hand, since $1 - (1+\varepsilon a)^{-1}$ commutes strongly with a and $a_\varepsilon = \varepsilon^{-1}(1 - (1+\varepsilon a)^{-1})$, $\varepsilon > 0$, we have

$$\big(1 - (1+\varepsilon a)^{-1}\big)\cdot a = a\cdot\big(1 - (1+\varepsilon a)^{-1}\big) = a - a_\varepsilon \in \widetilde{\mathcal{A}}_0\big[\|\cdot\|_0\big]^{\sim}[\tau]_{cq+}. \tag{4.6}$$

Using (4.5), (4.6) and the same arguments as in (4.4), we get that $\tau\text{-}\lim_{\varepsilon\downarrow 0} a_\varepsilon = a$.

(3) Let $a \in \widetilde{\mathcal{A}_0}[\tau]_{cq+} \cap (-\widetilde{\mathcal{A}_0}[\tau]_{cq+})$ and $\varepsilon > 0$ be sufficiently small. By (2) (see also Remark 4.6), we have

$$\widetilde{\mathcal{A}_0}\big[\|\cdot\|_0\big]^{\sim}[\tau]_{cq+} \ni a \cdot (1+\varepsilon a)^{-1} \underset{\tau}{\longrightarrow} a; \quad \text{in the same way } -a \cdot (1-\varepsilon a)^{-1} \underset{\tau}{\longrightarrow} -a.$$

Now the element

$$x_\varepsilon \equiv a \cdot (1+\varepsilon a)^{-1} - (-a) \cdot (1-\varepsilon a)^{-1} = 2a \cdot (1+\varepsilon a)^{-1}(1-\varepsilon a)^{-1}$$

belongs to $\widetilde{\mathcal{A}_0}[\|\cdot\|_0]_+$ by (1) and the functional calculus of commutative C^*-algebras. Similarly, $-x_\varepsilon = 2(-a) \cdot (1-\varepsilon a)^{-1}(1+\varepsilon a)^{-1} \in \widetilde{\mathcal{A}_0}[\|\cdot\|_0]_+$. Hence,

$$x_\varepsilon \in \widetilde{\mathcal{A}_0}\big[\|\cdot\|_0\big]_+ \cap \big(-\widetilde{\mathcal{A}_0}\big[\|\cdot\|_0\big]_+\big) = \{0\}, \quad \text{so that } a \cdot (1+\varepsilon a)^{-1} = -a \cdot (1-\varepsilon a)^{-1}.$$

Furthermore, by (2),

$$a = \tau\text{-}\lim_{\varepsilon \downarrow 0} a \cdot (1+\varepsilon a)^{-1} = \tau\text{-}\lim_{\varepsilon \downarrow 0} (-a) \cdot (1-\varepsilon a)^{-1} = -a, \quad \text{so } a = 0.$$

(4) Note that $y - a_\varepsilon = (y-a) + (a - a_\varepsilon) \in \widetilde{\mathcal{A}_0}[\tau]_{q+}$, since (by (4) and (2) resp.) the elements $y-a$, $a-a_\varepsilon$ belong to $\widetilde{\mathcal{A}_0}[\tau]_{q+}$ and the latter set is a positive wedge according to Lemma 4.5(2). On the other hand,

$$a_\varepsilon = a \cdot (1+\varepsilon a)^{-1} = (1+\varepsilon a)^{-1} \cdot a = \varepsilon^{-1}\big(1 - (1+\varepsilon a)^{-1}\big) \in \widetilde{\mathcal{A}_0}\big[\|\cdot\|_0\big].$$

Thus, taking under consideration the assumption (T_5) we conclude that

$$y - a_\varepsilon \in \widetilde{\mathcal{A}_0}[\tau]_{q+} \cap \widetilde{\mathcal{A}_0}\big[\|\cdot\|_0\big] = \widetilde{\mathcal{A}_0}\big[\|\cdot\|_0\big]_+,$$

which clearly gives $\|a_\varepsilon\|_0 \leqq \|y\|_0$, for every $\varepsilon > 0$. Applying (T_4), we show that $a \in \widetilde{\mathcal{A}_0}[\|\cdot\|_0]_+$. □

Definition 4.8. Let $\mathcal{A}_0[\|\cdot\|_0]$ be a unital C^*-normed algebra, τ a locally convex topology on $\mathcal{A}_0$ satisfying the conditions (T_1)–(T_5) (for (T_4), (T_5) see the previous proposition). Then,

- a quasi $*$-subalgebra $\mathcal{A}$ of the locally convex quasi $*$-algebra $\widetilde{\mathcal{A}_0}[\tau]$ over $\mathcal{A}_0$ containing $\widetilde{\mathcal{A}_0}[\|\cdot\|_0]$ is said to be a *locally convex quasi C^*-normed algebra over $\mathcal{A}_0$*.
- A locally convex quasi C^*-normed algebra $\mathcal{A}$ over $\mathcal{A}_0$ is said to be *normal* if $a \cdot y \in \mathcal{A}$ whenever $a \in \mathcal{A}$ and $y \in \widetilde{\mathcal{A}_0}[\|\cdot\|_0]$ commute strongly.
- A locally convex quasi C^*-normed algebra $\mathcal{A}$ over $\mathcal{A}_0$ is called a *locally convex quasi C^*-algebra* if $\mathcal{A}_0[\|\cdot\|_0]$ is a C^*-algebra.

Note that the condition (T_3) in the present paper is weaker than the condition

(T_3) $\forall \lambda \in \Lambda,\ \exists \lambda' \in \Lambda$: $p_\lambda(xy) \leqq \|x\|_0 p_{\lambda'}(y)$, $\forall x, y \in \mathcal{A}_0$ with $xy = yx$

in [8]. Nevertheless, results for locally convex quasi C^*-algebras in [8] are valid in the present paper for the wider class of locally convex C^*-normed algebras. It follows, by the very definitions, that *a locally convex quasi C^*-algebra is a normal locally convex quasi C^*-normed algebra*. A variety of examples of locally convex quasi C^*-algebras are given in [8, Sections 3 and 4]. Examples of locally convex quasi C^*-normed algebras are presented in Sections 6 and 7.

An easy consequence of Definition 4.8 and Lemma 4.3 is the following

Lemma 4.9. *Let $\mathcal{A}_0[\|\cdot\|_0]$ and τ be as in Definition* 4.8. *Then the following hold*:

(1) *$\widetilde{\mathcal{A}_0}[\tau]$ is a normal locally convex quasi C^*-normed algebra over $\mathcal{A}_0$.*
(2) *Suppose $\mathcal{A}$ is a commutative locally convex quasi C^*-normed algebra over $\mathcal{A}_0$. Then $\mathcal{A} \cdot \widetilde{\mathcal{A}_0}[\|\cdot\|_0] \equiv$ linear span of $\{a \cdot y$: $a \in \mathcal{A},\ y \in \widetilde{\mathcal{A}_0}[\|\cdot\|_0]\}$ is a commutative locally convex quasi C^*-algebra over $\widetilde{\mathcal{A}_0}[\|\cdot\|_0]$ under the multiplication $a \cdot y$ ($a \in \mathcal{A}$, $y \in \widetilde{\mathcal{A}_0}[\|\cdot\|_0]$). In particular, if $\mathcal{A}$ is normal, then $\mathcal{A}$ is a commutative locally convex quasi C^*-algebra over $\widetilde{\mathcal{A}_0}[\|\cdot\|_0]$.*

5. Commutative locally convex quasi C^*-normed algebras

In this section, we discuss briefly some results on the structure of a commutative locally convex quasi C^*-normed algebra $\mathcal{A}[\tau]$ and on a functional calculus for its quasi-positive elements, that are similar to those in [8, Sections 5 and 6].

Let $\mathcal{A}[\tau]$ be a commutative locally convex quasi C^*-normed algebra over $\mathcal{A}_0$ (see Definition 4.8). Then,

$$\mathcal{A}_0\big[\|\cdot\|_0\big] \subset \widetilde{\mathcal{A}_0}\big[\|\cdot\|_0\big] \subset \mathcal{A}[\tau] \subset \mathcal{A}[\tau] \cdot \widetilde{\mathcal{A}_0}\big[\|\cdot\|_0\big] \subset \widetilde{\mathcal{A}_0}[\tau],$$

where $\mathcal{A}_0[\|\cdot\|_0]$ is a commutative unital C^*-normed algebra and $\mathcal{A}[\tau]\cdot\widetilde{\mathcal{A}_0}[\|\cdot\|_0]$ is a commutative locally convex quasi C^*-algebra over the unital C^*-algebra $\widetilde{\mathcal{A}_0}[\|\cdot\|_0]$ according to Lemma 4.9(2). Thus, using some results of Sections 5, 6 in [8] for the latter algebra we obtain information for the structure of $\mathcal{A}[\tau]$.

Let W be a compact Hausdorff space, $\mathbb{C}^* = \mathbb{C}\cup\{\infty\}$, and let $\mathfrak{F}(W)_+$ be a set of $\mathbb{C}^*$-valued positive continuous functions on W, which take the value ∞ on at most a nowhere dense subset W_0 of W. The set

$$\mathfrak{F}(W) \equiv \bigl\{fg_0 + h_0\colon\ f\in\mathfrak{F}(W)_+ \text{ and } g_0, h_0 \in \mathcal{C}(W)\bigr\},$$

where $\mathcal{C}(W)$ is the C^*-algebra of all continuous $\mathbb{C}$-valued functions on W, is called *the set of $\mathbb{C}^*$-valued continuous functions on W generated by the wedge $\mathfrak{F}(W)_+$ and the C^*-algebra $\mathcal{C}(W)$*. Using [8, Definition 5.6] and $\mathfrak{F}(W)$ we get the following theorem, which is an application of Theorem 5.8 of [8] for the commutative locally convex quasi C^*-algebra $\mathcal{A}[\tau]\cdot\widetilde{\mathcal{A}_0}[\|\cdot\|_0]$ over the unital commutative C^*-algebra $\widetilde{\mathcal{A}_0}[\|\cdot\|_0]$, with $\mathcal{A}[\tau]_{q+}\cdot\widetilde{\mathcal{A}_0}[\|\cdot\|_0]$, in the place of $\mathfrak{M}(\mathcal{A}_0, \mathcal{A}[\tau]_{q+})$.

Theorem 5.1. *There exists a map Φ from $\mathcal{A}[\tau]_{q+}\cdot\widetilde{\mathcal{A}_0}[\|\cdot\|_0]$ onto $\mathfrak{F}(W)$, where W is the compact Hausdorff space corresponding to the Gel'fand space of the unital commutative C^*-algebra $\widetilde{\mathcal{A}_0}[\|\cdot\|_0]$, such that*:

(i) $\Phi(\mathcal{A}[\tau]_{q+}) = \mathfrak{F}(W)_+$ *and* $\Phi(\lambda a + b) = \lambda\Phi(a) + \Phi(b)$, $\forall a, b \in \mathcal{A}[\tau]_{q+}$, $\lambda \geqq 0$;
(ii) Φ *is an isometric $*$-isomorphism from* $\widetilde{\mathcal{A}_0}[\|\cdot\|_0]$ *onto* $\mathcal{C}(W)$;
(iii) $\Phi(ax) = \Phi(a)\Phi(x)$, $\Phi((\lambda a + b)x) = (\lambda\Phi(a) + \Phi(b))\Phi(x)$ *and* $\Phi(a(x_1 + x_2)) = \Phi(a)(\Phi(x_1) + \Phi(x_2))$, $\forall a, b \in \mathcal{A}[\tau]_{q+}$, $x, x_1, x_2 \in \mathcal{A}_0$ *and* $\lambda \geqq 0$.

• Further we consider a functional calculus for the quasi-positive elements of the commutative locally convex quasi C^*-normed algebra $\mathcal{A}[\tau]$ over $\mathcal{A}_0$. For this, we must extend the multiplication of $\mathcal{A}[\tau]$.

Let $a, b \in \mathcal{A}[\tau]_{q+}$. Then (see also [8, Definition 6.1]), a is called *left multiplier* of b if there are nets $\{x_\alpha\}_{\alpha\in\Sigma}$, $\{y_\beta\}_{\beta\in\Sigma'}$ in $(\mathcal{A}_0)_+$ such that $x_\alpha \underset{\tau}{\longrightarrow} a$, $y_\beta \underset{\tau}{\longrightarrow} b$ and $x_\alpha y_\beta \underset{\tau}{\longrightarrow} c$, where the latter means that the double indexed net $\{x_\alpha y_\beta\}_{(\alpha,\beta)\in\Sigma\times\Sigma'}$ converges to $c \in \mathcal{A}[\tau]$. Then, we set

$$a\cdot b := c = \tau\text{-}\lim_{\alpha,\beta} x_\alpha y_\beta,$$

where the multiplication $a\cdot b$ is well defined, in the sense that it is independent of the choice of the nets $\{x_\alpha\}_{\alpha\in\Sigma}$, $\{y_\beta\}_{\beta\in\Sigma'}$, as follows from the proof of Lemma 6.2 in [8] applying arguments of the proof of Proposition 4.7. In the sequel, we simply denote $a\cdot b$ by ab. In analogy to Definition 6.3 of [8], if $x, y \in \widetilde{\mathcal{A}_0}[\|\cdot\|_0]$ and $a, b \in \mathcal{A}[\tau]_{q+}$ with a left multiplier of b, we may define the product of the elements ax and by as follows:

$$(ax)(by) := (ab)xy.$$

The spectrum of an element $a \in \mathcal{A}[\tau]_{q+}$, denoted by $\sigma_{\widetilde{\mathcal{A}_0}[\|\cdot\|_0]}(a)$, is defined as in Definition 6.4 of [8].

So using Theorem 5.1, it is shown (cf., for instance, Lemma 6.5 in [8]) that for every $a \in \mathcal{A}[\tau]_{q+}$, one has that $\sigma_{\widetilde{\mathcal{A}_0}[\|\cdot\|_0]}(a)$ *is a locally compact subset of $\mathbb{C}^*$ and $\sigma_{\widetilde{\mathcal{A}_0}[\|\cdot\|_0]}(a) \subset \mathbb{R}_+\cup\{\infty\}$.*

According to the above, and taking into account the comments after Lemma 6.5 in [8] with $\widetilde{\mathcal{A}_0}[\|\cdot\|_0]$ in the place of $\mathcal{A}_0$, the next Theorem 5.2 provides a generalization of [8, Theorem 6.6] in the setting of commutative locally convex quasi C^*-normed algebras. In particular, Theorem 5.2 supplies us with a functional calculus for the quasi-positive elements of the commutative locally convex quasi C^*-normed algebra $\mathcal{A}[\tau]$.

Theorem 5.2. *Let $a \in \mathcal{A}[\tau]_{q+}$. Let a^n be well defined for some $n \in \mathbb{N}$. Then there is a unique $*$-isomorphism $f \to f(a)$ from $\bigcup_{k=1}^n \mathcal{C}_k(\sigma_{\widetilde{\mathcal{A}_0}[\|\cdot\|_0]}(a))$ [8, p. 540, (6.3)] into $\mathcal{A}[\tau]\cdot\widetilde{\mathcal{A}_0}[\|\cdot\|_0]$ such that*:

(i) *If $u_0(\lambda) = 1$, with $u_0 \in \bigcup_{k=1}^n \mathcal{C}_k(\sigma_{\widetilde{\mathcal{A}_0}[\|\cdot\|_0]}(a))$ and $\lambda \in \sigma_{\widetilde{\mathcal{A}_0}[\|\cdot\|_0]}(a)$, then $u_0(a) = 1$.*
(ii) *If $u_1(\lambda) = \lambda$ with $u_1 \in \bigcup_{k=1}^n \mathcal{C}_k(\sigma_{\widetilde{\mathcal{A}_0}[\|\cdot\|_0]}(a))$ and $\lambda \in \sigma_{\widetilde{\mathcal{A}_0}[\|\cdot\|_0]}(a)$, then $u_1(a) = a$.*
(iii) $(\lambda_1 f_1 + f_2)(a) = \lambda_1 f_1(a) + f_2(a)$, $\forall f_1, f_2 \in \mathcal{C}_k(\sigma_{\widetilde{\mathcal{A}_0}[\|\cdot\|_0]}(a))$ *and* $\lambda_1 \in \mathbb{C}$; $(f_1f_2)(a) = f_1(a)f_2(a)$, $\forall f_j \in \mathcal{C}_{k_j}(\sigma_{\widetilde{\mathcal{A}_0}[\|\cdot\|_0]}(a))$, $j = 1, 2$, *with* $k_1 + k_2 \leqq n$.
(iv) *Denoting with $\mathcal{C}_b(\sigma_{\widetilde{\mathcal{A}_0}[\|\cdot\|_0]}(a))$ the C^*-algebra of all bounded and continuous functions on $\sigma_{\widetilde{\mathcal{A}_0}[\|\cdot\|_0]}(a)$, the map $f \to f(a)$ restricted to the latter C^*-algebra is an isometric $*$-isomorphism, with values on on the closed $*$-subalgebra of $\widetilde{\mathcal{A}_0}[\|\cdot\|_0]$ generated by 1 and $(1+a)^{-1}$.*

Applying Theorem 5.2 and Proposition 4.7 in the proof of [8, Corollary 6.7] we get the following

Corollary 5.3. *Let $a \in \mathcal{A}[\tau]_{q+}$ and $n \in \mathbb{N}$. Then, there exists unique b in $\mathcal{A}[\tau]_{q+}\cdot\widetilde{\mathcal{A}_0}[\|\cdot\|_0]$ such that $a = b^n$. The unique element b is called quasi nth-root of a and we write $b = a^{\frac{1}{n}}$.*

6. Structure of noncommutative locally convex quasi C^*-normed algebras

Using the notation of [8, Section 4] (see also [3]), let $\mathcal{H}$ be a Hilbert space, $\mathcal{D}$ a dense subspace of $\mathcal{H}$ and $\mathcal{M}_0[\|\cdot\|_0]$ a unital C^*-normed algebra on $\mathcal{H}$, such that

$$\mathcal{M}_0\mathcal{D} \subset \mathcal{D}, \quad \text{but} \quad \widetilde{\mathcal{M}}_0\big[\|\cdot\|_0\big]\mathcal{D} \not\subset \mathcal{D}.$$

Then, the restriction $\mathcal{M}_0 \restriction \mathcal{D}$ of $\mathcal{M}_0$ to $\mathcal{D}$ is an O^*-algebra on $\mathcal{D}$, so that an element X of $\mathcal{M}_0$ may be regarded as an element $X \restriction \mathcal{D}$ of $\mathcal{M}_0\lceil\mathcal{D}$. Moreover, let

$$\mathcal{M}_0 \subset \mathcal{M} \subset \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H}),$$

where $\mathcal{M}$ is an O^*-vector space on $\mathcal{D}$, that is, a $*$-invariant subspace of $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$. Denote by $\mathcal{B}(\mathcal{M})$ the set of all bounded subsets of $\mathcal{D}[t_\mathcal{M}]$ ($t_\mathcal{M}$ is the graph topology on $\mathcal{M}$; see [14, p. 9]) and by $\mathcal{B}_f(\mathcal{D})$ the set of all finite subsets of $\mathcal{D}$. Then $\mathcal{B}_f(\mathcal{D}) \subset \mathcal{B}(\mathcal{M})$ and both of them are admissible in the sense of [8, p. 522].

We recall the topologies $\tau_{s^*}, \tau_*^u(\mathcal{B}), \tau_*^u(\mathcal{M})$ defined in [8, pp. 522–523]. More precisely, for an arbitrary admissible subset $\mathcal{B}$ of $\mathcal{B}(\mathcal{M})$, and any $\mathfrak{M} \in \mathcal{B}$ consider the following seminorm:

$$p_\dagger^{\mathfrak{M}}(X) := \sup_{\xi \in \mathfrak{M}} \big\{\|X\xi\| + \big\|X^\dagger\xi\big\|\big\}, \quad X \in \mathcal{M}.$$

We call the corresponding locally convex topology on $\mathcal{M}$ induced by the preceding family of seminorms, *strongly* $\mathcal{B}$-uniform topology* and denote it by $\tau_*^u(\mathcal{B})$. In particular, the strongly* $\mathcal{B}(\mathcal{M})$-uniform topology will be simply called *strongly* $\mathcal{M}$-uniform topology* and will be denoted by $\tau_*^u(\mathcal{M})$. In Schmüdgen's book [17], this topology is called *bounded topology*. The strongly* $\mathcal{B}_f(\mathcal{D})$-uniform topology is called *strong*-topology on* $\mathcal{M}$, denoted by τ_{s^*}. All three topologies are related in the following way:

$$\tau_{s^*} \preccurlyeq \tau_*^u(\mathcal{B}) \preccurlyeq \tau_*^u(\mathcal{M}).$$

Then, one gets that

$$\mathcal{M}_0\big[\|\cdot\|_0\big] \subset \widetilde{\mathcal{M}}_0\big[\|\cdot\|_0\big] \subset \widetilde{\mathcal{M}}_0\big[\tau_*^u\big] \subset \widetilde{\mathcal{M}}_0[\tau_{s^*}] \subset \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H}). \tag{6.1}$$

In this regard, we have now the following

Proposition 6.1. *Let $\mathcal{M}_0[\|\cdot\|_0]$, $\mathcal{M}$ be as before. Let $\mathcal{B}$ be any admissible subset of $\mathcal{B}(\mathcal{M})$. Then $\widetilde{\mathcal{M}}_0[\tau_*^u(\mathcal{B})]$ is a locally convex quasi C^*-normed algebra over $\mathcal{M}_0$, which is contained in $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$. In particular, $\widetilde{\mathcal{M}}_0[\tau_{s^*}]$ is a locally convex quasi C^*-normed algebra over $\mathcal{M}_0$. Furthermore, if $A \in \widetilde{\mathcal{M}}_0[\tau_*^u(\mathcal{B})]$ and $Y \in \widetilde{\mathcal{M}}_0[\|\cdot\|_0]$ commute strongly, then $A \,\square\, Y$ is well defined and*

$$A \,\square\, Y = A \cdot Y = Y \cdot A = Y \,\square\, A.$$

Proof. It is easily checked that $\widetilde{\mathcal{M}}_0[\tau_*^u(\mathcal{B})]$ and $\widetilde{\mathcal{M}}_0[\tau_{s^*}]$ are locally convex quasi C^*-normed algebras over $\mathcal{M}_0$. Suppose now that $A \in \widetilde{\mathcal{M}}_0[\tau_*^u(\mathcal{B})]$ and $Y \in \widetilde{\mathcal{M}}_0[\|\cdot\|_0]$ commute strongly. Then, there is a net $\{X_\alpha\}_{\alpha\in\Sigma}$ in $\mathcal{M}_0$ such that $X_\alpha Y = Y X_\alpha$, for all $\alpha \in \Sigma$ and $A = \tau_*^u(\mathcal{B}) - \lim_\alpha X_\alpha$. Since

$$\big(A^\dagger\xi\big|Y\eta\big) = \lim_\alpha\big(X_\alpha^\dagger\xi\big|Y\eta\big) = \lim_\alpha(\xi|X_\alpha Y\eta) = \lim_\alpha(\xi|Y X_\alpha\eta) = (\xi|YA\eta)$$

for all $\xi, \eta \in \mathcal{D}$, it follows that $A \,\square\, Y$ is well defined and $A \,\square\, Y = YA$. Furthermore, since

$$A \cdot Y = \tau_*^u(\mathcal{B}) - \lim_\alpha X_\alpha Y = \tau_{s^*} - \lim_\alpha X_\alpha Y,$$

we have

$$(A \cdot Y)\xi = \lim_\alpha X_\alpha Y\xi = \lim_\alpha Y X_\alpha\xi = YA\xi = (A \,\square\, Y)\xi$$

for each $\xi \in \mathcal{D}$. Hence, $A \cdot Y = A \,\square\, Y$. $\square$

Proposition 6.2. *$\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})[\tau_{s^*}]$ is a locally convex quasi C^*-normed algebra over $\mathcal{L}^\dagger(\mathcal{D})_b \equiv \{X \in \mathcal{L}^\dagger(\mathcal{D})\colon \overline{X} \in \mathcal{B}(\mathcal{H})\}$.*

Proof. Indeed, as shown in [3, Section 2.5], $\mathcal{L}^\dagger(\mathcal{D})_b$, is a C^*-normed algebra which is τ_{s^*} dense in $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$. Hence, $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ is a locally convex quasi C^*-normed algebra over $\mathcal{L}^\dagger(\mathcal{D})_b$. $\square$

Remark 6.3. The following questions arise naturally:

(1) What is exactly the C^*-algebra $\mathcal{L}^\dagger(\mathcal{D})_b^\sim[\|\cdot\|_0]$?
Under what conditions may one have the equality $\mathcal{L}^\dagger(\mathcal{D})_b^\sim[\|\cdot\|_0] = \mathcal{B}(\mathcal{H})$?
(2) Is $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ a locally convex quasi C^*-algebra under the strong* uniform topology τ_*^u?
More precisely, does the equality $\mathcal{L}^\dagger(\mathcal{D})_b^\sim[\tau_*^u] = \mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ hold?

We expect the answer to these questions to depend on the properties of the topology $t_\dagger \equiv t_{\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})}$ given on $\mathcal{D}$ and we conjecture positive answers in the case where $\mathcal{D} \equiv \mathcal{D}^\infty(T)$, with T a positive self-adjoint operator in a Hilbert space $\mathcal{H}$, and $\|\cdot\|_0$ the operator norm in $\mathcal{B}(\mathcal{H})$. We leave these questions open.

• In the rest of this section we consider conditions under which a locally convex quasi C^*-normed algebra is continuously embedded in a locally convex quasi C^*-normed algebra of operators.

So let $\mathcal{A}[\tau]$ be a locally convex quasi C^*-normed algebra over $\mathcal{A}_0$ and $\mathcal{D}$ a dense subspace in a Hilbert space $\mathcal{H}$. Let $\pi : \mathcal{A} \to \mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ be a $*$-representation. Then we have the following

Lemma 6.4. *Let $\mathcal{A}[\tau]$ be a locally convex quasi C^*-normed algebra over $\mathcal{A}_0$ and $\pi : \mathcal{A} \to \mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ a $(\tau, \tau_*^u(\mathcal{B}))$-continuous $*$-representation of $\mathcal{A}$. Then,*

(1) *π is a $*$-representation of the C^*-algebra $\widetilde{\mathcal{A}_0}[\|\cdot\|_0]$;*
(2) *$\pi(\mathcal{A})[\tau_*^u(\mathcal{B})]$ resp. $\pi(\mathcal{A})[\tau_{s^*}]$ are locally convex quasi C^*-normed algebras over $\pi(\mathcal{A}_0)$.*

Proof. (1) Since $\mathcal{A}_0 \subset \widetilde{\mathcal{A}_0}[\|\cdot\|_0] \subset \mathcal{A}$ and π is a $*$-representation of $\mathcal{A}$, it follows that

$$\pi(ay) = \pi(a) \,\square\, \pi(y), \quad \forall a \in \widetilde{\mathcal{A}_0}\big[\|\cdot\|_0\big],\ \forall y \in \mathcal{A}_0. \tag{6.2}$$

Now we show that

$$\pi(ab) = \pi(a) \,\square\, \pi(b), \quad \forall a, b \in \widetilde{\mathcal{A}_0}\big[\|\cdot\|_0\big]. \tag{6.3}$$

Indeed, let a, b be arbitrary elements of $\widetilde{\mathcal{A}_0}[\|\cdot\|_0]$. Then, there exists a sequence $\{y_n\}$ in $\mathcal{A}_0$ such that $b = \|\cdot\|_0 - \lim_{n\to\infty} y_n$. Hence, $ab = \|\cdot\|_0 - \lim_{n\to\infty} a y_n$.

Moreover, it is easily seen that π is also (τ, τ_{s^*})-continuous and so, by (6.2),

$$\begin{aligned}\big\langle \pi(b)\xi \,\big|\, \pi\big(a^*\big)\eta\big\rangle &= \lim_{n\to\infty}\big\langle \pi(y_n)\xi \,\big|\, \pi\big(a^*\big)\eta\big\rangle = \lim_{n\to\infty}\big\langle \pi(a)\,\square\,\pi(y_n)\xi \,\big|\, \eta\big\rangle \\ &= \lim_{n\to\infty}\big\langle \pi(a y_n)\xi \,\big|\, \eta\big\rangle = \big\langle \pi(ab)\xi \,\big|\, \eta\big\rangle,\end{aligned}$$

for every $\xi, \eta \in \mathcal{D}$. Thus, (6.3) holds.

For any $\xi \in \mathcal{D}$, we put

$$f(a) = \big\langle \pi(a)\xi \,\big|\, \xi\big\rangle, \quad a \in \widetilde{\mathcal{A}_0}\big[\|\cdot\|_0\big].$$

Then, by (6.3), f is a positive linear functional on the unital C^*-algebra $\widetilde{\mathcal{A}_0}[\|\cdot\|_0]$. Hence, we have

$$\big\|\pi(a)\xi\big\|^2 = f(a^*a) \leqslant f(1)\|a\|_0^2 = \|\xi\|^2\|a\|_0^2$$

for all $a \in \widetilde{\mathcal{A}_0}[\|\cdot\|_0]$, which implies that π is bounded. This completes the proof of (1).

(2) $\pi(\mathcal{A})$ is a quasi $*$-subalgebra of the locally convex quasi $*$-algebras $\widetilde{\pi(\mathcal{A})}[\tau_*^u(\mathcal{B})]$ and $\widetilde{\pi(\mathcal{A})}[\tau_{s^*}]$ over $\pi(\mathcal{A}_0)$. Furthermore, by (1), $\pi(\widetilde{\mathcal{A}_0}[\|\cdot\|_0])$ is a C^*-algebra and

$$\widetilde{\pi(\mathcal{A}_0)}\big[\|\cdot\|_0\big] = \pi\big(\widetilde{\mathcal{A}_0}\big[\|\cdot\|_0\big]\big) \subset \pi(\mathcal{A}). \qquad \square$$

Remark 6.5. Let $\mathcal{A}[\tau]$ be a locally convex quasi C^*-normed algebra over $\mathcal{A}_0$, and π a $(\tau, \tau_*^u(\mathcal{B}))$-continuous $*$-representation of $\mathcal{A}$, where $\mathcal{B}$ is an admissible subset in $\mathcal{B}(\pi(\mathcal{A}))$. Let $a \in \mathcal{A}$ be strongly commuting with $y \in \widetilde{\mathcal{A}_0}[\|\cdot\|_0]$. Then $\pi(a)$ commutes strongly with $\pi(y)$. The converse does not necessarily hold. So even if $\mathcal{A}[\tau]$ is normal, the locally convex quasi C^*-normed algebra $\pi(\mathcal{A})$ over $\pi(\mathcal{A}_0)$ is not necessarily normal.

We are going now to discuss the faithfulness of a (τ, τ_{s^*})-continuous $*$-representation of $\mathcal{A}$. For this, we need some facts on sesquilinear forms, for which the reader is referred to [8, p. 544]. We only recall that if

$$\mathcal{S}(\mathcal{A}_0) := \{\tau\text{-continuous positive invariant sesquilinear forms } \varphi \text{ on } \mathcal{A}_0 \times \mathcal{A}_0\},$$

we say that the set $\mathcal{S}(\mathcal{A}_0)$ is *sufficient*, whenever

$$a \in \mathcal{A} \text{ with } \tilde{\varphi}(a,a) = 0,\ \forall \varphi \in \mathcal{S}(\mathcal{A}_0), \quad \text{implies} \quad a = 0,$$

where $\tilde{\varphi}$ is the extension of φ to a τ-continuous positive invariant sesquilinear form on $\mathcal{A} \times \mathcal{A}$.

From the next results, Theorem 6.6 and Corollary 6.7 can be regarded as generalizations of the analogues of the Gel'fand–Naimark theorem, in the case of locally convex quasi C^*-algebras proved in [8, Section 7]. Theorem 6.6 is proved in the same way as [8, Theorem 7.3].

Theorem 6.6. *Let $\mathcal{A}[\tau]$ be a locally convex quasi C^*-normed algebra over a unital C^*-normed algebra $\mathcal{A}_0$. The following statements are equivalent*:

(i) *There exists a faithful (τ, τ_{s^*})-continuous $*$-representation of $\mathcal{A}$.*
(ii) *The set $S(\mathcal{A}_0)$ is sufficient.*

Corollary 6.7. *Suppose $S(\mathcal{A}_0)$ is sufficient. Then, the locally convex quasi C^*-normed algebra $\mathcal{A}[\tau]$ over $\mathcal{A}_0$ is continuously embedded in a locally convex quasi C^*-normed algebra of operators.*

We end this section with the study of a functional calculus for the commutatively quasi-positive elements (see Definition 4.4) of $\mathcal{A}[\tau]$.

Let $\mathcal{A}[\tau]$ be a locally convex quasi C^*-normed algebra over a unital C^*-normed algebra $\mathcal{A}_0[\|\cdot\|_0]$. If $a \in \mathcal{A}[\tau]_{cq+}$, then by Proposition 4.7(1), the element $(1+a)^{-1}$ exists and belongs to $\mathcal{U}(\widetilde{\mathcal{A}_0}[\|\cdot\|_0])$. Denote by $C^*(a)$ the maximal commutative C^*-subalgebra of the C^*-algebra $\widetilde{\mathcal{A}_0}[\|\cdot\|_0]$ containing the elements 1 and $(1+a)^{-1}$.

Lemma 6.8. *$\widetilde{C^*(a)}[\tau]$ is a commutative unital locally convex quasi C^*-algebra over $C^*(a)$ and $a \in \widetilde{C^*(a)}[\tau]_{q+}$.*

Proof. Since $C^*(a)$ is a unital C^*-algebra, we have only to check the properties (T_1)–(T_5). We show (T_1); the rest of them, as well as the fact that $a \in \widetilde{C^*(a)}[\tau]_{q+}$ are proved by the same way as in [8, Proposition 7.6 and Corollary 7.7]. From the condition (T_3) for $\mathcal{A}_0[\tau]$, we have that for all $\lambda \in \Lambda$, there exist $\lambda' \in \Lambda$ and $\gamma_\lambda > 0$ such that

$$p_\lambda(xy) \leqq \gamma_\lambda \|x\|_0 p_{\lambda'}(y), \quad \forall x, y \in C^*(a).$$

So, $C^*(a)[\tau]$ is a locally convex $*$-algebra with separately continuous multiplication. □

By Lemma 6.8 and Theorem 5.2 we can now obtain a functional calculus for the commutatively quasi-positive elements of the noncommutative locally convex quasi C^*-normed algebra $\mathcal{A}[\tau]$ (see also [8, Theorem 7.8, Corollary 7.9]).

Theorem 6.9. *Let $\mathcal{A}[\tau]$ be an arbitrary locally convex quasi C^*-normed algebra over a unital C^*-normed algebra $\mathcal{A}_0$ and $a \in \mathcal{A}[\tau]_{cq+}$. Suppose that a^n is well defined for some $n \in \mathbb{N}$. Then, there is a unique $*$-isomorphism $f \to f(a)$ from $\bigcup_{k=1}^n \mathcal{C}_k(\sigma_{C^*(a)}(a))$ into $\mathcal{A}[\tau] \cdot C^*(a)$ such that*:

(i) *If $u_0(\lambda) = 1$, with $u_0 \in \bigcup_{k=1}^n \mathcal{C}_k(\sigma_{C^*(a)}(a))$ and $\lambda \in \sigma_{C^*(a)}(a)$, then $u_0(a) = 1$.*
(ii) *If $u_1(\lambda) = \lambda$ with $u_1 \in \bigcup_{k=1}^n \mathcal{C}_k(\sigma_{C^*(a)}(a))$ and $\lambda \in \sigma_{C^*(a)}(a)$, then $u_1(a) = a$.*
(iii) *$(\lambda_1 f_1 + f_2)(a) = \lambda_1 f_1(a) + f_2(a)$, $\forall f_1, f_2 \in \bigcup_{k=1}^n \mathcal{C}_k(\sigma_{C^*(a)}(a))$ and $\lambda_1 \in \mathbb{C}$;*
$(f_1 f_2)(a) = f_1(a) f_2(a)$, $\forall f_j \in \mathcal{C}_{k_j}(\sigma_{C^*(a)}(a))$, $j = 1, 2$, with $k_1 + k_2 \leqq n$.
(iv) *The map $f \to f(a)$ restricted to $\mathcal{C}_b(\sigma_{C^*(a)}(a))$ is an isometric $*$-isomorphism of the C^*-algebra $\mathcal{C}_b(\sigma_{C^*(a)}(a))$ on the C^*-algebra $C^*(a)$.*

Using Theorem 6.9 and applying Corollary 5.3 for the commutative unital locally convex quasi C^*-algebra $\widetilde{C^*(a)}[\tau]$, we conclude the following

Corollary 6.10. *Let $\mathcal{A}[\tau]$ and $\mathcal{A}_0$ be as in Theorem* 6.9. *If $a \in \mathcal{A}[\tau]_{cq+}$ and $n \in \mathbb{N}$, there is a unique element $b \in \mathcal{A}[\tau]_{cq+} \cdot C^*(a)$, such that $a = b^n$. The element b is called commutatively quasi nth-root of a and is denoted by $a^{\frac{1}{n}}$.*

7. Applications

Locally convex quasi C^*-normed algebras arise, as we have discussed throughout this paper, as completions of a C^*-normed algebra with respect to a locally convex topology which satisfies a series of requirements. Completions of this sort actually occur in quantum statistics.

In statistical physics, in fact, one has to deal with systems consisting of a very large number of particles, so large that one usually considers this number to be *infinite*. One begins by considering systems living in a *local region* V (V is, for instance, a bounded region of $\mathbb{R}^3$ for gases or liquids, or a finite subset of the lattice $\mathbb{Z}^3$ for crystals) and requires that the set of local regions being directed, i.e., if V_1, V_2 are two local regions, then there exists a third local region V_3 containing both V_1 and V_2. The observables on a given bounded region V are supposed to constitute a C^*-algebra $\mathcal{A}_V$, where all $\mathcal{A}_V$'s have the same norm, and so the $*$-algebra $\mathcal{A}_0$ of local observables, $\mathcal{A}_0 = \bigcup_V \mathcal{A}_V$, is a C^*-normed algebra. Its uniform completion is, obviously, a C^*-algebra (more precisely, a quasi-local C^*-algebra) that in the original algebraic approach was taken as the observable algebra of the system. As a matter of fact, this C^*-algebraic formulation reveals to be insufficient, since for many models there is no way of including in this framework the thermodynamical limit of the local Heisenberg dynamics [6]. Then a possible procedure to follow in order to circumvent this difficulty is to define in $\mathcal{A}_0$ a new locally convex topology, τ, called, for obvious reasons, *physical topology*, in such a way that the dynamics in the thermodynamical limit belongs to the completion of $\mathcal{A}_0$ with respect to τ. For that purpose, a class of topologies for the $*$-algebra $\mathcal{A}_0$ of local observables of a quantum system was proposed by Lassner in [15,16]. We will sketch in what follows this construction. Let $\mathcal{A}_0$ be a C^*-normed algebra to be understood as the algebra of local observables described above; thus we will suppose that $\mathcal{A}_0 = \bigcup_{\alpha\in\Sigma} \mathcal{A}_\alpha$, where $\{\mathcal{A}_\alpha\}_{\alpha\in\Sigma}$ is a family of C^*-algebras labeled by a directed set of indices Σ. Assume that, for every $\alpha \in \Sigma$, π_α is a $*$-representation of $\mathcal{A}_0$ on a dense subspace $\mathcal{D}_\alpha$ of a Hilbert space $\mathcal{H}_\alpha$, i.e. each π_α is a $*$-homomorphism of $\mathcal{A}_0$ into the partial O^*-algebra $\mathcal{L}^\dagger(\mathcal{D}_\alpha, \mathcal{H}_\alpha)$ endowed, for instance, with the topology $\tau_*^u(\mathcal{L}^\dagger(\mathcal{D}_\alpha, \mathcal{H}_\alpha))$. We shall assume that $\pi_\alpha(x)\mathcal{D}_\alpha \subset \mathcal{D}_\alpha$, for every $\alpha \in \Sigma$ and $x \in \mathcal{A}_0$. Since every $\mathcal{A}_\alpha$ is a C^*-algebra, each π_α is a bounded and continuous $*$-representation, i.e. $\overline{\pi_\alpha(x)} \in \mathcal{B}(\mathcal{H}_\alpha)$, $\|\overline{\pi_\alpha(x)}\| \leqslant \|x\|_0$, for every $x \in \mathcal{A}_0$. So each π_α can be extended to the C^*-algebra $\widetilde{\mathcal{A}_0}[\|\cdot\|_0]$ (we denote the extension by the same symbol). The family is supposed to be *faithful*, in the sense that if $x \in \widetilde{\mathcal{A}_0}[\|\cdot\|_0]$, $x \neq 0$, then there exists $\alpha \in \Sigma$ such that $\pi_\alpha(x) \neq 0$. Let us further suppose that $\mathcal{D}_\alpha = \mathcal{D}^\infty(M_\alpha) = \bigcap_{n\in\mathbb{N}} \mathcal{D}(M_\alpha^n)$, where M_α is a self-adjoint operator. Without loss of generality we may assume that $M_\alpha \geqslant I_\alpha$, with I_α the identity operator in $\mathcal{B}(\mathcal{H}_\alpha)$. Under these assumptions, a *physical* topology τ can be defined on $\mathcal{A}_0$ by the family of seminorms

$$p_\alpha^f(x) = \|\pi_\alpha(x) f(M_\alpha)\| + \|\pi_\alpha(x^*) f(M_\alpha)\|, \quad x \in \mathcal{A}_0,$$

where $\alpha \in \Sigma$ and f runs over the set $\mathcal{F}$ of all positive, bounded and continuous functions $f(t)$ on $\mathbb{R}^+$ such that

$$\sup_{t\in\mathbb{R}^+} t^k f(t) < \infty, \quad \forall k = 0, 1, 2, \ldots.$$

Then, $\mathcal{A}_0[\tau]$ is a locally convex $*$-algebra with separately continuous multiplication (i.e. (T_1) holds). In order to prove that $\widetilde{\mathcal{A}_0}[\tau]$ is a locally convex quasi C^*-normed algebra, we need to show that (T_2)–(T_5) also hold. For (T_2), we have that, for every $\alpha \in \Sigma$,

$$p_\alpha^f(x) = \|\pi_\alpha(x) f(M_\alpha)\| + \|\pi_\alpha(x^*) f(M_\alpha)\| \leqslant 2\|f(M_\alpha)\|\|\pi_\alpha(x)\| \leqslant 2\|f(M_\alpha)\|\|x\|_0, \quad x \in \mathcal{A}_0.$$

The compatibility of τ with $\|\cdot\|_0$ follows easily from the closedness of the operators $f(M_\alpha)^{-1}$ and the faithfulness of the family $\{\pi_\alpha\}_{\alpha\in\Sigma}$ of *-representations.

The condition (R) does not hold, in general, but, on the other hand, if $x, y \in \mathcal{A}_0$ with $xy = yx$, we have

$$\begin{aligned}
p_\alpha^f(xy) &= \|\pi_\alpha(xy) f(M_\alpha)\| + \|\pi_\alpha\big((xy)^*\big) f(M_\alpha)\| \\
&= \|\pi_\alpha(xy) f(M_\alpha)\| + \|\pi_\alpha\big(x^* y^*\big) f(M_\alpha)\| \\
&\leqslant \|\pi_\alpha(x)\|\big(\|\pi_\alpha(y) f(M_\alpha)\| + \|\pi_\alpha\big(y^*\big) f(M_\alpha)\|\big) \\
&= \|\pi_\alpha(x)\| p_\alpha^f(y) \leqslant \|x\|_0 p_\alpha^f(y).
\end{aligned}$$

Hence (T_3) holds. As for (T_4), we begin with noticing that for every $\alpha \in \Sigma$, $\pi_\alpha(\mathcal{A}_0)$ is an O^*-algebra of bounded operators in $\mathcal{D}_\alpha$. Hence, its closure in $\mathcal{L}^\dagger(\mathcal{D}_\alpha, \mathcal{H}_\alpha)[\tau_*^u(\mathcal{L}^\dagger(\mathcal{D}_\alpha, \mathcal{H}_\alpha))]$ is a locally convex C^*-normed algebra of operators, by Proposition 6.1. Moreover, every π_α can be extended by continuity to $\widetilde{\mathcal{A}_0}[\|\cdot\|_0]$. The extension, that we denote by the same symbol, takes values in $\mathcal{L}^\dagger(\mathcal{D}_\alpha, \mathcal{H}_\alpha)[\tau_*^u(\mathcal{L}^\dagger(\mathcal{D}_\alpha, \mathcal{H}_\alpha))]$, since this space is complete. Now, if $\{x_\lambda\}$ is a net in $\mathcal{U}(\widetilde{\mathcal{A}_0}[\|\cdot\|_0])_+$ τ-converging to $x \in \widetilde{\mathcal{A}_0}[\|\cdot\|_0]$, then $x = x^*$ and $\pi_\alpha(x_\lambda) \to \pi_\alpha(x)$ in $\mathcal{L}^\dagger(\mathcal{D}_\alpha, \mathcal{H}_\alpha)[\tau_*^u(\mathcal{L}^\dagger(\mathcal{D}_\alpha, \mathcal{H}_\alpha))]$, for every $\alpha \in \Sigma$. Thus $\pi_\alpha(x) \geqslant 0$ and $\|\pi_\alpha(x)\| \leqslant 1$, for every $\alpha \in \Sigma$, since the same is true for every x_λ. By constructing a faithful representation π by the direct sum of π_α's, one easily realizes that $x \geqslant 0$ and $\|x\|_0 \leqslant 1$. The inclusion $\widetilde{\mathcal{A}_0}[\tau]_{q+} \cap \widetilde{\mathcal{A}_0}[\|\cdot\|_0] \subset \widetilde{\mathcal{A}_0}[\|\cdot\|_0]_+$ in condition (T_5) can be proved in similar fashion. The converse inclusion comes from Lemma 4.5. Thus, condition (T_5) holds.

Then we conclude the following:

Statement 7.1. $\mathcal{A} \equiv \widetilde{\mathcal{A}_0}[\tau]$ is a locally convex quasi C^*-normed algebra, which can be understood as the quasi $*$-algebra of the observables of the physical system.

A more concrete realization of the situation discussed above is obtained for the so-called BCS model. Let V be a finite region of a d-dimensional lattice Λ and $|V|$ the number of points in V. The local C^*-algebra $\mathcal{A}_V$ is generated by the Pauli operators $\vec{\sigma}_p = (\sigma_p^1, \sigma_p^2, \sigma_p^3)$ and by the unit 2×2 matrix e_p at every point $p \in V$. The $\vec{\sigma}_p$'s are copies of the Pauli matrices localized in p.

If $V \subset V'$ and $A_V \in \mathcal{A}_V$, then $A_V \to A_{V'} = A_V \otimes (\bigotimes_{p\in V'\setminus V} e_p)$ defines the natural imbedding of $\mathcal{A}_V$ into $\mathcal{A}_{V'}$.

Let $\vec{n} = (n_1, n_2, n_3)$ be a unit vector in $\mathbb{R}^3$, and put $(\vec{\sigma} \cdot \vec{n}) = n_1\sigma^1 + n_2\sigma^2 + n_3\sigma^3$. Then, denoting as $Sp(\vec{\sigma} \cdot \vec{n})$ the spectrum of $\vec{\sigma} \cdot \vec{n}$, we have $Sp(\vec{\sigma} \cdot \vec{n}) = \{1, -1\}$. Let $|\vec{n}\rangle \in \mathbb{C}^2$ be a unit eigenvector associated with 1.

Let now denote by $\mathbf{n} := \{\vec{n}_p\}_{p\in\Lambda}$ an infinite sequence of unit vectors in $\mathbb{R}^3$ and $|\mathbf{n}\rangle = \bigotimes_p |\vec{n}_p\rangle$ the corresponding unit vector in the infinite tensor product $\mathcal{H}_\infty = \bigotimes_p \mathbb{C}^2_p$. We put $\mathcal{A}_0 = \bigcup_V \mathcal{A}_V$ and $\mathcal{D}^0_{\mathbf{n}} = \mathcal{A}_0|\mathbf{n}\rangle$ and we denote the closure of $\mathcal{D}^0_{\mathbf{n}}$ in $\mathcal{H}_\infty$ by $\mathcal{H}_{\mathbf{n}}$. As we saw above, to any sequence $\mathbf{n}$ of three-vectors there corresponds a state $|\mathbf{n}\rangle$ of the system. Such a state defines a realization $\pi_{\mathbf{n}}$ of $\mathcal{A}_0$ in the Hilbert space $\mathcal{H}_{\mathbf{n}}$. This representation is faithful, since the norm completion $\mathcal{A}_S$ of $\mathcal{A}_0$ is a simple C^*-algebra. A special basis for $\mathcal{H}_{\mathbf{n}}$ is obtained from the *ground* state $|\mathbf{n}\rangle$ by *flipping* a finite number of spins using the following strategy:

Let $\vec{n}$ be a unit vector in $\mathbb{R}^3$, as above, and $|\vec{n}\rangle$ the corresponding vector of $\mathbb{C}^2$. Let us choose two other unit vectors $\vec{n}^1, \vec{n}^2$ so that $(\vec{n}, \vec{n}^1, \vec{n}^2)$ form an orthonormal basis of $\mathbb{R}^3$. We put $\vec{n}_\pm = \frac{1}{2}(\vec{n}^1 \pm i\vec{n}^2)$ and define $|m, \vec{n}\rangle := (\vec{\sigma} \cdot \vec{n}_-)^m|\vec{n}\rangle$ $(m = 0, 1)$. Then we have

$$(\vec{\sigma} \cdot \vec{n})|m, \vec{n}\rangle = (-1)^m|m, \vec{n}\rangle \quad (m = 0, 1).$$

Thus, the set $\{|\mathbf{m}, \mathbf{n}\rangle = \bigotimes_p |m_p, \vec{n}_p\rangle;\ m_p = 0, 1,\ \sum_p m_p < \infty\}$ forms an orthonormal basis in $\mathcal{H}_{\mathbf{n}}$.

In this space we define the unbounded self-adjoint operator $M_{\mathbf{n}}$ by

$$M_{\mathbf{n}}|\mathbf{m}, \mathbf{n}\rangle = \left(1 + \sum_p m_p\right)|\mathbf{m}, \mathbf{n}\rangle. \tag{7.1}$$

$M_{\mathbf{n}}$ counts the number of the flipped spins in $|\mathbf{m}, \mathbf{n}\rangle$ with respect to the ground state $|\mathbf{n}\rangle$. Now we put

$$\mathcal{D}_{\mathbf{n}} = \bigcap_k \mathcal{D}\left(M^k_{\mathbf{n}}\right).$$

The representation $\pi_{\mathbf{n}}$ is defined on the basis vectors $\{|\mathbf{m}, \mathbf{n}\rangle\}$ by

$$\pi_{\mathbf{n}}\left(\sigma^i_p\right)|\mathbf{m}, \mathbf{n}\rangle = \sigma^i_p|m_p, \vec{n}_p\rangle \otimes \left(\prod_{p'\neq p} \otimes|m_{p'}, \vec{n}_{p'}\rangle\right) \quad (i = 1, 2, 3).$$

This definition is then extended in obvious way to the whole space $\mathcal{H}_{\mathbf{n}}$. It turns out that $\pi_{\mathbf{n}}$ is a *bounded* representation of $\mathcal{A}_0$ in the Hilbert space $\mathcal{H}_{\mathbf{n}}$. For more details we refer to [20,11]. Hence, the procedure outlined above applies, showing that a natural framework for discussing the BCS model is, indeed, provided by locally convex quasi C^*-normed algebras considered in this paper. We argue that an analysis similar to that of [11] can be carried out also in the present context, so that for suitable finite volume hamiltonians, the thermodynamical limit of the local dynamics can be appropriately defined in $\widetilde{\mathcal{A}_0}[\tau]$.

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