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On the Outer Automorphism Groups of Triangular Alternation Limit Algebras

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Let A denote the alternation limit algebra, studied by Hopenwasser and Power and by Poon, which is the closed direct limit of upper triangular matrix algebra determined by refinement embeddings of multiplicity r_k and standard embeddings of multiplicity s_k. It is shown that the quotient of the isometric automorphism group by the approximately inner automorphisms is the abelian group $\mathbb{Z}^d$, where d is the number of primes that are divisors of infinitely many terms of each of the sequences (r_k) and (s_k). This group is also the group of automorphisms of the fundamental relation of A.

1. INTRODUCTION

In Hopenwasser and Power [HF] and in Poon [Po] the alternation limit algebras described below were classified. In this note we determine the quotient $\mathrm{Out}_{\mathrm{isom}}\, A = \mathrm{Aut}_{\mathrm{isom}}\, A/I(A)$ for these algebras where $\mathrm{Aut}_{\mathrm{isom}}\, A$ is the group of isometric algebra automorphisms and $I(A)$ is the normal subgroup of Aut A of approximately inner automorphisms. An automorphism σ is said to be approximately inner if there exists a sequence (b_k) of invertible elements such that $\alpha(a) = \lim_k b_k a b_k^{-1}$ for all a in A.

Let (r_k), (s_k) be sequences of positive intergers. Write $T(r_k, s_k)$ for the Banach algebra limit of the system

$$\mathbb{C} \to T_{r_1} \to T_{r_1 s_1} \to T_{r_1 s_1 r_2} \to \cdots,$$

where T_n is the algebra of upper triangular $n \times n$ matrices and where the embeddings are unital and are alternately of refinement type ($\rho(a) = (a_{ij} 1_t)$, with 1_t the $t \times t$ identity) and of standard type ($\sigma(a) = a \oplus \cdots \oplus a$, t times).

THEOREM 1. *$\mathrm{Out}_{\mathrm{isom}}(T(r_k, s_k)) = \mathbb{Z}^d$, where d is the number of primes p that are divisors of infinitely many terms of each of the sequences (r_k) and (s_k). (If $d = \infty$ interpret $\mathbb{Z}^d$ as the countably generated free abelian group.)*

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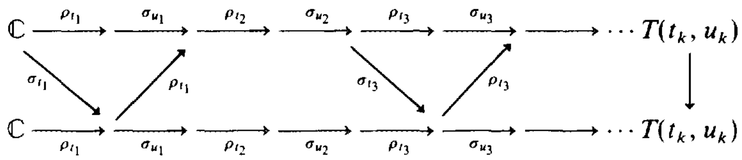
The proof uses the methods of [HP]. A major step is to characterize the automorphism group of the fundamental relation, or semigroupoid, which is associated with an alternation algebra. This order-topological result is of independent interest and is stated and proved separately below.

Let r and s be the generalised integers $r_1 r_2 \ldots$ and $s_1 s_1 \ldots$, respectively, and suppose that p is a prime satisfying the condition in the statement of the theorem. Then p^∞ divides r and s. Thus we can arrange new formal products $r = t_1 t_2 \ldots$, $s = u_1 u_2 \ldots$, with $t_k = u_k = p$ for all odd k. As noted in [HP], because of the commutation of refinement and standard embeddings, we can easily display a commuting zig zag diagram to show that $T(r_k, s_k)$ and $T(t_k, u_k)$ are isometrically isomorphic. However, with the new formal product we can construct one of the generators of $\mathrm{Out}_{\mathrm{isom}}\, A$. Consider the automorphism α determined by the following commuting diagram, where the matrix algebras are omitted for notational enconomy:



It will be shown that α provides a nonzero coset and that the totality of such cosets provides a generating set for the isometric outer automorphism group.

2. Proof of Theorem 1

Let X, or $X(r_k, s_k)$, be the Cantor space

$$X = \prod_{k=-\infty}^{-1} \{1, \ldots, s_{-k}\} \times \prod_{k=1}^{\infty} \{1, \ldots, r_k\},$$

where we have fixed the sequences (r_k) and (s_k). Define the equivalence relation $\tilde{R}$ on X to consist of the pairs (x, y) of points $x = (x_k)$, $y = (y_k)$, in X with $x_k = y_k$ for all large enough and small enough k. $\tilde{R}$ carries a natural locally compact Hausdorff topology (giving it the structure of an approximately finite groupoid). Write R, or $R(r_k, s_k)$ for the antisymmetric topologised subrelation of $\tilde{R}$ consisting of pairs (x, y) in R for which x preceeds y in the lexicographic order. Thus $(x, y) \in R$ if and only if $(x, y) \in \tilde{R}$ and either $x = y$, or for the smallest k for which $x_k \neq y_k$ we have $x_k < y_k$.

An automorphism of $R(r_k, s_k)$ is a binary relation isomorphism (implemented by a bijection α of the underlying space X) which is a homeomorphism for the (relative groupoid) topology of $R(r_k, s_k)$. Necessarily α is a homeomorphism of X.

THEOREM 2. *The group of automorphisms of the topological binary relation $R(r_k, s_k)$ is $\mathbb{Z}^d$, where d is the number of primes which divide infinitely many terms of each of the sequences (r_k) and (s_k).*

Proof. Let $\overline{\mathcal{O}(x)}$ denote the closure of the R-orbit of the point x in X. Here $\mathcal{O}(x) = \{y: (y, x) \in R\}$. Recall from [HP] that the pair of points x, x^+ is called a *gap pair* if $x^+ \notin \overline{\mathcal{O}(x)}$ and

$$\overline{\mathcal{O}(x^+)} = \overline{\mathcal{O}(x)} \cup \{x\}.$$

Furthermore x, x^+ is a gap pair if and only if

(1) there exists n such that $x_m = 1$ for all $m \leqslant n$,

(2) there exists p such that $x_q = r_q$ for all $q \geqslant p$.

Also if p is the smallest integer for which (2) holds (with $r_p = s_{-p}$ if p is negative), then x^+ is given by

$$(x^+)_j = \begin{cases} x_j & \text{if } j < p-1 \\ x_{p-1}+1 & \text{if } j = p-1 \\ 1 & \text{if } j \geqslant p. \end{cases}$$

The usefulness of this for our purpose is that an automorphism α of R necessarily maps gap pairs to gap pairs and so the coordinate description of these pairs leads ultimately to a coordinate description of α.

Let α be an automorphism of R. Consider the (left) gap points $x_* = (..., 1, 1, \hat{1}, r_1, r_2, ...)$ where $\hat{1}$ indicates the coordinate position for s_1. Then $\alpha(x_*)$ is necessarily a (left) gap point; thus

$$\alpha(x_*) = (..., 1, 1, z_{-t+1}, z_{-t}, ..., z_{t-1}, r_t, t_{t+1} ...)$$

for some positive integer t. We have

$$\overline{\mathcal{O}(x_*)} = \{x = (..., 1, \hat{1}, x_1, x_2, ...): x_k \leqslant r_k \text{ for all } k\},$$

$$\overline{\mathcal{O}(\alpha(x_*))} = \{y = (... 1, w', y_t, y_{t+1}, ...)\},$$

where $y_k \leqslant r_k$ for all $k \geqslant t$ and where w' is any word of length $2t-2$ which precedes (or is equal to) the word $w = z_{-t+1}, z_{-t}, \dots, z_{t-1}$ in the lexicographic order. Restating this, we have natural homeomorphisms

$$\overline{\mathcal{O}(x_*)} \approx \prod_{k=1}^{\infty} \{1, \dots, r_k\}$$

$$\overline{\mathcal{O}(\alpha(x_*))} \approx \{1, \dots, n\} \times \prod_{k=t}^{\infty} \{1, \dots, r_k\},$$

where n is the number of words w'. Moreover, these identifying homeomorphisms induce isomorphisms between the restrictions $R|\overline{\mathcal{O}(x_*)}$ and $R|\overline{\mathcal{O}(\alpha(x_*))}$ and the unilateral relations R_1 and R_2, respectively, where $R_1 = R(r_k, u_k)$, with $u_k = 1$ for all k, and $R_2 = R(r'_k, u_k)$, with u_k as before, $r'_1 = n$, and $r'_k = r_{k+t-2}$ for $k = 2, 3, \dots$. Since α induces an isomorphism between the restrictions, we obtain an induced isomorphism β between R_1 and R_2. It is well known that this means that $r = r'$ where $r = r_1 r_2 \dots$ and $r' = r'_1 r'_2 \dots$ are generalised integers. (See [P2] for example). Thus we obtain the necessary condition that the integer n is a divisor of the generalised integer r.

We now improve on this necessary condition.

The isomorphism between $R|\overline{\mathcal{O}(x_*)}$ and $R|\overline{\mathcal{O}(\alpha(x_*))}$ is given explicitly by

$$\alpha\colon (\dots 1, \hat{1}, x_1, x_2, \dots) \to (\dots 1, w', y_t, y_{t+1}, \dots),$$

where

$$\frac{\|w'\| - 1}{n} + \sum_{k=1}^{\infty} \frac{(y_{t+k-1} - 1)}{n m_{t+k-1}} m_{t-1} = \sum_{k=1}^{\infty} \frac{x_k - 1}{m_k}, \tag{1}$$

where $\|w'\|$ is the cardinality of the set of points in the order interval from the $(2t-2)$-tuple $(1, 1, \dots, 1)$ to w', and where $m_k = r_1 r_2 \dots r_k$ for $k = 1, 2 \dots$. The identity (1) follows from the fact that there are unique canonical R-invariant probability measures on $\overline{\mathcal{O}(x_*)}$ and on $\overline{\mathcal{O}(\alpha(x_*))}$ and the quantities in (1) are the measures of the subsets $\overline{\mathcal{O}(\alpha(x))}$ and $\overline{\mathcal{O}(x)}$, respectively.

To verify these facts one must recall how the topology of a topological binary relation is defined. In the case of $R_1 = R|\overline{\mathcal{O}(x_*)}$ fix two words

$$(x_1, x_2, \dots, x_l) \leqslant (x'_1, x'_2, \dots, x'_l)$$

in lexicographic order. Then the set E of pairs

$$((x_1, x_2, \dots, x_l, z_{l+1}, z_{l+2}, \dots), (x'_1, x'_2, \dots, x'_l, z_{l+1}, z_{l+2}, \dots))$$

is, by definition, a basic open and closed subset of the topology. Note that for this set, the left and right coordinate projection maps, $\pi_l: E \to \overline{\mathcal{O}(x_*)}$, $\pi_r :\to \overline{\mathcal{O}(x_*)}$ are injective. In the language of groupoids, E is a G-set. If λ is a Borel measure such that $\lambda(\pi_l(E)) = \lambda(\pi_r(E))$ for all closed and open G-sets E, then λ is said to be R-invariant. It is easy to see that this requirement forces λ to be the product measure $\lambda_1 \times \lambda_2 \times \ldots$, where λ_k is the uniformly distributed probability measure on $\{1, \ldots, r_k\}$. (One can also bear in mind that R-invariant measures are also $\tilde{R}$-invariant, where $\tilde{R}$ is the topological equivalence relation (i.e., groupoid) generated by R, and that the $\tilde{R}$-invariant measures correspond to traces on the C^*-algebra of $\tilde{R}$. In our context $C^*(\tilde{R})$ is UHF, and the R-invariant measure corresponds to the unique trace.)

Let $v(x)$ denote the right hand quantity of (1). Then the coordinates for $\alpha(x)$ are calculated from the identity (1), bearing in mind that the ambiguity arising from the equality $v(x) = v(x^+)$, for a gap pair x, x^+, is resolved by the known correspondence of left and right gap points.

Note that if x is in $\overline{\mathcal{O}(x_*)}$, and $\alpha(x) = y = (y_k)$, and $\|w'\| = 1$ (so that $y_{-t+1}, y_{-t}, \ldots, y_t$ are equal to 1), then , by (1),

$$v(\alpha(x)) = \sum_{k=1}^{\infty} \frac{y_k - 1}{m_k} = \sum_{k=1}^{\infty} \frac{y_{t+k-1} - 1}{m_{t+k-1}} = \frac{nv(x)}{m_{t-1}}.$$

We have obtained the identity $v(\alpha(x)) = cv(x)$, with $c = n/m_{t-1}$, for all points x in $\overline{\mathcal{O}(x_*)}$ for which $v(x)$ is small. In fact, because of the R-invariance of the measures on $\overline{\mathcal{O}(x_*)}$ and $\overline{\mathcal{O}(\alpha(x_*))}$, which we call λ_1 and λ_2, respectively, it follows that $v(\alpha(x)) = cv(x)$ for all points x for which $\alpha(x) \in \overline{\mathcal{O}(x_*)}$. To be more precise about this, consider the left gap points

$$g = (\ldots 1, \hat{1}, 1, \ldots, 1, r_{l+1}, \ldots),$$
$$x = (\ldots 1, \hat{1}, w, r_l, r_{l+1}, \ldots),$$
$$x' = (\ldots 1, \hat{1}, w, r_l - 1, r_{l+1}, \ldots),$$

where w is some word $w_1, w_2, \ldots, w_{l-1}$. Note that the set

$$E = \{((\ldots 1, \hat{1}, w, r_l, z_{l+1}, z_{l+2}, \ldots), (\ldots 1, \hat{1}, \ldots, 1, z_{l+1}, z_{l+2}, \ldots)) : z_j \leqslant r_j\}$$

has $\pi_l(E) = \overline{\mathcal{O}(x)} \backslash \overline{\mathcal{O}(x')}$ and $\pi_r(E) = \overline{\mathcal{O}(g)}$, and so $v(g) = v(x) - v(x')$. Since α preserves orbits and G-sets we also deduce that

$$\begin{aligned} v(\alpha(g)) &= \lambda_1(\overline{\mathcal{O}(\alpha(g))}) = \lambda_1(\pi_r((\alpha \times \alpha)(E))) \\ &= \lambda_1(\pi_l((\alpha \times \alpha)(E))) = \lambda_1(\overline{\mathcal{O}(\alpha(x))} \backslash \overline{\mathcal{O}(\alpha(x'))}) \\ &= v(\alpha(x)) - v(\alpha(x')). \end{aligned}$$

Thus, if we choose l large, so that we know that $v(\alpha(g)) = cv(g)$, we deduce that

$$v(\alpha(x)) - v(\alpha(x')) = v(\alpha(g)) = cv(g) = c(v(x) - v(x')),$$

from which it follows that $v(\alpha(x)) = c(v(x))$ for general points x with $\alpha(x)$ in $\overline{\mathscr{C}(x')}$.

We can similarly extend this identity to points in the set

$$X_0 = \{(y_k) \in X : \exists k_0 \text{ such that } y_k = 1 \text{ for all } k \leqslant k_0\}$$

and the extension of v given by

$$v(y) = \sum_{k=1}^{\infty} (y_{-k} - 1)\, s_0 s_1 \dots s_{k-1} + \sum_{k=1}^{\infty} \frac{y_k - 1}{m_k}$$

for y in X_0, where $s_0 = 1$. The range of v on the gap points of X_0 is the additive cone of rationals of the form l/m_k for some $k = 1, 2, \dots$ and some natural number l. The identity $v(\alpha(x)) = cv(x)$ for x in X_0 shows that multiplication by c is a bijection of the cone. From this we obtain the necessary condition that c has the form

$$c = p_1^{a_1} \dots p_d^{a_d}$$

where $a_i \in \mathbb{Z}$, $1 \leqslant i \leqslant d$, and where $p_1, \dots p_d$ are primes which divide infinitely many terms of the sequence (r_k).

We now improve further on this condition by considering the fact that α is a homeomorphism of X and is determined by its restriction to X_0.

Suppose, by way of contradiction, that $a_1 \neq 0$ and that p_1 does not divide infinitely many terms of the sequence (s_k). We may assume that $a_1 > 0$. (c depends only on α and so we may replace α by α^{-1} if $a_1 < 0$.) By relabelling we may also assume that p_1 divides no terms of the sequence. Without loss of generality assume that $s_1 > 1$ and consider the proper clopen subset E of points $y = (y_k)$ in X with $y_{-1} = 1$. We show that $\alpha(E)$ is dense, which is the desired contradiction. Observe that the range of v on $E \cap X_0$ is the union of the intervals $[ks_1, ks_1 + 1]$ for $k = 0, 1, 2, \dots$. Pick x in X_0 arbitrarily, pick j large, and consider the countable set

$$F_j(x) = \{x' \in X_0 : x' = (x'_k) \text{ and } x'_k = x_k \text{ for all } k \geqslant -j\}.$$

The range of v on $F_j(x)$ is an arithmetic progression of period $s_1 s_2 \dots s_j$. In view of the identity $v(\alpha(y)) = cv(y)$, the range of v on $\alpha(E) \cap X_0$ is the union of the intervals $[cks_1, cks_1 + c]$, which is an arithmetic progression of intervals of period cs_1. It follows from our hypothesis on p_1 that one of these intervals contains a point in $v(F_j(x))$, and so $\alpha(E)$ meets $F_j(x)$. Since the intersection of the sets $F_1(x), F_2(x), \dots$ is the singleton x, it follows that x lies in the closure of $\alpha(E)$. Since X_0 is dense it follows that $\alpha(E)$ is dense as desired.

We have now shown that if α is an automorphism of $R = R(r_k, s_k)$, then $\nu(\alpha(x)) = c\nu(x)$ for all x in X_0 where c has the form $c = p_1^{a_1} p_2^{a_2} \dots p_d^{a_d}$ where $a_1, \dots, a_d$ are integers and where $p_1, \dots p_d$ are primes which divide infinitely many terms of (r_k) and of (s_k). It is also clear from the above that for each such c there is at most one automorphism α satisfying the identity $\nu(\alpha(x)) = c\nu(x)$. It follows that the map

$$\alpha \to (a_1, \dots, a_d)$$

is an injective group homomorphism from Aut R to $\mathbb{Z}^d$. (d may be infinite.) It remains to show that this map is surjective. One way to do this is to start with c of the required form above and to show that the bijection of X_0 induced by multiplication by c (that is, the bijection α satisfying $\nu(\alpha(x)) = c\nu(x)$) does extend to an order preserving homeomorphism of X which defines an automorphism of R. Another way, which we now follow, is to make the connection between $R(r_k, s_k)$ and $T(r_k, s_k)$, and to determine generators of Aut R in terms of commuting diagrams, as we indicated after the statement of Theorem 1.

Consider the diagram

$$\begin{array}{ccccccccc}
\mathbb{C} & \xrightarrow{\rho_{r_1}} & M_{r_1} & \xrightarrow{\sigma_{s_1}} & M_{s_1} \otimes M_{r_1} & \xrightarrow{\rho_{r_2}} & M_{s_1} \otimes M_{r_1} \otimes M_{r_2} & \xrightarrow{\sigma_{s_2}} & \cdots B \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
\mathbb{C} & \xrightarrow{\rho_{r_1}} & T_{r_1} & \xrightarrow{\sigma_{s_1}} & T_{s_1 r_1} & \xrightarrow{\rho_{r_2}} & T_{s_1 r_1 r_2} & \xrightarrow{\sigma_{s_2}} & \cdots A
\end{array}$$

The vertical maps are inclusions, where $T_{s_1 r_1 r_2}$, for example, is realised in terms of the lexicographic order on the indices (i, j, k) of the minimal projections $e_{ii} \otimes e_{jj} \otimes e_{kk}$ in $M_{s_1} \otimes M_{r_1} \otimes M_{r_2}$. (For more detail concerning this discussion, read the introduction of [HP].) The maximal ideal space of the diagonal C^*-algebra $A \cap A^*$ is naturally identified with the space X. Indeed, $x = (x_k)$ in X corresponds to the point in the intersection of the Gelfand supports of the projections

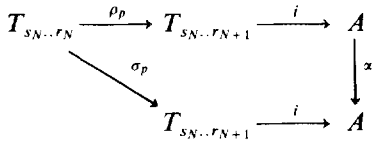
$$e(x, N) = e_{x_{-N, -N}} \otimes \dots \otimes e_{x_{-1, -1}} \otimes e_{x_{1,1}} \otimes \cdots \otimes e_{x_{N,N}}$$

for $N = 1, 2, \dots$. Furthermore, (x, y) belongs to $R = R(r_k, s_k)$ if and only if for all large N there is a matrix unit in the appropriate upper triangular matrix algebra with initial projection $e(y, N)$ and final projection $e(x, N)$. (In fact R is the fundamental relation of the limit algebra A.)

Suppose now that $r_k = s_k = p$ for all odd k and let α be the isometric automorphism of $T(r_k, s_k)$ determined by the diagram given in the introduction. Let α also denote the induced automorphism of R. We prove that $\nu(\alpha(x)) = p^{-1}\nu(x)$, completing the proof of the theorem.

Let us calculate $\alpha(e(x, N))$, where N is even, $x = (\dots, 1, \hat{1}, 2, 1, \dots)$, and where we abuse notation somewhat and write $e(x, N)$ for the image of

$e(x, N)$ in the limit algebra. Let $d(N) = s_N \dots s_1 r_1 \dots r_N$, and let $e(x, N)$ occupy position $a(N)$ in the lexicographic ordering of the $d(N)$ matrix units. Consider the following part of the diagram defining α:



Then

$$\rho_p(e(x, N)) = \sum_{k=1}^{p} e(x, N) \otimes e_{kk}.$$

On the other hand $\sigma_p(e(x, N))$ is the summation of the diagonal matrix units in positions $a(N)$, $a(N) + d(N), \dots, a(N) + (p-1)\, d(N)$ in the lexicographic order. Let these projections correspond to the matrix unit tensors with subscripts $z^{(i)} = (z^{(i)}_{-N}, \dots, z^{(i)}_{N+1})$ for $1 \leqslant i \leqslant p$, and denote the projections themselves by $f_1, \dots, f_p$, respectively. It follows (from the partial diagram above) that the homeomorphism $\alpha: X \to X$ maps the support of $e(x, N)$ onto the union of the supports of $f_1, \dots, f_p$. Denote these supports by $E(x, N)$, $F_1, \dots, F_p$ respectively. Since X_0 is invariant for α,

$$\alpha(E(x, N) \cap X_0) = \bigcup_{k=1}^{p} F_k \cap X_0.$$

Note that x is the unique point in $E(x, N) \cap X_0$ with the property that if $y \in E(x, N) \cap X_0$ and $(x, y) \in \tilde{R}$ then $(x, y) \in R$. The point in the union of $F_1 \cap X_0, \dots, F_p \cap X_0$ with this minimum property is the point

$$u = (\dots 1\ 1\ z^{(1)}_{-N}, \dots, z^{(1)}_{N+1}, 1, 1, \dots)$$

and so $\alpha(x) = u$. Finally, one can verify that $\nu(x) = p^{-1}$ and $\nu(u) = p^{-2}$, as desired. ∎

Recall that the fundamental relation $R(A)$ of a canonical triangular subalgebra A of an AF C^*-algebra B is the topological binary relation on the Gelfand space $M(A \cap A^*)$ induced by the partial isometries of A which normalise $A \cap A^*$. (See [P4].) In [HP] we identified $R(A)$, for $A = T(r_k, s_k)$, with $R(r_k, s_k)$. (This identification is also effected in the proof above by virtue of the fact that a matrix unit system determines $R(A)$.) Let β be an isometric automorphism of A. Then β induces an automorphism of $R(A)$ (because $\beta(A \cap A^*) = A \cap A^*$ and β maps the normalizer onto itself). Thus β determines an automorphism of $R(r_k, s_k)$ and so by the last theorem there is an isometric automorphism α of A such that $\gamma = \alpha^{-1} \circ \beta$ induces the trivial automorphism of $R(r_k, s_k)$. This means that γ is an isometric automorphism with γ equal to the identity map on $A \cap A^*$.

LEMMA. *Let γ be an automorphism of $T(r_k, s_k)$ which is the identity on the diagonal subalgebra (and which is not necessarily isometric). Then γ is approximately inner.*

Proof. Let $A = T(r_k, s_k)$ and let $A_1 \to A_2 \to \dots$ be the direct system defining A. The hypothesis is that $\gamma(c) = c$ for all c in $C = A \cap A^*$. This ensures that $\gamma(\tilde{A}_n) = \tilde{A}_n$ where $\tilde{A}_n$ is the subalgebra generated by A_n and C. To see this, recall from Lemma 1.2 of [P1] that there are contractive maps $P_n: A \to \tilde{A}_n$ which are defined in terms of limits of sums of compressions by projections in C, and so, for a in $\tilde{A}_n$, $\gamma(a) = \gamma(P_n(a)) = P_n(\gamma(a))$. The restriction automorphism $\gamma | \tilde{A}_n$ is necessarily inner. Indeed identify $\tilde{A}_n$ with $T_r \otimes D$, for appropriate r, where D is an abelian approximately finite C^*-algebra and let $u_i \in D$, $1 \leqslant i \leqslant r-1$, be the invertible elements such that $\gamma(e_{i,i+1}) = e_{i,i+1} \otimes u_i$. Also set $u_0 = 1$. Then it follows that $\gamma(a) = u^{-1} au$, where

$$u = \sum_{i=1}^{r} e_{i,i} \otimes u_0 u_1 \dots u_{r-1}.$$

Furthermore, since $\gamma(e_{1,r}) = e_{1,r} \otimes u_0 u_1 \dots u_{r-1}$, it follows that $\|u\| \leqslant \|\gamma\|$. Similarly $\|u^{-1}\| \leqslant \|\gamma^{-1}\|$. The inner automorphisms $Ad u^{-1}$, for varying n, thus form a uniformly bounded sequence which converge pointwise on each A_n, and so determine an approximately inner automorphism. ∎

It follows from the lemma and the preceding discussion that

$$\mathrm{Aut}_{\mathrm{isom}}\, A/I(A) = \mathrm{Aut}\, R(A) = \mathbb{Z}^d.$$

Remark 1. Suppose that $\delta \in \mathrm{Aut}\, A$. Then δ determines a scaled group homomorphism $\delta_*: K_0(A) \to K_0(A)$ which preserves the algebraic order on the scale $\Sigma(A)$ of $K_0(A)$. Thus, by [P3, Theorem 3.2], there is an isometric algebra automorphism of A, ϕ say, with $\phi_* = \delta_*$. In particular $\psi = \phi^{-1} \circ \delta$ has ψ_* trivial. This means that if P: $A \to A \cap A^*$ is the diagonal expectation, then $P(\psi(e)) = e$ for each projection e in $A \cap A^*$. Thus to show that $\mathrm{Aut}\, A/I(A) = \mathbb{Z}^d$ it remains only to show that such automorphisms ψ are approximately inner.

Remark 2. There are approximately inner automorphisms of alternation algebras which are not inner. To see this, consider the standard limit algebra $A = \varinjlim(T_{2^n}, \sigma)$. Let λ be a unimodular complex number and let $d_n = \lambda e_{1,1} + \lambda^2 e_{2,2} + \cdots + \lambda^{2^n} e_{2^n, 2^n}$. Then $d_n a d_n^{-1} = d_m a d_m^{-1}$ if $a \in T_{2^n}$ and $m > n$, from which it follows that $\alpha(a) = \lim_n (d_n a d_n^{-1})$ is an isometric approximately inner automorphism.

Suppose now that α is inner, and $\alpha(a) = gah^{-1}$ for some invertible g in A. Since $\alpha(c) = c$ for all c in the massa C it follows that $g \in C$. In particular

$\|\alpha-\beta\| \leqslant \frac{1}{2}$ for some inner automorphism β of the form $\beta(a)=hah^{-1}$ where, for some large enough n, $h \in T_{2^n} \cap (T_{2^n})^*$. However, in T_{2^m}, for large m, the diagonal element h has matrix entries which are periodic with period 2^n. One can now verify that if λ is chosen so that no power of order 2^k is unity then for large enough m there exist matrix units $e \in T_{2^m}$ such that $\|\lambda e - heh^{-1}\| > \frac{1}{4}$. In this case then α fails to be inner.

Remark 3. Let (x, y) be a point in $R(C^*(T(r_k, s_k)))$ with $x=(..., x_{-2}, x_{-1}, x_1, x_2, ...)$, $y=(..., y_{-2}, y_{-1}, y_1, y_2, ...)$. Then, although $v(x)$ and $v(y)$ may be infinite, we may define $d(x, y)$ as the sum

$$\sum_{k=1}^{\infty} (y_{-k}-x_{-k}) s_0 s_1 \dots s_{k-1} + \sum_{k=1}^{\infty} \frac{y_k - x_k}{r_1 r_2 \dots r_k}$$

because only finitely many terms are nonzero. Since $d(x, y)=d(x, z)+d(z, y)$, and $(x, y) \in R(r_k, s_k)$ if and only if $d(x, y) \geqslant 0$, it follows that $d(x, y)$ is a continuous real valued cocycle determining $A(r_k, s_k)$ as an analytic subalgebra of $C^*(A(r_k, s_k))$. See [V], where some special cases are discussed as well as some general aspects of analyticity.

Note added in proof. Unfortunately the proof of the classification of alternation algebras given in [HP] and [P4] appears to be incomplete. (It is not clear, in [P4], whether q can be chosen with the desired properties.) However, the present paper is independent of [HP] and the arithmetic progression argument above can be adapted, in the case of an isomorphism α between two alternation algebras, to show that the supernatural numbers for the standard multiplicities are finitely equivalent.

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