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Steiner Quadruple Systems All of Whose Derived Steiner Triple Systems Are Nonisomorphic

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1. INTRODUCTION

A *Steiner quadruple system* (or more simply a *quadruple system*) is a pair $(Q, \mathscr{B})$ where Q is a finite set and $\mathscr{B}$ is a collection of 4-subsets of Q (called *blocks*) such that any 3-subset of Q belongs to exactly one block of $\mathscr{B}$. The number $|Q|$ is called the order of the quadruple system $(Q, \mathscr{B})$. It has been proved by Hanani in 1960 [4] that the spectrum for quadruple systems consists of all positive integers $n \equiv 2$ or $4 \pmod 6$. If $(Q, \mathscr{B})$ is a quadruple system and x is any element in Q we will denote by Q_x the set $Q \backslash \{x\}$ and the set of all triples $\{a, b, c\}$ such that $\{x, a, b, c\} \in \mathscr{B}$ by $\mathscr{B}(x)$. It is a routine matter to see that $(Q_x, \mathscr{B}(x))$ is a Steiner triple system called a *derived triple system* (DTS) of the quadruple system $(Q, \mathscr{B})$.

A very interesting problem is the determination of the number of nonisomorphic DTSs of a given quadruple system. It has been shown in [9] that there are exactly four nonisomorphic quadruple systems of order 14. Two of these quadruple systems have all 14 DTSs isomorphic while the other two have 2 nonisomorphic DTSs (the maximum number possible for order 14 since there are exactly two nonisomorphic triple systems of order 13). In [6], the results in [9] and the ordinary direct product have been used to construct an infinite class of quadruple systems having at

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least two nonisomorphic DTSs. Subsequently in [7] a different construction for quadruple systems was given which produced for any positive integer t a quadruple system having at least t nonisomorphic DTSs. Unfortunately, the size of the quadruple system compared to t is quite large: For example, if $t = 8$ then the quadruple system is of order 400 [7].

In this paper (i) we give a construction for quadruple systems in which the number of nonisomorphic DTSs can be rapidly computed (provided, of course, the quadruple system is not too large); (ii) we use this construction to obtain a quadruple system of order 20 having all 20 of its derived triple systems pairwise nonisomorphic (the first known example of a quadruple system with the property that all of its DTSs are pairwise nonisomorphic: a quadruple system with this property will henceforth be called *heterogeneous*); and finally (iii) we use this result coupled with a recursive construction to obtain an infinite class of heterogeneous quadruple systems.

2. Computation of the Number of Nonisomorphic DTSs

Let $(X, \mathscr{B})$ and $(Y, \mathscr{C})$ be any two quadruple systems of order n where $X \cap Y = \varnothing$. Let $\mathscr{F} = \{F_1, F_2, ..., F_{n-1}\}$ and $\mathscr{G} = \{G_1, G_2, ..., G_{n-1}\}$ be any two 1-factorizations of K_n (the complete n-graph) based on X and Y respectively, and let α be any permutation on the set $\{1, 2, ..., n-1\}$. Define a collection of blocks $\mathscr{A}$ on $Q = X \cup Y$ as follows:

(1) Any block belonging to $\mathscr{B}$ or $\mathscr{C}$ belongs to $\mathscr{A}$, and

(2) If $x_1, x_2 \in X$ and $y_1, y_2 \in Y$ then $\{x_1, x_2, y_1, y_2\} \in \mathscr{A}$ if and only if $[x_1, x_2] \in F_i$, $[y_1, y_2] \in G_j$ and $i\alpha = j$.

It is a routine matter to see that $(Q, \mathscr{A})$ is a quadruple system. It is important to note that there need be no relationship between $(X, \mathscr{B})$ and $(Y, \mathscr{C})$, that $\mathscr{F}$ and $\mathscr{G}$ can be any 1-factorizations, and that α can be any permutation. We denote this quadruple system by $[X \cup Y](\mathscr{B}, \mathscr{C}, \mathscr{F}, \mathscr{G}, \alpha)$.

If x is any element in X the DTS$(X_x, \mathscr{B}(x))$ of $(X, \mathscr{B})$ is a subsystem of the DTS$(Q_x, \mathscr{A}(x))$. (A similar statement holds if $x \in Y$.) If the quadruple system $(Q, \mathscr{A})$ has the property that for every $x \in Q$, the only subsystem of $(Q_x, \mathscr{A}(x))$ of order $n-1$ is $(X_x, \mathscr{B}(x))$ or $(Y_x, \mathscr{C}(x))$ as the case may be, then we will say that $(Q, \mathscr{A})$ is *component-simple*. For example, if we take $|X| = |Y| = 10$ then $(Q, \mathscr{A})$ is component-simple as a consequence of the fact that a triple system of order 19 can have at most one subsystem of order 9.

Now let $(X, \mathscr{B})$ and $(Y, \mathscr{C})$ be disjoint quadruple systems of order n

and $\mathscr{F} = \{F_1, F_2, \ldots, F_{n-1}\}$ and $\mathscr{G} = \{G_1, G_2, \ldots, G_{n-1}\}$ any two non-isomorphic automorphism-free 1-factorizations of K_n on X and Y respectively [8]. Let α be any permutation on $\{1, 2, \ldots, n-1\}$ and for each $x \in X$ denote by xf_i the unique element in X such that $[x, xf_i] \in F_i$. Similarly, for each $y \in Y$, denote by yg_i the unique element in Y such that $[y, yg_i] \in G_i$. If x and y are any two elements in X, define the mapping $\beta_{xy}: X_x \to X_y$ by $\beta_{xy}(xf_i) = yf_i$. If x and y belong to Y define the mapping $\gamma_{xy}: Y_x \to Y_y$ by $\gamma_{xy}(xg_i) = yg_i$.

THEOREM 1. *Let $(X, \mathscr{B})$, $(Y, \mathscr{C})$, $\mathscr{F}$, $\mathscr{G}$, and α be as above, and let the quadruple system $(Q, \mathscr{A}) = [X \cup Y](\mathscr{B}, \mathscr{C}, \mathscr{F}, \mathscr{G}, \alpha)$ be component-simple. If $x, y \in X$ then $(Q_x, \mathscr{A}(X))$ and $(Q_y, \mathscr{A}(y))$ are isomorphic if and only if β_{xy} is an isomorphism. If $x, y \in Y$, then $(Q_x, \mathscr{A}(x))$ and $(Q_y, A(y))$ are isomorphic if and only if γ_{xy} is an isomorphism. Finally, if $x \in X$ and $y \in Y$ then $(Q_x, \mathscr{A}(x))$ and $(Q_y, A(y))$ are nonisomorphic.*

Proof. Let x and y belong to X and assume λ to be an isomorphism of $(Q_x, \mathscr{A}(x))$ onto $(Q_y, \mathscr{A}(y))$. Sinze $(Q, \mathscr{A})$ is component-simple the only subsystem of $(Q_x, \mathscr{A}(x))$ of order $n-1$ is $(X_x, \mathscr{B}(x))$ and the only subsystem of $(Q_y, \mathscr{A}(y))$ of order $n-1$ is $(X_y, \mathscr{B}(y))$. Therefore, λ must map X_x onto X_y and therefore Y onto Y. Hence, λ induces an automorphism of $\mathscr{G}$, and since $\mathscr{G}$ is automorphism-free λ must be the identify mapping on Y. Hence, if $i\alpha = j$ then λ must map each triple of the form $\{xf_i, c, d\}$, $[c, d] \in G_j$ onto a triple of the form $\{t, c, d\} \in \mathscr{A}(y)$. But if this is so then $t = yf_i$ by construction. Therefore, λ reduces to β_{xy} on X_x so that β_{xy} must be an isomorphism from X_x onto X_y. A similar argument shows that if $x, y \in Y$ then $(Q_x, \mathscr{A}(x))$ and $(Q_y, \mathscr{A}(y))$ are isomorphic if and only if γ_{xy} is an isomorphism of Y_x onto Y_y.

Finally, suppose $x \in X$ and $y \in Y$, and let λ be an isomorphism of $(Q_x, \mathscr{A}(x))$ onto $(Q_y, \mathscr{A}(y))$. Since $(Q, \mathscr{A})$ is component-simple, λ must map X_x onto Y_y and therefore X onto Y. This induces an isomorphism of the 1-factorization $\mathscr{F}$ onto the 1-factorization $\mathscr{G}$. Since $\mathscr{F}$ and $\mathscr{G}$ are nonisomorphic by assumption we have a contradiction. Hence $(Q_x, \mathscr{A}(x))$ and $(Q_y, \mathscr{A}(y))$ are nonisomorphic. This completes the proof of the theorem.

3. A HETEROGENEOUS QUADRUPLE SYSTEM OF ORDER 20

Let $X = \{1, 2, \ldots, 10\}$, $Y = \{1', 2', \ldots, 10'\}$, and let $(X, \mathscr{B})$ and $(Y, \mathscr{C})$ be the quadruple systems of order 10 given as follows: $\mathscr{B}$ consists of the 30 quadruples $\{i, i+1, i+3, i+4\}$, $\{i, i+1, i+2, i+6\}$, $\{i, i+2$ $i+4, i+7\}$ (mod 10):

1	2	4	5	1	2	3	7	1	3	5	8
2	3	5	6	2	3	4	8	2	4	6	9
3	4	6	7	3	4	5	9	3	5	7	10
4	5	7	8	4	5	6	10	1	4	6	8
5	6	8	9	1	5	6	7	2	5	7	9
6	7	9	10	2	6	7	8	3	6	8	10
1	7	8	10	3	7	8	9	1	4	7	9
1	2	8	9	4	8	9	10	2	5	8	10
2	3	9	10	1	5	9	10	1	3	6	9
1	3	4	10	1	2	6	10	2	4	7	10

$\mathscr{C}$ is obtained from $\mathscr{B}$ by replacing each symbol i by i'. Further, let $\mathscr{F}$ and $\mathscr{G}$ be the following two nonisomorphic automorphism-free 1-factorizations of K_{10} [3]:

$\mathscr{F}$: F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9
1 2	1 3	1 4	1 5	1 6	1 7	1 8	1 9	1 10
3 4	2 4	2 3	2 6	2 7	2 5	2 10	2 8	2 9
5 6	5 7	5 8	3 7	3 9	3 10	3 6	3 5	3 8
7 8	6 9	6 10	4 10	4 8	4 9	4 7	4 6	4 5
9 10	8 10	7 9	8 9	5 10	6 8	5 9	7 10	6 7

$\mathscr{G}$: G_1	G_2	G_3	G_4	G_5	G_6	G_7	G_8	G_9
1′ 2′	1′ 3′	1′ 4′	1′ 5′	1′ 6′	1′ 7′	1′ 8′	1′ 9′	1′ 10′
3′ 4′	2′ 4′	2′ 3′	2′ 6′	2′ 7′	2′ 8′	2′ 5′	2′ 10′	2′ 9′
5′ 6′	5′ 7′	5′ 8′	3′ 7′	3′ 9′	3′ 10′	3′ 6′	3′ 5′	3′ 8′
7′ 8′	6′ 9′	6′ 10′	4′ 10′	4′ 8′	4′ 6′	4′ 9′	4′ 7′	4′ 5′
9′ 10′	8′ 10′	7′ 9′	8′ 9′	5′ 10′	5, 9′	7′ 10′	6′ 8′	6′ 7′

Take α to be the identity mapping on $\{1, 2, \ldots, 9\}$, and form the quadruple system $(Q, \mathscr{A}) = [X \cup Y](\mathscr{B}, \mathscr{C}, \mathscr{F}, \mathscr{G}, \alpha)$.

Claim. $(Q, \mathscr{A})$ is a heterogeneous quadruple system. To verify this, we need only check to see that none of the mappings β_{xy} or γ_{xy} is an isomorphism from $(X_x, \mathscr{B}(x))$ onto $(X_y, \mathscr{B}(y))$ or from $(Y_x, \mathscr{C}(x))$ onto $(Y_y, \mathscr{C}(y))$, as the case may be. In Tables I–III we list all 10 DTSs of $(X, \mathscr{B})$ and $(Y, \mathscr{C})$, respectively, and all mappings β_{xy} and γ_{xy}. The derived triple systems of $(Y, \mathscr{C})$ are obtained from the derived triple systems of $(X, \mathscr{B})$ by replacing each symbol i by i'.

TABLE I

Mappings β_{ij} (Choose the Two Rows Headed i and j)

1	2	3	4	5	6	7	8	9	10
2	1	4	3	6	7	8	5	10	9
3	4	1	2	7	9	10	6	5	8
4	3	2	1	10	8	6	9	7	5
5	6	7	8	1	10	9	2	3	4
6	5	9	10	2	1	4	3	8	7
7	8	5	9	3	2	1	10	4	6
8	7	10	5	9	4	2	1	6	3
9	10	6	7	8	3	5	4	1	2
10	9	8	6	4	5	3	7	2	1

TABLE II

Mappings γ_{ij} (Choose the Two Rows Headed i' and j')

1′	2′	3′	4′	5′	6′	7′	8′	9′	10′
2′	1′	4′	3′	6′	7′	5′	10′	8′	9′
3′	4′	1′	2′	7′	9′	10′	6′	5′	8′
4′	3′	2′	1′	10′	8′	9′	7′	6′	5′
5′	6′	7′	8′	1′	10′	2′	9′	3′	4′
6′	5′	9′	10′	2′	1′	8′	3′	4′	7′
7′	8′	5′	9′	3′	2′	1′	4′	10′	6′
8′	7′	10′	5′	9′	4′	6′	1′	2′	3′
9′	10′	6′	7′	8′	3′	4′	5′	1′	2′
10′	9′	8′	6′	4′	5′	3′	2′	7′	1′

TABLE III

Derived Triple Systems of $(X, \mathscr{B})$

	1			2	
2 4 5	7 8 10	3 6 9	1 4 5	3 9 10	6 7 8
2 3 7	5 9 10	4 6 8	1 3 7	4 6 9	5 8 10
2 6 10	3 5 8	4 7 9	1 8 9	3 5 6	4 7 10
2 8 9	5 6 7	3 4 10	1 6 10	3 4 8	5 7 9
	3			4	
1 2 7	4 5 9	6 8 10	1 2 5	3 6 7	8 9 10
1 5 8	4 6 7	2 9 10	1 3 10	5 7 8	2 6 9
1 4 10	2 5 6	7 8 9	1 6 8	2 7 10	3 5 9
1 6 9	2 4 8	5 7 10	1 7 9	2 3 8	5 6 10
	5			6	
1 2 4	6 8 9	3 7 10	1 5 7	2 4 9	3 8 10
1 6 7	3 4 9	2 8 10	1 2 10	3 4 7	5 8 9
1 9 10	2 3 6	4 7 8	1 4 8	2 3 5	7 9 10
1 3 8	4 6 10	2 7 9	1 3 9	4 5 10	2 7 8
	7			8	
1 2 3	4 5 8	6 9 10	1 7 10	5 6 9	2 3 4
1 8 10	3 4 6	2 5 9	1 2 9	4 5 7	3 6 10
1 5 6	3 8 9	2 4 10	1 3 5	2 6 7	4 9 10
1 4 9	2 6 8	3 5 10	1 4 6	2 5 10	3 7 9
	9			10	
1 2 8	6 7 10	3 4 5	1 3 4	6 7 9	2 5 8
1 5 10	3 7 8	2 4 6	1 7 8	2 3 9	4 5 6
1 4 7	5 6 8	2 3 10	1 5 9	3 6 8	2 4 7
1 3 6	4 8 10	2 5 7	1 2 6	4 8 9	3 5 7

The reader can now easily check that none of the 90 mappings β_{xy}, γ_{xy} are isomorphisms. For example, to check that $(Q_4, \mathscr{A}(4))$ and $(Q_9, \mathscr{A}(9))$ are nonisomorphic it suffices to show that β_{49} is not an isomorphism from $(X_4, \mathscr{B}(4))$ onto $(X_9, \mathscr{B}(9))$:

From Table I for β_{xy} we have

$$\beta_{49} = \begin{pmatrix} 3 & 2 & 1 & 10 & 5 & 6 & 9 & 7 & 5 \\ 10 & 6 & 7 & 8 & 3 & 5 & 4 & 1 & 2 \end{pmatrix}.$$

From the table of DTSs of $(X, \mathscr{B})$, $\{1, 2, 5\} \in \mathscr{B}(4)$. Since the image of this triple under β_{49} is $\{2, 6, 7\} \notin \mathscr{B}(9)$, the mapping β_{49} is not an isomorphism of $(X_4, \mathscr{B}(4))$ onto $(X_9, \mathscr{B}(9))$ and so the DTSs $(Q_4, \mathscr{A}(4))$ and $(Q_9, \mathscr{A}(9))$ are nonisomorphic. The check for the remaining 89 mappings goes just as quickly as this example. Since none of the mappings β_{xy}, γ_{xy} are isomorphisms it follows that $(Q, \mathscr{A})$ has all of its DTSs pairwise non-isomorphic and thus is a heterogeneous quadruple system.

4. An Infinite Class of Heterogeneous Quadruple Systems

For unexplained notions and results on 1-factorizations of the complete graph we refer the reader to [8].

LEMMA 2. *Let $n \equiv 1$ or $2 \pmod 3$, $n \geqslant 5$. Then there exists two nonisomorphic 1-factorizations of K_{2n} neither of which contains a sub-1-factorization of index 2.*

Proof. The well-known type of 1-factorizations of K_{2n} discovered and studied by many authors (see, e.g., [1, 5]) and denoted sometimes by $GF(K_{2n})$ has no sub-1-factorization of index 2 for $n \geqslant 3$ (cf. [8]). On the other hand, a result in [2] states that there exists a Steiner triple system of every order v having no nontrivial subsystems (i.e., no subsystems other than those of order 1, 3, and v). The corresponding Steiner 1-factorization of K_{v+1} corresponding to such a Steiner triple system of order v clearly has no sub-1-factorization of index 2 whenever $v \geqslant 9$. Observing that for $n > 2$, the 1-factorization $GF(K_{2n})$ is never a Steiner 1-factorization completes the proof.

THEOREM 3. *If there exists a heterogeneous quadruple system of order n then there exists a heterogeneous quadruple system or order $2n$.*

Proof. Let $(Q, \mathscr{B})$ be a heterogeneous quadruple system of order n. Since there are no heterogeneous quadruple systems of order $n \leqslant 14$

we may assume $n \geqslant 16$. Let $(Q_1, \mathscr{B}_1)$, $(Q_2, \mathscr{B}_2)$ be two disjoint copies of $(Q, \mathscr{B})$, i.e., $Q_1 \cap Q_2 = \varnothing$. Let $\mathscr{F} = \{F_1, F_2, ..., F_{n-1}\}$ and $\mathscr{G} = \{G_1, G_2, ..., G_{n-1}\}$ be two nonisomorphic 1-factorizations of K_{2n} based on Q_1 and Q_2, respectively, with neither $\mathscr{F}$ nor $\mathscr{G}$ containing a sub-1-factorization of index 2 (Lemma 2 guarantees existence of such 1-factorizations $\mathscr{F}$ and $\mathscr{G}$). Construct, as in section 2, the quadruple system $(Q^*, \mathscr{B}^*) = [Q_1 \cup Q_2](\mathscr{B}_1, \mathscr{B}_2, \mathscr{F}, \mathscr{G}, \alpha)$ where α is any permutation on $\{1, 2, ..., n-1\}$. We claim that $(Q^*, \mathscr{B}^*)$ is a heterogeneous quadruple system. To verify this we observe first that $(Q^*, \mathscr{B}^*)$ is component-simple as both 1-factorizations $\mathscr{F}$ and $\mathscr{G}$ were assumed not to contain sub-1-factorizations of index 2 (cf. [8]).

Let now r, s be any two distinct elements of Q^*, and let $(Q_r^*, \mathscr{B}^*(r))$, $(Q_s^*, \mathscr{B}^*(s))$ be the corresponding derived triple systems. Assume λ to be an isomorphism mapping $(Q_r^*, \mathscr{B}^*(r))$ onto $(Q_s^*, \mathscr{B}^*(s))$. Consider two cases:

Case 1. Both r, s belong to the same set Q_i for $i \in \{1, 2\}$, say, to Q_1. Then the unique subsystem $(Q_{1r}, \mathscr{B}_1(r))$ of order $n-1$ of $(Q_r^*, \mathscr{B}^*(r))$ is a derived triple system of $(Q_1, \mathscr{B}_1)$, and a similar statement holds for s. Then λ must map $(Q_{1r}, \mathscr{B}_1(r))$ onto $(Q_{1s}, \mathscr{B}_1(s))$ which contradicts the fact that $(Q_1, \mathscr{B}_1)$, is a heterogeneous quadruple system.

Case 2. r, s belong to different sets Q_i, say, $r \in Q_1$, $s \in Q_2$. Then λ must map again $(Q_{1r}, \mathscr{B}_1(r))$ onto $(Q_{2s}, \mathscr{B}_2(s))$, and consequently, λ must map $\mathscr{G}$ onto $\mathscr{F}$ which contradicts our assumption that $\mathscr{F}$ and $\mathscr{G}$ are nonisomorphic 1-factorizations.

This completes the proof of the theorem.

COROLLARY 4. *A heterogeneous quadruple system exists for every order* $n = 10 \cdot 2^k$ *where* k *is a positive integer.*

Clearly, there is no heterogeneous quadruple system of order $n \leqslant 14$, and its existence for order 16 is in doubt. However, we conjecture that a heterogeneous quadruple system exists for every admissible order $n \geqslant 20$.

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