Note

On Packing Unequal Rectangles in the Unit Square

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This paper improves a previous bound, due to Meir and Moser in [J. Combin. Theory 5 (1968), 126–134] concerning the smallest square into which all of the rectangles of size $1/n \times 1/(n + 1)$, $n = 1, 2, 3, \ldots$ can be packed. © 1994 Academic Press, Inc.

Leo Moser noted that $\sum_{n=1}^{\infty} 1/n(n + 1) = 1$ and asked if the rectangles $1/n \times 1/(n + 1)$, $n = 1, 2, 3, \ldots$ can be packed into the unit square [1–4]. Meir and Moser showed that they can be packed into a square of side $31/30$ [1]. This result is improved by showing that they can be packed into a square of side $133/132$. The problem of finding the smallest $\varepsilon \geq 0$ such that all the rectangles can be packed into a square of side $1 + \varepsilon$ is still unsolved. Whether $\varepsilon > 0$ or $\varepsilon = 0$ is an open question.

THEOREM. All the rectangles of size $1/n \times 1/(n + 1)$, $n = 1, 2, 3, \ldots$ can be packed into a square of side $133/132$.

Proof. Let the rectangle of size $1/n \times 1/(n + 1)$ be represented by $Q_n$. Figure 1 shows how the inequality

$$\frac{1}{2n} + \frac{1}{2n + 1} < \frac{1}{n} \quad (1)$$

is used to pack the rectangles

$$\{Q_{n-1}\}, \{Q_{2n-1}, Q_{2n}\}, \{Q_{4n-1}, Q_{4n}, Q_{4n+1}, Q_{4n+2}\},$$

$$\{Q_{8n-1}, Q_{8n}, \ldots, Q_{8n+6}\}, \ldots$$
Figure 1

into a rectangle of size

$$\frac{1}{n} \times \left\{ \frac{1}{n - 1} + \frac{1}{2n - 1} + \frac{1}{4n - 1} + \frac{1}{8n - 1} + \cdots \right\}, \quad (2)$$

which is itself contained in the rectangle $R_{n-1}$, of size $1/n \times 2/(n - 1)$.

Similarly, we can pack the rectangles

$$\{Q_n\}, \{Q_{2n+1}, Q_{2n+2}\}, \{Q_{4n+3}, Q_{4n+4}, Q_{4n+5}, Q_{4n+6}\},$$
$$\{Q_{8n+7}, Q_{8n+8}, \cdots, Q_{8n+14}\}, \cdots$$

into the rectangle $R_n$, of size $1/(n + 1) \times 2/n$, and the rectangles

$$\{Q_{n+1}\}, \{Q_{2n+3}, Q_{2n+4}\}, \{Q_{4n+7}, Q_{4n+8}, Q_{4n+9}, Q_{4n+10}\},$$
$$\{Q_{8n+15}, Q_{8n+16}, \cdots, Q_{8n+22}\}, \cdots$$

into the rectangle $R_{n+1}$. Continuing in this manner, we can pack the rectangles

$$\{Q_{2n-2}\}, \{Q_{4n-3}, Q_{4n-2}\}, \{Q_{8n-5}, Q_{8n-4}, Q_{8n-3}, Q_{8n-2}\}, \cdots$$

into the rectangle $R_{2n-2}$. Hence we have shown that all the rectangles $Q_i$, $i = n - 1, n, n + 1 \ldots$ can be packed into the $n$ rectangles $\{R_{n-1}, R_n, \ldots, R_{2n-2}\}$. Therefore, if we can pack

$$\{Q_1, Q_2, \ldots, Q_{n-2}, R_{n-1}, R_n, \ldots, R_{2n-2}\}$$

into a square of side $x$, we can certainly pack $Q_i$, $i = 1, 2, 3, \ldots$ into the same square. Letting $n := n + 1$, the problem is now one of packing the
$2n$ rectangles

$$\{Q_1, Q_2, \ldots, Q_{n-1}, R_n, R_{n+1}, \ldots, R_{2n}\}$$

into as small a square as possible. Note that the total area of these $2n$ rectangles is given by

$$\sum_{r=1}^{n-1} \frac{1}{r(r+1)} + 2 \sum_{r=n}^{2n} \frac{1}{r(r+1)} = 1 + \frac{1}{n(2n+1)},$$

which is still fairly close to 1 for moderately small values of $n$. So we have included an area of only $1/n(2n + 1)$ of unused space in the argument so far.

Taking $n = 15$, Figure 2 completes the proof of the theorem by showing how $\{Q_1, Q_2, \ldots, Q_{14}, R_{15}, R_{16}, \ldots, R_{30}\}$ can be packed into a square of side $133/132$ ($Q_1$ is aligned below $Q_2$, $Q_3$, and $Q_4$). The dimensions of the enclosing rectangle are vertically (measured to $C$),

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{11} + \frac{1}{12} = 1 + \frac{1}{132},$$

and horizontally (measured to $B$),

$$\frac{1}{14} + \frac{1}{25} + \frac{1}{13} + \frac{1}{26} + \frac{1}{5} + \frac{1}{7} + \frac{1}{10} + \frac{1}{19} + \frac{1}{27} + \frac{1}{30} + \frac{1}{22} + \frac{1}{14} + \frac{1}{21} + \frac{1}{20}
< 1 + \frac{1}{139}.$$

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![Figure 2](image-url)
The vertical dimensions at the other edges (D, E, F, and G) significantly overlapping the top of the unit square are given by

\[
\begin{align*}
\text{(D)} & \quad \frac{1}{2} + \frac{1}{3} + \frac{1}{9} + \frac{1}{16}, \\
\text{(E)} & \quad \frac{1}{2} + \frac{1}{5} + \frac{1}{9} + \frac{1}{18} + \frac{2}{29} + \frac{2}{28}, \\
\text{(F)} & \quad \frac{1}{2} + \frac{1}{5} + \frac{1}{9} + \frac{1}{18} + \frac{1}{24} + \frac{1}{15} + \frac{1}{31}, \\
\text{(G)} & \quad \frac{1}{2} + \frac{1}{5} + \frac{1}{7} + \frac{1}{17} + \frac{2}{19},
\end{align*}
\]

each of which is less than 133/132. The horizontal dimension measured to edge A is less than 140/139.

Improvements on this result may be possible by taking a larger value of \(n\). Also, note that the length of the rectangle \(R_{n-1}\) can be successively shortened from \(2/(n-1)\) to

\[
\frac{(4n-3)/(n-1)(2n-1)},
\]
\[
\frac{(16n^2 - 17n + 4)/(n-1)(2n-1)(4n-1)}{\ldots}
\]

by truncating (with remainder) the sequence of (2).

This method of packing can of course be applied to other problems of a similar nature. For example, the question is asked in [4] if the squares of sides \(1/2, 1/3, 1/4, \ldots\) can be packed into some rectangle of area \((\pi^2/6) - 1\). If we let the square of side \(1/n\) be represented by \(S_n\) and the rectangle of size \(1/n \times 2/n\) be represented by \(T_n\) then a similar packing argument, using inequality (1), shows that we can pack all the \(S_i, i = 2, 3, 4, \ldots\) into \(\{S_2, S_3, \ldots, S_n, T_{n+1}, T_{n+2}, \ldots, T_{2n+1}\}\). So if we can pack \(\{S_2, S_3, \ldots, S_n, T_{n+1}, T_{n+2}, \ldots, T_{2n+1}\}\) into a rectangle of size \(a \times b\), then we can pack all the \(S_i, i = 2, 3, 4, \ldots\) into the same rectangle. Now the total area of \(\{S_2, S_3, \ldots, S_n, T_{n+1}, T_{n+2}, \ldots, T_{2n+1}\}\) is given by

\[
\sum_{r=2}^n \frac{1}{r^2} + 2 \sum_{r=n+1}^{2n+1} \frac{1}{r^2} < \left(\frac{\pi^2}{6} - 1\right) + \frac{1}{2n(n+1)}.
\]

Therefore the surplus area included in the argument so far is \(< 1/2n(n+1)\). Hence, by this method, it should be possible to find a packing of \(S_i, i = 2, 3, 4, \ldots\) into some rectangle of area close to \((\pi^2/6) - 1\), for moderately small values of \(n\). Indeed, taking \(n = 15\) Fig. 3 illustrates the theorem.

**Theorem.** All the squares of sides \(1/n, n = 2, 3, 4\ldots\) can be packed into a rectangle of area 47/72.
Note that
\[
\frac{47}{72} < \left( \frac{\pi^2}{6} - 1 \right) + \frac{1}{127},
\]
and that the dimensions of the enclosing are $5/6 \times 47/60$, given by
\[
\left( \frac{1}{2} + \frac{1}{3} \right) \times \left( \frac{1}{3} + \frac{1}{4} + \frac{1}{5} \right).
\]
The width to point A is
\[
\frac{1}{2} + \frac{1}{7} + \frac{1}{14} + \frac{2}{29} = \left( \frac{1}{3} + \frac{1}{4} + \frac{1}{5} \right) - \frac{1}{12180}.
\]

REFERENCES