Fretting contact of two elastic solids with graded coatings under torsion

Tie-jun Liu a,⁎, Yue-sheng Wang b, Yong-ming Xing a

a Mechanical Department, College of Sciences of Inner Mongolia University of Technology, Huhhot, Inner Mongolia, 010051, China
b Institute of Engineering Mechanics, Beijing Jiaotong University, Beijing 100044, China

A R T I C L E   I N F O

Article history:
Received 8 October 2011
Received in revised form 5 February 2012
Available online 24 February 2012

Keywords:
Fretting contact
Friction
Functionally graded coating
Singular integral equations
Hankel integral transform
Torsion

A B S T R A C T

In this paper, the fretting contact problem for two elastic solids with graded coatings is investigated. We assume a conventional axisymmetric Hertzian contact takes place between two elastic solids under the action of the normal pressure. The application of the torque produces an annulus of slip. It is assumed that the surface shear traction within the contact area is limited by Coulomb's friction law and the torsion angel was produced within the central adhesion zone as a rigid body. The linear multi-layer model is used to model the functionally graded coating with arbitrarily varying shear modulus. This model divides the coating into a series of sub-layers with the elastic modulus varying linearly in each sub-layer and continues on the sub-surfaces. By using the transfer matrix method and Hankel integral transform technique, this problem is formulated as the solution of the Cauchy singular integral equations. The contact tractions are calculated by solving the equations numerically. The results show that the appropriate gradual variation of the shear modulus can significantly alter the contact tractions. Therefore, graded coatings may have potential applications in improving the resistance to fretting contact damage at the contact surfaces.

© 2012 Elsevier Ltd. All rights reserved.

1. Introduction

Functional graded materials (FGMs) are a new kind of non-homogeneous composites which consist of a gradual change in the volume fraction of constituents from one location to the other in a component. Used as coatings, it was proved that appropriate gradual variation of the elastic modulus could significantly alter the stresses around the indenter and lead to suppression of Hertzian cracking at the edge of the contact region (Giannakopoulos et al., 1997). So control of gradients in mechanical properties offers opportunities for the design of surfaces with resistance to contact deformation and damage that cannot be realized in conventional homogeneous materials (Suresh, 2001). In the past few years, some researchers paid attention to the contact problems of functionally graded materials (FGMs). Pender and Thompson (2001), Pender et al. (2001), Jorgensen et al. (1998) and Krumova et al. (2001) presented the experimental investigations on indentation testing methods to characterize the local properties of FGMs such as the elastic modulus, yield strength, strain hardening coefficient, hardness and fracture toughness, etc. Aizikovich et al. (2010) suggested the evaluation technique for the shear modulus of a functionally-graded coating by torsion experiments. Guler and Erdogan (2004) solved the contact problem of a functionally graded coating with the material properties varying as an exponential function. Recently, Ke and Wang (2006, 2007a,b,c) solved the two-dimensional frictionless and frictional contact problems using the linear multi-layered model (Wang and Gross, 2000). Sequentially they (Ke and Wang, 2007a,b, 2010) applied the linear multi-layered model to solve the two-dimensional fretting contact of functionally graded coated half-spaces. The axisymmetric problems of a graded half-space subjected to a concentrated load or to flat, spherical and conical indenters were considered by Giannakopoulos and Suresh (1997a,b). Liu and Wang applied the linear multi-layered model (Liu et al., 2008) or assumed the elastic modulus varying as the exponent function (Liu and Wang, 2008) to solve the axisymmetric frictionless contact problems of functionally graded coatings. Aizikovich et al. (2002) consider the contact problem for the impression of spherical indenter into a non-homogeneous (both layered and functionally graded) elastic half-space. Recently, they (Aizikovich et al., 2008) solved an elastic half-space with a gradient elastic coating by using the analytical-numerical method of solving the Neumann boundary-value problem.

In studies by Ke and Wang (2007a,b, 2010) the cyclic force was applied tangentially and in the contact plane. In practical cases, fretting contact may take place under the action of cyclic torque loads. This gives rise to a well-defined partial slip problem which has not gained much attention in the problem of fretting fatigue. Although this problem is attractive from an experimental point of view and may well simulate certain practical situations such as torsional loading of a bolted connection or inter-stand contact in wire ropes under tension (Hills and Nowell, 1994). Lubkin...
and stick region size are given. The contact tractions are limited by friction surrounds the adhered central disk. Deresiewicz (1954) extended the Lubkin’s solution to the case in which the normal force is held constant while the torsional couple is oscillating. Hills and Sackfield (1986) presented the complete stress field beneath an axisymmetric Hertzian contact undergoing twist. However, it can be seen from the published literature, the fretting contact problem under torsion including the effects of material non-homogeneity has not yet been studied up to present.

In this paper we will first consider the contact problem under a monotonically increasing torque, and then the fretting contact between two FGM coatings under a cyclic torque. The linear multi-layered model is used to simulate the FGM coating with arbitrarily varying shear modulus. By using the Hankel integral transform technique and transfer matrix method, the problem is reduced to Cauchy singular integral equations. The contact pressures are calculated by solving the equations numerically. The relation between the twisting torque and stick region size as well as between the twisting angel and stick region size are given. The contact tractions are calculated. The relations between the twisting torque/angel and stick region size are given.

2. Formulation of the problem

Fig. 1 shows two contact bodies with functionally graded coatings which are pressed together by an applied load \( P \) and subjected to a subsequent twisting torque \( T \). We assume that the contact bodies are indented by the normal load \( P \) to form a contact region with radius \( r \leq a \). Then, the application of the torque produces an annulus of slip. Within the slip annulus the shear stress is limited by friction. After the torque has been applied up to a maximum value \( T^* \) and leaving a stick zone with the radius of \( b \), the torque is reduced and lead to a negligible slip over a thin annulus \( c \leq r \leq b \). The fundamental solution in different stages is as follows:

2.1. Two elastic solids with graded coatings pressed together by a normal load

Consider the problem shown in Fig. 2a. An arbitrary distributed axisymmetric normal load, \( p(r) \), and radial load, \( q(r) \), acts on the surface of a functionally graded coated half-space. The coordinate system was located at the surface. The half-space is homogeneous with the shear modulus \( \mu' \). The shear modulus of the functionally graded coating can be described by an arbitrary continuous function of \( z \), \( \mu(z) \), with boundary values \( \mu(0) = \mu_0 \). In the present paper, we assume that the Poisson’s ratios for both coating and half-space are a constant with the same value, \( \nu = 1/3 \). The linear multi-layer model (Liu et al. 2008) divides the functional graded coating into \( N \) sub-layers as shown in Fig. 2b. The shear modulus \( \mu(z) \) in each sub-layer is assumed to take the following form:

\[
\mu(z) \approx \mu_j(z) = C_j(1 + z/b_j) = C_j \left( \frac{Z}{b_j} \right), \quad h_1 \leq z \leq b_{j-1}, \quad j = 1, 2, \ldots, N
\]

where \( z^* = z^*/b_0 \) and \( \mu_j \) is equal to the real value of the shear modulus at the sub-surfaces, \( z = h_j \), i.e., \( \mu(z_j) = \mu_j \), which leads to

\[
b_j = \frac{\mu_j \cdot h_j - \mu_j \cdot h_{j-1}}{\mu_j - \mu_{j-1}}, \quad C_j = \frac{\mu_j}{1 + h_j/b_j} \tag{2a, b}
\]

According to Liu et al. (2008), we obtain the normal surface displacement component, \( w_i(r) \), and the radial surface displacement component, \( u_i(r) \), as follows (see Eq. (28) in Liu et al. (2008)):

\[
w_i(r) = \int_0^\infty (W_i) J_0(sr)s ds = \int_0^\infty \{ M_{11}(s, h_0) g_i(s) + M_{12}(s, h_0) g_2(s) \} J_0(sr)s ds
\]

\[
= \int_0^a p(t)t \int_0^\infty sM_{11}(s, h_0) J_0(st) J_0(sr) ds dt + \int_0^a q(t)t \times \int_0^\infty sM_{12}(s, h_0) J_1(st) J_0(sr) ds dt
\]

\[
u_i(r) = \int_0^\infty (\bar{u}_i) J_1(sr)s ds = \int_0^\infty \{ M_{21}(s, h_0) g_i(s) + M_{22}(s, h_0) g_2(s) \} J_1(sr)s ds dr
\]

\[
= \int_0^a p(t)t \int_0^\infty sM_{21}(s, h_0) J_0(st) J_1(sr) ds dt + \int_0^a q(t)t \times \int_0^\infty sM_{22}(s, h_0) J_1(st) J_1(sr) ds dt \tag{3a, b}
\]

where

\[
\left[ \begin{array}{cc}
M_{11}(s, h_0) & M_{12}(s, h_0) \\
M_{21}(s, h_0) & M_{22}(s, h_0)
\end{array} \right] = \frac{1}{2h_0} B_3 M C,
\]

\[
M = [T_1(h_0 + b_1)][V_1][B][T_1(h_0 + b_1)][V_1]^{-1},
\]

\[
B_3 = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{bmatrix}, \quad B = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} \tag{4}
\]

‘\( M \)’ is the transfer matrix of the inhomogeneous multiple layered medium. [\( T_1(h_0 + b_1) \)] is the coefficient matrix in the first layer. [\( V_1 \)] is the product of the coefficient matrix. The superscript ‘-1’ denote the inverse of the matrix.
Considering the asymptotic behavior of the Bessel functions for large arguments (Andrews et al., 2000), one may prove
\[ \lim_{s \to \infty} \begin{bmatrix} sM_{11}(s, h_0) & sM_{12}(s, h_0) \\ sM_{21}(s, h_0) & sM_{22}(s, h_0) \end{bmatrix} = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = \begin{bmatrix} 2/(3\mu_0) & 1/(6\mu_0) \\ 1/(6\mu_0) & 2/(3\mu_0) \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \\ x_2 & x_1 \end{bmatrix}. \tag{5} \]
where \( x_0 \) is the limit of \( sM_i(s, h_0) \) when \( s \) tends to the positive infinity.

Then Eq. (3) may be written as
\[
\begin{align*}
 w_1(r) &= \sum_{j=1}^{2} \int_{0}^{a} p_j(t) t \int_{0}^{\infty} [sM_{ij}(s, h_0) - x_0] J_{j-1}(st) J_0(sr) \, ds \, dt + x_{ij} \times \int_{0}^{a} p_j(t) t \int_{0}^{\infty} H_{j-10}(r, t) \, ds \, dr,
 u_1(r) &= \sum_{j=1}^{2} \int_{0}^{a} p_j(t) t \int_{0}^{\infty} [sM_{ij}(s, h_0) - x_0] J_{j-1}(st) J_1(sr) \, ds \, dt + x_{ij} \int_{0}^{a} p_j(t) t \int_{0}^{\infty} H_{j-11}(r, t) \, ds \, dt. \tag{6a, b} \end{align*}
\]
where \( H_j(r, t) = \int_{0}^{\infty} J_j(st) J_{j-1}(sr) \, ds \), \( p_1(t) = p(t) \) and \( p_2(t) = q(t) \). Deriving Eq. (5) with respect to \( r \) and extending the definition of the unknown functions, \( p(r) \) and \( q(r) \), into the range \(-a \leq r \leq 0\), we can get
\[
\begin{align*}
 m_1(r) &= \frac{1}{2} \int_{-a}^{a} \{ p(t)[I_{11}(r, t) + q(t)[I_{12}(r, t)] \} \, dt + \frac{x_1}{\pi} \int_{-a}^{a} \frac{p(t)}{t-r} \, dt \\
 &\quad + \frac{x_1}{\pi} \int_{-a}^{a} p(t) H_1(r, t) \, dt - x_2 q(r), \tag{7a}
 m_2(r) &= \frac{1}{2} \int_{-a}^{a} \{ q(t)[I_{22}(r, t) + p(t)[I_{21}(r, t)] \} \, dt \\
 &\quad + \frac{x_1}{\pi} \int_{-a}^{a} \frac{q(t)}{t-r} \, dt + \frac{x_1}{\pi} \int_{-a}^{a} q(t) H_2(r, t) \, dt + x_2 p(r), \tag{7b}
\end{align*}
\]
where
\[
\begin{align*}
 m_1(r) &= \frac{\partial w_1(r)}{\partial r}, \quad m_2(r) = \frac{1}{r} \frac{\partial}{\partial r} (ru_1(r)), \\
 H_i(r, t) &= \left( h_i(r, t) - 1 \right) / (t-r); \\
 I_{0j}(r, t) &= (-1)^j \int_{0}^{\infty} (sM_0(s, h_0) - x_0) sJ_{j-1}(sr) J_{j-1}(st) \, ds \quad (i=1, 2; j=1, 2). 
\end{align*}
\]
write the equations as

t_1(r,t) = \begin{cases} \frac{|t/r|E(t/r)}{(t/r)^2 E(r/t)} - \frac{(t/r^2)}{E(t/r)} K(t/r), & (|t| < |r|), \\ \frac{E(t/r)}{(t/r^2 E(r/t))}, & (|t| > |r|). \end{cases}

t_2(r,t) = \begin{cases} \frac{(t/r^2)}{|(t/r)|} K(t/r) + \frac{(r/t)}{E(t/r)} E(t/r), & (|t| < |r|), \\ \frac{E(t/r)}{(t/r^2 E(r/t))}, & (|t| > |r|). \end{cases}

(8)

with \(K(\cdot)\) and \(E(\cdot)\) being the complete elliptic integrals of the first and second kinds, respectively. For a homogeneous half-space without the functionally graded coatings, the first term in Eq. (7) vanishes; then we can get the results presented in book by Hill and Nowell [1994].

As the two bodies are pressed together, deformation must occur so that the deformed bodies will conform within the contact region. For body A we have

\[ \frac{\partial \omega_i^A(r)}{\partial r} = \frac{1}{2} \int_a^a \{ p(t) | \{ t | t | p_t^A(r_t, t) + q(t) | t | q_t^A(r_t, t) \} dt + \frac{\omega_i^A}{\pi} \int_a^a \{ \frac{p(t)}{t - r} \} dt + \frac{\omega_i^A}{\pi} \int_a^a \frac{p(t)}{t - r} H(t) \{ t | H(r) \} dt \]  

\[ + \frac{\omega_i^A}{\pi} \int_a^a \frac{q(t)}{t - r} H(t) \{ t | H(r) \} dt \]  

(9a, b)

Similar arguments apply to processing of body B, and we can now write the equations as

\[ \frac{\partial \omega_i^B(r)}{\partial r} = \frac{1}{2} \int_a^a \{ p(t) | \{ t | t | p_t^B(r_t, t) + q(t) | t | q_t^B(r_t, t) \} dt + \frac{\omega_i^B}{\pi} \int_a^a \{ \frac{p(t)}{t - r} \} dt \]  

\[ + \frac{\omega_i^B}{\pi} \int_a^a \frac{q(t)}{t - r} H(t) \{ t | H(r) \} dt \]  

(10a, b)

Fig. 3. Stick/slip arrangement in the contact region during contact of two similar elastic solids (a) after normal loading alone is applied, (b) after applying a Torque, (c) after reducing infinitesimally torque, (d) after reducing a finite torque.

\[ F_2(r) = w_2p(r) + \frac{\omega_i^A}{\pi} \int_a^a \{ \frac{p(t)}{t - r} \} dt + \frac{\omega_i^A}{\pi} \int_a^a \{ q(t)H(r) \} dt \]  

\[ + \frac{1}{r} \int_a^a \{ q(t) | t | q_t^A(r_t, t) + p(t) | t | p_t^A(r_t, t) \} dt \]  

(11a, b)

where

\[ F_1(r) = \frac{\omega_i^A}{\pi} \int_a^a \{ \frac{p(t)}{t - r} \} dt, \quad F_2(r) = \frac{1}{r} \int_a^a \{ q(t)H(r) \}, \]  

(12)

with \( F_1(r) \) and \( F_2(r) \) being the normal and radial relative displacements components, respectively, and \( E(\cdot) \) being the complete elliptic integrals of the first and second kinds, respectively.

2.2. Application of a monotonically increasing torque

Next, we assumed two bodies are pressed together by normal load \( P \) to form an adhesion region with radius \( r < a \) as shown in Fig. 3a. A monotonically increasing torque \( T \) is applied to produce the annulus of slip. The linear multi-layer model is employed. The shear modulus in each sub-layer is assumed to take the linear form:

\[ \mu(z) \approx \mu_j(z) = \mu_j(\alpha_j + a_j z), \quad h_j < z < h_{j+1}, \quad j = 1, 2, \ldots, N \]  

(13)

where \( \alpha_j = c_1/\mu_j, \quad a_j = c_1/(\mu_j h_j) \), and \( \mu_j \) is equal to the real shear modulus at the sub-interface \( z = h_j \), i.e., \( \mu_j = \mu(h_j) \).

So we have

\[ a_j = h_{j+1} - h_j/\mu_{j+1} \mu_j, \quad a_j = h_{j+1} - h_j/\mu_{j+1} \mu_j. \]  

(14a, b)

Making the use of the Hankel inverse transformation and the transfer matrix method, we can obtain the surface displacement component (Liu and Wang, 2009)

\[ u_{01}(r, h_0) = \int_0^a \tau(t) \int_0^a sN(s, h_0)J_1(t s)J_1(r s) ds \]  

(15)

where \( N(s, h_0) = B_2[T(\{ h_0 \})] | V_1 | [B_1[T(\{ h_0 \})] | V_1 |]^{-1} \) (see Eq. (23) in Liu and Wang (2009)) with \( B_2 = [1, 0] \).

Considering the asymptotic behavior of \( \lim_{a \to \infty} sN(s, h_0) = 1/\mu_j \).

Then (15) may be written as

\[ u_{01}(r, h_0) = \int_0^a \tau(t)I(r, t) dt + 1/\mu_j \int_0^a \tau(t)H_1(r, t) dt \]  

(17)

where

\[ I(r, t) = \int_0^\infty (sN(s, h_0) - 1/\mu_j)J_1(t s)J_1(r s) ds \]  

\[ H_{11}(r, t) = \int_0^\infty J_1(t)J_1(s r) ds = \begin{cases} \frac{1}{2} | K(\cdot) - E(\cdot) |, \quad (t < r) \\ \frac{1}{2} | K(\cdot) - E(\cdot) |, \quad (t > r) \end{cases} \]  

(18)

\[ f(r) = \int_0^a \tau(t)Q(r, t) dt + \frac{1}{\pi \mu_j} \int_0^a \tau(t) \]  

(19)
where
\[ Q(r, t) = tf'(r, t) + \frac{1}{\pi \mu_0} \left( -t - r + 2\theta_2(r, t) \right). \]
\[ f(r) = \frac{1}{r} \frac{\partial (tu_0(r, h_0))}{\partial r}. \]  
(20)

with
\[ I'(r, t) = \int_0^\infty (sN(r, h_0) - 1/\mu_0) \beta(t) \beta(t) \, ds. \]
\[ h_2(r, t) = \begin{cases} tE(t/r) + \frac{\nu t^2}{r^2} K(t/r) & t < r \\ E(t/r) & t > r. \end{cases} \]
(21)

For body A we have
\[ f_a(r) = \frac{1}{\pi \mu_0} \int_0^a \frac{\tau(t)}{t-r} \, dt + \int_0^a \tau(t) Q_A(r, t) \, dt. \]
(22)

Similar arguments apply to the processing of body B; and we can write
\[ f_b(r) = -\frac{1}{\pi \mu_0} \int_0^a \frac{\tau(t)}{t-r} \, dt + \int_0^a \tau(t) Q_B(r, t) \, dt. \]
(23)

Finally, we can get
\[ F(r) = \frac{1}{\pi \mu_0} \int_0^a \frac{\tau(t)}{t-r} \, dt + \int_0^a \tau(t) Q_{AB}(r, t) \, dt, \]
where
\[ F(r) = f_a(r) - f_b(r), \quad Q_{AB}(r, t) = Q_A(r, t) - Q_B(r, t) = t(h_2(r, t) - h_1(r, t)) + \frac{2}{\pi \mu_0} \left( -t - r + 2\theta_2(r, t) \right). \]
(25)

Because the normal stress on the contact surface is zero as \( r \to a \), slip must be expected to occur, starting at the boundary \( r = a \) and presumably progressing inward to a radius \( r = b < a \) (Hills and Nowell, 1994). In the circular annulus \( b < r < a \), the circumferential component of the traction is assumed to fall to the maximum available (Fig. 3b). This value is taken as the product of a constant friction coefficient \( f \) and the pressure \( p(r) \). Then we have (Lubkin, 1951)
\[ \tau(r) = fp(r), \quad b \leq r \leq a. \]
(26)

We represent the circumferential shear traction as
\[ \tau(r) = \tau'(r) + fp(b)b, \quad r < b, \]
where \( \tau'(r) \) is an unknown function which vanishes at \( r = 0 \) and \( b \). Substituting Eqs. (26) and (27) into Eq. (24), we get
\[ \frac{1}{\pi \mu_0} \int_0^b \frac{\tau'(t)}{t-r} \, dt + \int_0^b \tau'(t) Q_{AB}(r, t) \, dt = U(r), \]
(28)

where
\[ U(r) = F(r) - \frac{1}{\pi \mu_0} \int_0^b \frac{fp(t)}{t-r} \, dt - \int_0^b \frac{fp(t)}{t-r} Q_{AB}(r, t) \, dt - fp(b) \frac{1}{\pi \mu_0} \int_0^b \frac{t}{t-r} \, dt - \frac{fp(b)}{b} \int_0^b t Q_{AB}(r, t) \, dt. \]
(29)

2.3. Application of a cyclic torque

Assume the cyclic torque varies between the limits \( \pm T' \) as shown in Fig. 3. Base on the above analysis, we suppose that a steadily increasing torque has been applied up to a maximum value \( T' \), leaving a stick zone with the radius of \( b \). The torque is now reduced infinitesimally. This will lead to an opposite direction surface displacement in the slip annulus and give rise to adhesion over the entire contact as shown in Fig. 3c. A further reduction in the torque to a value \( T \) now permits back slip over a thin annulus \( c \leq r < a \) (Fig. 3d). In this annulus, \( \tau(r) = -fp(r) \) so that the change of the tangential traction is \(-2fp(r)\). Since no additional slip occurs in the region of \( r < c \), the change of the displacement therein must be equal to that of the rigid-body rotation. Thus the change in the traction due to the reduction of \( T \) is
\[ \tau_c(r) = \begin{cases} 2fp(r), & c < r \leq a \\ 2\left( \tau'(r) + fp(c) \right), & r < c. \end{cases} \]
(30)

where \( \tau' \) satisfies
\[ \frac{1}{\pi \mu_0} \int_0^c \frac{\tau'(t)}{t-r} \, dt + \int_0^c \tau'(t) Q_{AB}(r, t) \, dt = U_c(r), \]
(31)

with
\[ U_c(r) = F(r) - \frac{1}{\pi \mu_0} \int_0^c \frac{fp(t)}{t-r} \, dt - \int_0^c \frac{fp(t)}{t-r} Q_{AB}(r, t) \, dt \]
\[ - \frac{fp(c)}{c} \frac{1}{\pi \mu_0} \int_0^c \frac{t}{t-r} \, dt - \frac{fp(c)}{c} \int_0^c t Q_{AB}(r, t) \, dt. \]
(32)

The resultant traction denoted by \( \tau_c(r) \) accompanying a reduction in \( T \) is obtained by adding the initial traction \( \tau'(r) \), Eq. (19), to the change \( \tau_c(r) \), resulting in
\[ \tau(r) = \begin{cases} -fp(r), & c < r \leq a \\ fp(r) - \tau_c(r), & b < r < c \\ \tau(r) - \tau_c(r), & r < b. \end{cases} \]
(33)

3. Solution of singular integral equations

If two contact bodies are elastically similar, they will initially adhere over the entire contact (Hills and Nowell, 1994). We consider the contact between two elastically similar bodies which consist of the same homogeneous materials coated by the same FGM layers with the same thickness \( h_1 = h_2 = h \). Then, we will find
\[ \frac{\partial \psi_1}{\partial r} = -\frac{\partial \psi_2}{\partial r}, \quad \frac{1}{r} \frac{\partial \psi_2}{\partial r} = \frac{1}{r} \frac{\partial \psi_1}{\partial r} \]
(34a, b, c, d)
\[ f_1 = -f_1, \quad f_2 = f_2. \]
(35a, b, c, d)

Therefore, Eq. (11) reduces to
\[ F_1(r) = \frac{\psi_1}{\pi} \int_a^b \frac{p(t)}{t-r} \, dt + \frac{\psi_1}{\pi} \int_a^b p(t) H_1(t, r) \, dt \]
\[ + \frac{1}{2} \int_a^b p(t) |t|^2 H_1(t, r) \, dt. \]
(36a, b)

It is seen that Eqs. (36a) and (36b) are uncoupled. Thus the solutions for the normal tractions may be obtained independently. It is understood that there is no relative radial displacement between the same contacting bodies and thus the radial shearing traction is zero. This is exactly the case we are considering here (i.e., the contact between two same homogeneous half-spaces coated by the same FGM layers with the same thickness). In addition, we should ensure the equilibrium with the external force, \( P \), by writing
Express and the equation corresponding to knowns in Eq. (39). In practice because it is trivial. Finally the present results and Lubkin's results (1951). There are

\[ N \]

Table 1

Comparison of \( T \) and \( \rho \) related to the stick region (b/a) with the comparison between the present results and Lubkin's results (1951).

<table>
<thead>
<tr>
<th>b/a</th>
<th>( \mu^2 \sigma_b \rho / P )</th>
<th>( T / P a )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Lubkin's</td>
<td>Present</td>
</tr>
<tr>
<td>0.2</td>
<td>0.4692</td>
<td>0.4719</td>
</tr>
<tr>
<td>0.3</td>
<td>0.3656</td>
<td>0.3665</td>
</tr>
<tr>
<td>0.4</td>
<td>0.2885</td>
<td>0.2887</td>
</tr>
<tr>
<td>0.5</td>
<td>0.2257</td>
<td>0.2250</td>
</tr>
<tr>
<td>0.6</td>
<td>0.1716</td>
<td>0.1713</td>
</tr>
<tr>
<td>0.7</td>
<td>0.1234</td>
<td>0.1231</td>
</tr>
<tr>
<td>0.8</td>
<td>0.0794</td>
<td>0.0791</td>
</tr>
<tr>
<td>0.9</td>
<td>0.0385</td>
<td>0.0380</td>
</tr>
</tbody>
</table>

\[ \text{Fig. 4. Torque loading history.} \]

\[ P = \pi \int_{-a}^{a} p(t)|t| \, dt. \]  

(37)

To solve Eq. (36a), we use the method developed by Civelek (1972). Express \( p(x) \) as

\[ p(x) = f_1(x) \sqrt{1 - x^2}, \]  

(38)

where \( f_1(x) \) is an unknown function. Then (36a) can be cast to a discretized form

\[ \frac{1}{a} \frac{a y_l}{x_l} \sum_{k=1}^{N} \left( \frac{1-x_l^2}{N+1} \right) f_1(x_l) \left[ \frac{1}{2} \log |x_k - y_l| + \frac{1}{a} L(y_l, x_l) \right] + \frac{\phi[f(x_{1:1}), f(x)]}{2y_l}, \]  

(39)

where

\[ x_k = \cos[k\pi/(N+1)], \quad k = 1, 2, \ldots, N \]

\[ y_l = \cos[(2l-1)/2(N+1)], \quad l = 1, 2, \ldots, N, N+1 \]

\[ L(y_l, x_l) = \frac{q^2}{2} |x_l f_1(y_l, x_l) + \frac{1}{\pi} \left( \frac{1}{y_l} - \log |x_k - y_l| - \log |x_k - y_l| \right) \]

\[ \phi[f(x_{1:1}), f(x)] = \frac{f(x_{1:1})}{\pi} \left( (1-x_l^2)^{1/2} \log |1-y_l^2|^{1/2} - \eta \right) \]

\[ - \eta \left( 1-x_l^2 \log |x_k - y_l| \right) \]

\[ + \frac{f(x)}{\pi} \left( (1-y_l^2)^{1/2} \log |1-y_l^2|^{1/2} - \eta \right) \]

\[ - \eta \left( 1-x_l^2 \log |x_k - y_l| \right) \]

(40)

There are \( N_1 + 1 \) possible collocation points to determine \( N_1 \) unknowns in Eq. (39). In practice \( N_1 \) will be chosen as an even integer and the equation corresponding to \( I = N_1/2 + 1 \) will be ignored because it is trivial. Finally \( P \) is easily computed by Eq. (37).

\[ \text{Fig. 5. (a) Distribution of the normal contact traction when } P/\mu h^2 = 4.0 \times 10^{-4} \text{ in the contact radius for some selected values of } \mu_0 \mu \rho; (b) the normal load } P \text{ versus the contact radius } a \text{ for some selected values of } \mu_0 \mu \rho. \text{ The lines are from the present model and the scattered symbols from the exponential model.} \]

For the present case of the contact between two same elastic bodies, we have

\[ f_A(r) = -f_B(r), \quad f_A'(r, t) = -f_B'(r, t) = f'(r, t), \]

\[ Q_A(r) = -Q_B(r) = Q(r, t). \]  

(41)

According to Lubkin (1951), in the central adhesion zone \( r < b \) particles will move tangentially as a rigid body with \( \mu \beta \) (where \( \beta \) is the unknown torsion angle). Then, Eq. (28) reduces to

\[ \frac{1}{\pi \mu_0} \int_{0}^{b} \frac{\tau^*(t)}{t - r} \, dt + \int_{0}^{b} \tau^*(t) Q(r, t) \, dt = U(r), \]  

(42)

where

\[ U(r) = 2\beta - \frac{1}{\pi \mu_0} \int_{0}^{b} \frac{fp(t)}{t - r} \, dt - \int_{0}^{b} \frac{fp(t)}{b} Q(r, t) \, dt \]

\[ - \frac{fp(b)}{b} \frac{1}{\pi \mu_0} \int_{0}^{b} \frac{t - r}{t - r} \, dt - \frac{fp(b)}{b} \int_{0}^{b} r Q(r, t) \, dt. \]  

(43)

If we employ the substitutions

\[ t = (1 + \eta)b/2, \quad r = (1 + \zeta)b/2. \]  

(44a, b)

then Eq (42) may be rewritten as

\[ \frac{1}{\pi \mu_0} \int_{-1}^{1} \frac{1}{\eta - \zeta} \, d\eta + \int_{1}^{1} b \frac{1}{2} \tau^*(\eta) Q(\zeta, \eta) \, d\eta = U(\zeta). \]  

(45)
where
\[ U(r) = 2\beta - \frac{1}{\pi \mu_0} \int_b^a \frac{f_p(t)}{t-r} \, dr - \int_b^a f_p(t)Q_2(r,t) \, dr \]
\[ - \frac{f_p(b)}{b} \frac{1}{\pi \mu_0} \int_b^r \frac{t}{t-r} \, dt - \frac{f_p(b)}{b} \int_b^r tQ_2(r,t) \, dt. \] (46)

Finally, the equilibrium condition states
\[ T = \int_0^{2\pi} \tau(r)r^2 \, d\theta = 2\pi \int_0^a \tau(r)r^2 \, dr \]
\[ = 2\pi \int_0^b \tau'(r)r^2 \, dr + 2\pi \int_b^a \frac{f_p(b)}{b} \int_0^r r^2 \, dr + 2\pi \int_b^a \tau'(r)r^2 \, dr \]
\[ = \frac{\pi b^3}{4} \int_1^a \tau'((\eta + 1)^2 \, d\eta + \frac{\pi f_p(b)b^3}{2} + 2\pi \int_b^a \tau'(r)r^2 \, dr. \] (47)

After \( p(r) \) at a given contact radius \( a \) are known, we can use the Erdogan-Gupta's method (Erdogan and Gupta, 1972) to solve Eq (45). Express \( \tau'((\eta) \sqrt{1 - \eta^2} \). (48)

where \( f_2(\eta) \) is an unknown function. Then we get
\[ \sum_{n=1}^{N_2} \frac{\pi(1 - \eta_m^n)}{N_2 + 1} f_2(\eta_m) \left[ \frac{1}{\eta_m - \xi_n} + \frac{b}{2} \mu_0 Q(\xi_n, \eta_m) \right] = \mu_0 U(\xi_n). \] (49)

where \( \eta_m = \cos \left[ \frac{\pi n}{N_2 + 1} \right], \quad \xi_n = \cos \left[ \frac{\pi(2n - 1)(N_2 + 1)}{2(N_2 + 1)} \right], \]
\( n = 1, 2, \ldots, N_2 + 1 \) and \( N_2 \) is the total number of the discrete points of \( f_2(\eta_m) \). If the stick region \( b \) is given, then there are \( N_2 + 1 \)
equations for $N + 1$ unknowns $f_2(g_1), f_2(g_2), \ldots, f_2(g_N)$ and $\beta$. The torque $T$ is easily obtained from Eq. (47).

Similarly, Eq. (31) is described by

$$
\frac{1}{\pi \mu_0} \int_{-\eta}^{\eta} \frac{\tau^{(\eta)}}{\eta - \xi} d\eta + \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{c}{2} \tau^{(\eta)} Q_2(\xi, \eta) d\eta = U_c(\xi), 
$$

with

$$
U_c(r) = 2\beta - \frac{1}{\pi \mu_0} \int_{-T}^{T} \frac{fp(t)}{t - r} dt - \int_{-T}^{T} \frac{fp(t)}{t - r} dt Q_2(r, t) dt 
- \frac{fp(c)}{c} \int_{-T}^{T} \frac{t}{t - r} dt - \frac{fp(c)}{c} \int_{-T}^{T} t Q_2(r, t) dt.
$$

We can use the Erdogan-Gupta’s method to solve Eq. (50). Finally, from Eqs (45) and (50) we obtain the tangential traction $\tau(r)$ (Eq. (33)) under the action of a cyclic torque.

Fig. 9. Distribution of the shear traction during cycled torsional loading for some selected values of $\mu_0/\mu$; (a) for $T = T^*$, corresponding to point A in Fig. 3; and (b) for $T = 0.45T^*$, corresponding to point B in Fig. 3; and (c) for $T = 0$, corresponding to point C in Fig. 3; and (d) for $T = -0.45T^*$, corresponding to point D in Fig. 3; and (e) for $T = -T^*$, corresponding to point E in Fig. 3. The lines are from the present model and the scattered from the direct solution.
4. Numerical results and discussion

Numerical results will be presented in this section. In calculation, we take 30 collocation points, i.e., \( N_1 = N_2 = 30 \) in Eqs. (45) and (50). As a check of the method, we first calculate the contact stress on the surface of a homogeneous half-space by a rigid spherical indenter using the present model for comparison with Lubkin’s (1951) analytical results. The results for \( R/h = 10 \) are listed in Table 1 which shows that the present results are in good agreement with those obtained by Lubkin (1951).

Then we present the numerical results for the case of a graded coatings loaded by a spherical indenter under an applied normal load \( P \) and a torque \( T \) (Fig. 4). We first consider the shear modulus of the coating varying in the exponential form:

\[
\mu(z) = \mu_0 e^{az},
\]

where \( a = (1/h) \log(\mu_0/\mu^*) \). The merit of doing so lines in fact that for such a form of material properties, axisymmetric contact problem can be solved in a way similar to that used by Liu and Wang (2008) (the exponential model) and thus comparison can be made between the results obtain by the present linear multi-layered model and the exponential model so as to check the correctness of the present method. Liu et al. (2008) discussed how many sub-layers are necessary to obtain the sufficiently accurate results and indicated that \( N = 4 \) or 6 works well. So in all numerical examples, we choose \( N = 6 \), that is, divide the coating into six sub-layers. It should be indicated that all computations are performed by fixing \( R/h = 10 \).

Fig. 5a shows the effects of the stiffness ratio \( \mu_0/\mu^* \) on the normal contact traction distribution under the normal load \( P/\mu^* h^2 = 4.0 \times 10^{-4} \). Note that the contact radius \( a/h \) is equal to 0.140, 0.0998 or 0.0702 corresponding to \( \mu_0/\mu^* = 1/3, 1 \) or 3, respectively. With the decrease of \( \mu_0/\mu^* \), the normal contact traction distribution becomes more even. The relations of \( P \) versus \( a \) for the spherical indenter are depicted in Fig. 5b for some selected values of the stiffness ratio \( \mu_0/\mu^* \). It is shown that, with the increase of \( \mu_0/\mu^* \), a larger applied normal load is needed to create the same contact region.

Fig. 6 demonstrates the effects of the stiffness ratio \( \mu_0/\mu^* \) on the relation between a monotonically applied torque \( T \) and \( b/\alpha \) (Fig. 6a) as well as between the stick region \( b/\alpha \) and the twisting angle \( \beta \) (Fig. 7b) for \( a/h = 0.1 \). It is shown that, with the increase of \( \mu_0/\mu^* \), the smaller twisting torque and twisting angle are needed to create the same stick region size \( b/\alpha \), if the contact radius is fixed. It is noted that the stick region will shrink to zero and slipping will take place over the entire contact region when \( T \) is large enough.

Fig. 8 presents the effects of the stiffness ratio \( \mu_0/\mu^* \) on the relation between the stick region size \( c/\alpha \) and the twisting torque \( T \) as
well as between the twisting angle and stick region size for some selected values of the stiffness ratio $\frac{l_0}{l}$ with $a/h = 0.1$ under a decreasing torque after the stick region increases to $b/a = 0.4$ by a monotonically increasing torque. It is shown that, when the torque is completely reversed from $T$ to $-T$ and the twist angle is completely reversed from $\beta$ to $-\beta$, the stick region $c/a$ decreases from 1 to 0.4. Fig. 5 also shows that the effects of the stiffness ratio $\frac{l_0}{l}$ on the relation between a applied torque $T$ and $b/a$ (Fig. 8a) as well as between the stick region ($b/a$) and the twisting angle $\beta$ (Fig. 8b) are the same as in the case of a monotonically applied torque.

Fig. 9 illustrates the effects of the stiffness ratio $\frac{l_0}{l}$ on the shear traction for the initial 3/4 cycle of the torque load (see Fig. 4) including the first loading and the subsequent unloading stages when $P/\mu^2 = 4.0 \times 10^{-4}$. We can observe that the peak circumferential shear traction appears at the interface of stick/slip and the circumferential shear tractions are zero at the center and edge of the contact region. With the increase of $\frac{l_0}{l}$, the magnitude of the shearing traction increases and becomes more uneven in the stick region. Fig. 9 also shows that the stick region decreases with the increase of $\frac{l_0}{l}$. Fig. 9a shows the effects of the stiffness ratio $\frac{l_0}{l}$ on the shear traction distribution for $T = T^*$ corresponding to point A in Fig. 4. Fig. 9b–e plot the effects of the stiffness ratio $\frac{l_0}{l}$ on the shear traction distribution corresponding to points B, C, D and E in Fig. 4, respectively. Firstly, we can find that reverse slip near the edges of the contact occurs during unloading, where the circumferential shear traction becomes $q(r) = -fp(r)$. With the decrease of $T$, both reverse slip region and reverse circumferential shear traction increase, but the circumferential shear traction in the stick region decreases. Comparing Fig. 9a and e, we find that, when the load is completely reversed from $T^*$ to $-T^*$, the circumferential shear traction distribution and the slip region are also completely reversed. We can observe that the position of the peak circumferential shear traction distribution in the stick region is unchanged during the unloading stage. An important phenomenon is also observed in Fig. 9a–e, that is, the localized maximum or minimum of the shear traction varies as $\frac{l_0}{l}$ changes. This behavior provides a way for us to change the distribution of the contact traction by adjusting the stiffness ratio of the coating surface.

Using the exponential model, we examine the effects of Poisson ratio $\nu$ on the normal and circumferential shear traction distributions for $P/\mu^2 = 4.0 \times 10^{-4}$ and $T = T^*$ (corresponding to point A in Fig. 4). Fig. 10 shows that normal and circumferential shear traction distributions slightly increase with the Poisson ratio $\nu$ increasing.

Next we consider the shear modulus of the coating varying in the following form:

$$\mu(z) = \mu_0 + (\mu_c - \mu_0) |z/h|^n,$$

where $n$ is a positive constant characterizing the gradual variation of the shear modulus. In the following calculation we use the present model to solve the contact problems by choosing the number of the sub-layer to be 6.
corresponding to points A (Fig. 13a) and D (Fig. 13b) in Fig. 3 under stick region size twisting torque and twisting angle are needed to create the same facts can be found from the numerical results: For us to change the distribution of the contact traction by adjusting the gradient of the coating while remaining the stiffness of the coating surface unchanged.

5. Concluding remarks

In this paper, we solve fretting contact between two elastic solids with graded coatings which are subjected to a normal force, \( P \), and a cyclic twisting torque, \( \tau \). The linear multi-layered model and the exponential model are used to model the functional graded materials with arbitrarily varying shear modulus. The following facts can be found from the numerical results:

1. The smaller twisting torque and twisting angle is needed to create the same stick region size \( b/a \) with the increase of \( \mu_0 / \mu^* \) or decrease of gradient parameter \( n \), if the contact radius is fixed.

2. The stiffness ratio, \( \mu_0 / \mu^* \), has a significant effect on the distribution of the circumferential shear traction. With the increase of \( \mu_0 / \mu^* \), the magnitude of the shearing traction increases and becomes more uneven in the stick region. The distribution of contact stress can be improved by adjusting the gradient of the coating.

3. The results also provide us a method to modify the distribution of the contact traction by adjusting the gradient of the coating even remaining the stiffness of the coating surface unchanged.

Finally, the results of the numerical solution provided a guidance for the fretting experiment under torsion.

Acknowledgements

Support by China National Natural Science Foundation under Grant Nos. 10572019 and 11002065 are gratefully acknowledged. The second author is grateful to the support of NSFC under Grant No. 10872026.

References