Hilbert functions, Hilbert–Samuel quasi-polynomials with respect to $f$-good filtrations, multiplicities

H. Dichi$^a$*, D. Sangaré$^b$

$^a$Université Blaise Pascal, Mathématiques pures, Complexe Scientifique des Cézeaux, 63177 Aubière Cedex, France
$^b$IUFM de Bourg en Bresse, 40 rue Général Delestraint, BP 153, 01004 Bourg en Bresse, France

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Abstract

In this paper we prove a generalization of the Hilbert Theorem in terms of quasi-polynomial functions. As an application we give an extension of the Hilbert–Samuel Theorem to finitely generated $A$-modules filtered by $f$-good filtrations, where $f$ is a noetherian filtration on a noetherian ring $A$. We deduce a multiplicity function on the category of finitely generated $A$-modules with values in the set $\mathbb{Q}$ of rational numbers and we show that this multiplicity behaves like that of ideals. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

The Hilbert function of a graded $A$-module $M$, its generating function and the Hilbert polynomial of $M$ have been extensively studied since the famous paper of Hilbert: Über die Theorie der algebraischen Formen, Math. Ann. 36 (1890) 473–534. In particular the coefficients and the degree of the Hilbert polynomial of $M$ play an important role in Algebraic Geometry. The recent development of Computer Science have enhanced the importance of this subject. The purpose of this paper is, in a different direction, to extend the Hilbert–Samuel Theorem to finitely generated $A$-modules filtered by $f$-good filtrations, where $f$ is a noetherian filtration on a noetherian ring $A$. This

* Corresponding author. E-mail: dichi@ucfma.univ-bpclermont.fr.
enables us to define a multiplicity function on the category of finitely generated \( A \)-modules with values in the field \( \mathbb{Q} \) of rational numbers which behaves like that of ideals. The main result of Section 2 is Theorem 2.7 which is a generalization of the Hilbert Theorem in terms of quasi-polynomial functions. In Section 3, we give at the beginning preliminary results on modules which are filtered by \( f \)-good filtrations, a concept which was introduced by Ratliff and Rush [6] and which is a generalization of the concept of \( I \)-good filtration. This notion has proved to be a useful tool to answer many questions in the theory of filtrations. We prove Theorem 3.4 which is the main result by applying Theorem 2.7 to associated graded modules with respect to \( f \)-good filtrations. This paper ends with Theorem 3.10 which is the semi-local version of Theorem 3.4.

2. Hilbert functions of graded modules

Definition 2.1. Let \( A = \bigoplus_{n > 0} A_n \) be a positively graded noetherian ring. Then \( A_0 \) is a noetherian ring and \( A \) is a finitely generated \( A_0 \)-algebra of the form \( A = A_0[x_1, \ldots, x_r] \), where each \( x_i \) is homogeneous of degree \( k_i \geq 1 \). Let \( M = \bigoplus_{n \in \mathbb{Z}} M_n \) be a finitely generated positively graded \( A \)-module. Then each homogeneous component \( M_n \) of \( M \) is a finitely generated \( A_0 \)-module. From here, we will assume that \( A_0 \) is an artinian ring. Let \( l_{A_0}(\cdot) \) denote the length function on the \( A \)-modules. Then \( l_{A_0}(M_n) \) is finite for all \( n \).

(1) The numerical function \( H(M, \cdot) : \mathbb{Z} \to \mathbb{Z} \) defined as \( H(M, n) = l_{A_0}(M_n) \) for all \( n \in \mathbb{Z} \), is called the Hilbert function of \( M \). We put \( H^*(M, n) = \sum_{j \leq n} H(M, j) \) for all \( n \). The numerical function \( H^*(M, \cdot) \) defined by the above formula is called the cumulative Hilbert function of \( M \).

(2) A numerical function \( \varphi : \mathbb{Z} \to \mathbb{Z} \) is said to be of polynomial type of degree \( d \) if there exists a polynomial \( F \in \mathbb{Q}[X] \) of degree \( d \) such that \( \varphi(n) = F(n) \) for all \( n \gg 0 \), with the convention that the zero polynomial has degree \(-1\).

(3) Given a numerical function \( \varphi : \mathbb{Z} \to \mathbb{Z} \), the difference operator \( \Delta \) is defined by \( \Delta \varphi(n) = \varphi(n + 1) - \varphi(n) \) for all \( n \in \mathbb{Z} \). We set \( \Delta^0 \varphi = \varphi \) and for \( i \geq 1 \), \( \Delta^i = \Delta \circ \Delta \circ \Delta \circ \cdots \circ \Delta \), \( i \) times. We omit the proof of the following Lemma which is similar to Lemma 4.1.2 of Bruns and Herzog [3].

Lemma 2.2. Let \( \varphi : \mathbb{Z} \to \mathbb{Z} \) be a numerical function and \( d \geq 0 \) an integer. Then the following assertions are equivalent:

(i) \( \Delta^d \varphi(n) = c \), \( c \) is a constant, for all \( n \gg 0 \).

(ii) \( \varphi \) is of polynomial type of degree \( \leq d \).

The following theorem is well known, see for example [3], Theorem 4.1.3 and Remark 4.1.6.

Theorem 2.3 (Hilbert). Let \( A = \bigoplus_{n \geq 0} A_n \) be a positively graded noetherian ring. Assume that \( A = A_0[x_1, \ldots, x_r] \), where each \( x_i \) is homogeneous of degree 1 and that \( A_0 \)
is an Artinian local ring. Let $M = \bigoplus_{n \in \mathbb{Z}} M_n$ be a finitely generated positively graded $A$-module of dimension $d$. Then the Hilbert function $H(M, -)$ (resp. the cumulative Hilbert function $H^*(M, -)$) is of polynomial type of degree $d - 1$ (resp. $d$).

**Remark 2.4.** With the same notation as in 2.1, if for some $i = 1, 2, \ldots, r$, $k_i = \deg x_i \geq 2$, then the above Theorem is no longer true, see Example 3.7. But in this case the functions $H(M, -)$ and $H^*(M, -)$ still agree with periodic polynomials in some sense as shown in what follows.

**Definition 2.5.** Let $k$ and $d$ be integers with $k \geq 1$ and $d \geq -1$. A function $\varphi : \mathbb{Z} \to \mathbb{Z}$ is called a quasi-polynomial function of period $k$ and degree $d$ if there exists a necessarily unique sequence $F = (F_0, F_1, \ldots, F_{k-1})$ of polynomials $F_j \in \mathbb{Q}[X]$ such that for $j = 0, 1, \ldots, k - 1$,

(i) $\varphi(n) = F_j(n)$ for all $n \gg 0$ with $n \equiv j \pmod{k}$ and

(ii) $\max_{0 \leq j \leq k-1} \deg F_j = d$.

$F$ is called the quasi-polynomial associated with $\varphi$ and $d$ is also called the degree of $F$. It may so happen that all the polynomials $F_j$ have the same leading coefficient in which case it is called the leading coefficient of the quasi-polynomial $F$.

The following Theorem is a generalization of the Hilbert’s Theorem 2.3. A different proof can be found in [1], Theorem 2.2. In the first part of our theorem, we give a slight improvement of the proof of Theorem 4.1.3 of [3] since we do not assume the ring $A_0$ to be local. The advantage of our theorem relatively to Theorem 2.2 of [1] is that we express the degree of the quasi-polynomial associated with $H(M, -)$ (resp. $H^*(M, -)$) in terms of the Krull dimension $M$. In our proof, we will use the following lemma which is a graded version of the well known result. We give the proof of this lemma for the reader’s convenience.

**Lemma 2.6.** Let $A = \bigoplus_{n \geq 0} A_n$ be a positively graded noetherian ring of finite dimension $d$ and let $x$ be a nonzero divisor homogeneous element of $A$ such that $\deg x > 0$. Then $\dim A/xA = d - 1$.

**Proof.** Since $x$ is a nonzero divisor in $A$, $x$ does not belong to any minimal prime ideal of $A$. Consequently, $\dim A/xA \leq d - 1$. Conversely, since $A$ is noetherian, there exists a graded prime ideal $\varnothing$ of $A$ such that $d = \dim A/\varnothing$. Actually, $\varnothing$ is a maximal graded ideal. Set $\varnothing_0 = \varnothing \cap A_0$ and $A_+ = \bigoplus_{n \geq 1} A_n$. Then $\varnothing = \varnothing_0 \oplus A_+$. Since $\deg x > 0$, $x \in \varnothing$. We have $\dim A/xA \geq \dim (A/xA)_{\varnothing/\varnothing_0} = \dim A_+/xA_+ = \dim A_+ - 1$, the last equality coming from [2], Ch. 8, Section 3, Corollary 2, since $(A/\varnothing, \varnothing A/\varnothing)$ is a noetherian local ring such that $\frac{1}{x}$ is a nonzero divisor in $A_+\varnothing$, and hence the conclusion follows. \qed

**Theorem 2.7.** Let $A = \bigoplus_{n \geq 0} A_n$ be a positively graded noetherian ring of finite Krull dimension. Assume that $A = A_0[x_1, \ldots, x_r]$, where each $x_i$ is homogeneous of degree
\(k_i \geq 1\) and that \(A_0\) is an artinian ring. Let \(M = \bigoplus_{n \in \mathbb{Z}} M_n\) be a finitely generated positively graded \(A\)-module and let \(d = \dim M\). If \(k - \text{LCM}(k_1, k_2, \ldots, k_r)\) then:

1. The Hilbert function \(H(M, -)\) of \(M\) is a quasi-polynomial function of period \(k\) and degree \(d - 1\). More precisely, there exists a sequence \(F = (F_0, F_1, \ldots, F_{k-1})\) of polynomials \(F_j \in \mathbb{Q}[X]\) such that
   
   (i) \(H(M, n) = H_j(n)\) for all \(n \gg 0\) with \(n \equiv j \pmod{k}\),
   
   (ii) \(\max_{0 \leq j \leq k-1} \deg F_j = d - 1 = \deg F_0\)

2. The cumulative Hilbert function \(H^*(M, -)\) of \(M\) is a quasi-polynomial function of period \(k\) and degree \(d\). More precisely there exists a sequence \(G = (G_0, G_1, \ldots, G_{k-1})\) of polynomials \(G_j \in \mathbb{Q}[X]\) such that
   
   (i) \(H^*(M, n) = G_j(n)\) for all \(n \gg 0\) with \(n \equiv j \pmod{k}\),
   
   (ii) All the polynomials \(G_j\) have the same degree equal to \(d\) and the same leading coefficients.

**Proof.** (1) We proceed by induction on \(d = \dim M\). To this end the following preliminary remark will be useful. Suppose that \(M \neq (0)\). Then there exists a chain of graded submodules of \(M\) of the form \(M_0 = (0) \subset M_1 \subset \cdots \subset M_s = M\), and graded prime ideals \(P_i \subset A\) such that \(M_i/M_{i-1} \cong (A/P_i)(a_i)\) for \(i = 1, 2, \ldots, s\) where \(a_i \in \mathbb{Z}\) for all \(i\). Then \(\text{Ass}_A M \subset \{P_1, P_2, \ldots, P_s\} \subset \text{Supp}_A M\). Therefore \(\dim M = \sup_{1 \leq i \leq s} \dim A/P_i\). Since the function \(H(A/P_i, -)\) is additive we have\(H(M, n) = \sum_{i=1}^{s} H((A/P_i)(a_i), n)\). Since \(H(A/P_i, n) \geq 0\) for all \(n\), each polynomial associated with the function \(H(A/P_i, -)\) is the zero polynomial or has positive leading coefficient and the degree of the sum of such polynomials is equal to the maximum of their degrees. We may therefore assume that \(M = A/P\), where \(P\) is a graded prime ideal of \(A\). If \(d = \dim M = 0\), then \(P\) is a maximal graded ideal. Let \(\varnothing_0 = P \cap A_0\) and \(A_+ = \bigoplus_{n \geq 1} A_n\). Then \(P = \varnothing_0 + A_+\), hence \(A/P = A_0/\varnothing_0\) and \(H(A/P, n) = 0\) for all \(n \geq 1\). It then suffices to take \(F_j = 0\) for \(j = 0, 1, \ldots, k - 1\) and \(F = (F_0, F_1, \ldots, F_{k-1})\). We have \(\deg F = -1 = \deg F_0\). Suppose now that \(d = \dim M > 0\) and suppose that the assertion has been proved for all finitely generated graded \(A\)-modules of dimension \(\leq d - 1\). Let \(N\) be a finitely generated graded \(A\)-module with \(\dim N = d\). By the above remark we may suppose that \(N = A/Q\), where \(Q\) is a graded prime ideal of \(A\). Then there exists \(j = 1, 2, \ldots, r\) such that \(Q\) does not contain \(A_{k_j}\), since \(A_0\) is artinian. Let \(a \in A_{k_j}\), \(a \notin Q\). Write \(k = \eta k_j\). Take \(x = a + Q\) and \(y = x^q\). Then \(y\) is homogeneous of degree \(k\) in \(A/Q\) and \(y \neq 0\). Consider the exact sequence

\[(S) : 0 \rightarrow (A/Q)(-k) \rightarrow A/Q \rightarrow A/(y, Q) \rightarrow 0,\]

where the mapping \((A/Q)(-k) \rightarrow A/Q\) consists in multiplication by \(y\). This sequence gives the following equation:

\[H(A/Q, n) = H(A/Q, n - k) + H(A/(y, Q), n)\]

for all \(n\).

Replacing \(n\) by \((n + 1)k + j\), we have

\[(E) \quad H(A/Q, (n + 1)k + j) = H(A/Q, nk + j) = H(A/(y, Q), (n + 1)k + j).\]

By Lemma 2.6, \(\dim A/(y, Q) = d - 1\). So our induction hypothesis implies that \(H(A/(y, Q), -)\) is a quasi-polynomial function of period \(k\) and degree \(d - 2\). Denote \(K = (K_0, K_1, \ldots, K_{k-1})\) as the quasi-polynomial associated with this function. Then for all \(n \gg 0\),

\[K_j((n + 1)k + j) = H(A/Q, (n + 1)k + j) - H(A/Q, nk + j),\]

with \(\deg K_j \leq d - 2\).
for \( j = 1, 2, \ldots, k - 1 \) and \( \deg K_0 = d - 2 \). Consider the function \( \varphi_j : \mathbb{Z} \to \mathbb{Z} \) defined as
\[
\varphi_j(n) = H(A/Q, nk + j), \quad j = 0, 1, \ldots, k - 1.
\]
Then \( K_j((n + 1)k + j) = \Delta \varphi_j(n) \), for all \( n > 0 \), where \( A \) is the difference operator. If \( d - 2 \geq 0 \), then \( \Delta^{d-1} \varphi_j(n) = \Delta^{d-2} K_j((n + 1)k + j) = c \), \( c \) is a constant, for all \( n > 0 \). Hence by Lemma 2.2, \( \varphi_j \) is of polynomial type of degree \( \leq d - 1 \) and there exists a polynomial \( T_j \in \mathbb{Q}[X] \) such that \( \deg T_j \leq d - 1 \) and
\[
H(A/Q, nk + j) = \varphi_j(n) T_j(n) \quad \text{for all } n > 0.
\]
It follows that \( H(A/Q, n) = T_j(n) \), for all \( n > 0 \), \( n \equiv j \mod(k) \), where \( T_j \in \mathbb{Q}[X] \) and \( \deg T_j \leq d - 1 \). If \( d = 1 \), then \( \dim A/(y, Q) = 0 \), \( y \) being as defined above and following the first part of the proof, \( H(A/(y, Q), -) \) is a quasi-polynomial function of degree \(-1\), and by
\[
(E), \quad H(A/(y, Q), (n + 1)k + j) = H(A/Q, nk + j)
\]
for \( j = 0, 1, 2, \ldots, k - 1 \) and for all \( n > 0 \). Therefore, \( H(A/Q, -) \) is a quasi-polynomial function of degree \( \leq 0 = d - 1 \). It remains to show that the maximum of the degrees is reached when \( j = 0 \). If \( d - 2 \geq 0 \), then \( \Delta^{d-1} \varphi_0(n) = \Delta^{d-2} K_0((n + 1)k) = c \), \( c \) is a constant \( \neq 0 \) since \( \deg K_0 = d - 2 \). Then by Lemma 4.1.2 of [3], \( \varphi_0 \) is of polynomial type of degree \( d - 1 \) and \( \deg F_0 = \deg T_0 = d - 1 \). If \( d = 1 \), then (E) implies, for all \( n > 0 \):
\[
H(A/Q, nk) = H(A/(y, Q), p_k) = l(A_0/A_0 \cap Q) + c,
\]
where \( c \geq 0 \) is a constant since \( H(A/(y, Q), n) = 0 \) for all \( n > 0 \). Finally, \( H(A/Q, nk) = c' \), \( c' \) constant \( \neq 0 \) for all \( n > 0 \) since \( l(A_0/A_0 \cap Q) \neq 0 \). Hence \( \deg F_0 = d - 1 \) and the proof of (1) is complete.

(2) The cumulative Hilbert function \( H^*(M, -) \) is defined as follows: \( H^*(M, n) = \sum_{i \leq n} H(M, i) \). Set \( n = kq + j \), \( j = 0, 1, \ldots, k - 1 \). Then \( H^*(M, n) = \sum_{i=0}^{k-1} \sum_{j=0}^{q} H(M, ik + t) - \sum_{i=j+1}^{k-1} H(M, qk + t) \). \( \Delta^d H^*(M, qk + j) = \sum_{i=0}^{k-1} \Delta^d H(M, ik + t) - \sum_{i=j+1}^{k-1} \Delta^d H(M, qk + t) \). By (1) \( H(M, -) \) is a quasi-polynomial function of period \( k \) and degree \( d - 1 \). It follows that, for \( t = j + 1, j + 2, \ldots, k - 1 \), the function \( q \to H(M, qk + t) \) is of polynomial type of degree \( \leq d - 1 \), hence \( \Delta^d H(M, qk + t) = 0 \) for each \( t \) and for all \( q \gg 0 \). Furthermore, for the same reason, the function \( q \to H^*(M, ik + t) \) is of polynomial type of degree \( \leq d \). Its degree is equal to \( d \) if \( t = 0 \). Consequently, \( \sum_{i=0}^{k-1} \Delta^d (\sum_{j=0}^{q} H(M, ik + t)) = c \), \( c \) is a constant. This constant \( c \) is easily seen to be \( \neq 0 \). Then by Lemma 4.1.2 of [3], \( q \to H^*(M, qk + j) \) is of polynomial type of degree \( d \) for all \( j = 0, 1, \ldots, k - 1 \). It follows that \( n \to H^*(M, n) \) is a quasi-polynomial function of period \( k \) and that the quasi-polynomial \( G = (G_0, G_1, \ldots, G_{k-1}) \) associated to this function is such that \( \deg G_j = d \) for all \( j = 0, 1, \ldots, k - 1 \). On the other hand, \( H^*(M, qk + j + 1) = H^*(M, qk + j) + H(M, qk + j + 1) \). It follows that \( G_{j+1}(qk + j + 1) = G_j(qk + j) + F_{j+1}(qk + j + 1) \), for \( j = 0, 1, \ldots, k - 2 \) and for all \( q \gg 0 \), where \( F = (F_0, F_1, \ldots, F_{k-1}) \) is the quasi-polynomial associated with the function \( H(M, -) \). This shows that all the polynomials \( G_j \) have the same leading coefficient and hence the proof is complete. \( \square \)

3. Hilbert Samuel functions of well filtered modules

**Definition 3.1.** (1) By a filtration on the commutative ring \( A \) we mean a family
\[
f = (I_n)_{n \in \mathbb{Z}}
\]
of ideals of \( A \) such that \( I_0 = A, I_{n+1} \subset I_n \) and \( I_n I_m \subset I_{n+m} \) for all \( n, m \in \mathbb{Z} \). For any ideal \( I \) of \( A \), the filtration \( f_I = (I^n) \), where \( I^n = A \) for all \( n \leq 0 \), is called the \( I \)-adic filtration.
(2) In the same way, a filtration on the \textit{A}-module \textit{M} is a family \( \Phi = (M_n)_{n \in \mathbb{Z}} \) of submodules of \textit{M} such that \( M_0 = M \) and \( M_{n+1} \subseteq M_n \) for all \( n \in \mathbb{Z} \).

(3) Two filtrations \( f = (I_n) \) on \textit{A} and \( \Phi = (M_n) \) on \textit{M} are said to be compatible if \( I_p M_q \subseteq M_{p+q} \) for all \( p, q \in \mathbb{Z} \). In this case \( \Phi \) is said to be \( f \)-good if there exists an integer \( N \geq 1 \) such that \( M_n = \sum_{p = 1}^{N} I_{n-p} M_p \) for all \( n \geq N \). For a given ideal \( I \) of \textit{A}, if \( f_I \) denotes the \( I \)-adic filtration, then \( f_I \)-good filtrations on \textit{M} are simply called \( I \)-good. See [6] and [4] for more information on \( f \)-good filtrations.

Let \( f = (I_n) \) be a filtration on \textit{A}. The graded ring associated with \( f \) is defined as \( G_f(A) = \bigoplus_{n \geq 0} I_n/I_{n+1} \). The multiplication in \( G_f(A) \) is induced by the following one: \( (a + I_{p+1})(b + I_{q+1}) = ab + I_{p+q+1} \), for all \( p, q \in \mathbb{N} \), where \( a \in I_p \) and \( b \in I_q \). If \( \Phi = (M_n) \) is a filtration on the \textit{A}-module \textit{M} which is compatible with the filtration \( f = (I_n) \) on \textit{A}, we put \( G_\Phi(M) = \bigoplus_{n \geq 0} M_n/M_{n+1} \). Then \( G_\Phi(M) \) is a graded-module if we define scalar multiplication as \( (a + I_{p+1})(x + M_{q+1}) = ax + I_{p+q+1} \), for all \( p, q \in \mathbb{N} \), \( a \in I_p \), \( x \in M_q \). This graded module is said to be associated with \( \Phi \). In particular if \( f = f_I = (I^n) \) and \( \Phi = (I^n M) \), where \( I \) is an ideal of \textit{A} then we write \( G_f(A) = G_I(A) \), \( G_\Phi(M) = G_I(M) \).

(4) A filtration \( f = (I_n) \) on \textit{A} is said to be noetherian if its Rees ring \( \Re(A, f) = \sum_{n \in \mathbb{Z}} I_n X^n \) is noetherian. Following [5], this can be equivalent represented as: There exists an integer \( m \geq 1 \) such that \( I_{m+j} = I_m I_j \) for all \( j \geq m \). Note that if \( f \) is noetherian, then \( G_f(A) \) is noetherian since \( G_f(A) \simeq \Re(A, f)/(1/X) \Re(A, f) \).

In the sequel \( f \) will denote a noetherian filtration on a noetherian ring \textit{A} and \( \Phi \) a \( f \)-good filtration on a finitely generated \textit{A}-module \textit{M}.

**Lemma 3.2.** Under the above conditions, \( G_\Phi(M) \) is a finitely generated \( G_f(A) \)-module.

**Proof.** By Definition 3.1 (3), there exists an integer \( N \geq 1 \) such that \( M_n = \sum_{p = 1}^{N} I_n-p M_p \), for all \( n \geq N \). Let \( m \) be defined as in (4) above. Note that \( mN \) plays the same role as \( m \) and \( N \). So we may suppose that \( m = N \). If \( n \geq 2m \), then \( M_{m+n}/M_{m+n+1} = \sum_{p = 1}^{m} (I_n M_{n-p} M_p)/M_{m+n+1} = \sum_{p = 1}^{m} (I_n M_p + M_{m+p+1})/M_{m+n+1} \). Each \( M_p \) is finitely generated as a submodule of \( M \), and of each \( I_m \) also. Therefore \( G_\Phi(M) \) is generated as \( G_f(A) \)-module by \( M_0/M_1, \ldots, M_{2m}/M_{2m+1} \). \( \square \)

**Lemma 3.3.** Let \( A \) be a noetherian ring, \( f = (I_n) \) a noetherian filtration on \textit{A}, \textit{M} a finitely generated \textit{A}-module, \( \Phi = (M_n) \) a filtration on \textit{M} which is \( f \)-good. Assume that the \textit{A}-module \( M/I_1 M \) is of finite length. Then \( M/M_n \) and \( M_n/M_{n+1} \) are of finite lengths for all \( n \).

**Proof.** We have \( M_0 = M \) and \( I_n M_0 \subseteq M_n \) for all \( n \). Hence \( l_A(M/M_n) \leq l_A(M/I_n M) \), for all \( n \), where \( l_A(-) \) denotes the length function for the \textit{A}-modules. For any ideal \( I \) of \textit{A} let \( V(I) \) denote the set of all prime ideals of \textit{A} containing \( I \) and for any \textit{A}-module \textit{M} and let \( \text{Supp} M \) denote the support of \textit{M}. Then \( \text{Supp}(M/I_1 M) = \text{Supp} M \cap V(I_1) = \text{Supp} M \cap V(I_1) = \text{Supp}(M/I_1 M) \subseteq \text{Max} A \) the set of all maximal ideals of \textit{A}. Hence
It follows that $l(M/M_n) < \infty$, by [2], Chap. 4, Section 2, No. 5, Proposition 7. It follows that $l_A(M/M_n) < \infty$ and $l_A(M_n/M_{n+1}) < \infty$ for all $n$, since $M_n/M_{n+1}$ is a submodule of $M/M_{n+1}$. □

The next theorem is a generalization to filtrations of the theorem of Samuel. It is the non-graded version of Theorem 2.7 which is obtained by applying this theorem to associated graded rings with respect to filtrations.

**Theorem 3.4.** Let $A$ be a noetherian ring of finite dimension, $f = (I_n)$ a noetherian filtration on $A$, $M$ a finitely generated $A$-module, $\Phi = (M_n)$ a filtration on $M$. Assume that $\Phi$ is $f$-good and that $l_A(M/I_1M)$ is finite. Then the numerical functions $n \mapsto l_A(M_n/M_{n+1})$ and $n \mapsto l_A(M/M_n)$ are quasi-polynomial functions. In addition the degree and the leading coefficient of the quasi-polynomial associated with the function $n \mapsto l_A(M/M_n)$ depend only on $M$ and $f$ and not on the $f$-good filtration $\Phi$.

**Proof.** We put $K = \text{Ann}_AM$, $B = A/K$ and for each $n$, $J_n = (I_n + K)/K$. Then $B$ is a noetherian ring and $g = (J_n)$ is a noetherian filtration on $B$. $M$ is a finitely generated $B$-module and $\text{Ann}_BM = (0)$. Therefore, $\text{Supp}_BM = \text{Spec} B$. Let $H = G_y(B) = \bigoplus_{n \geq 0} H_n$, where $H_n = J_n/J_{n+1}$ for all $n$. Then $H_0 = B/J_1$ is a noetherian ring. On the other hand, since $J_1M = I_1M$ and $l_B(M/J_1M)$ is finite, we have $V_B(J_1) = V_B(J_1) \cap \text{Spec} B = \text{Supp}_BM/J_1M$ which is contained in $\text{Max} B$. Hence $H_0 = B/J_1$ is an artinian ring. The filtration $\Phi = (M_n)$ being $f$-good, is also $g$-good. Then by Lemma 3.2, $G_\Phi(M) = \bigoplus_{n \geq 0} M_n/M_{n+1}$ is a finitely generated positively graded $G_y(B)$-module. $\dim G_\Phi(B)$ is finite since $\dim A$ is finite. Then it follows from Theorem 2.7 that the Hilbert function $H(G_\Phi(M), -)$ and the cumulative Hilbert function $H^*(G_\Phi(M), -)$ of the $G_y(B)$-module $G_\Phi(M)$ are both quasi-polynomial functions. This shows the first part of the theorem since, for all $n$, $H(G_\Phi(M), n) = l_A(M_n/M_{n+1})$ and $H^*(G_\Phi(M), n) = l_H(M/M_{n+1}) = l_A(M/M_{n+1})$. As for the last part, since $G_f(A)$ is a noetherian graded ring, it is of the form $G_f(A) = (A/I_1)[z_1, z_2, \ldots, z_r]$ where each $z_i$ is homogeneous of degree $k_i \geq 1$. Let $k = \text{LCM}(k_1, k_2, \ldots, k_r)$. The filtration $fM = (I_nM)/M$ of $M$ is $f$-good since $f$ is noetherian. Then there exists an integer $N \geq 1$ which may be taken as a multiple of $k$, such that $I_nM \subseteq M_n \subseteq I_{n-N}M \subseteq M_{n-N}$, for all $n \geq N$. Therefore,

(*) \quad l_A(M/M_{n-N}) \leq l_A(M/I_{n-N}M) \leq l_A(M/M_n) \leq l_A(M/I_nM)$, for all $n \geq N$.

Let $G = (G_0, G_1, \ldots, G_{k-1})$ (resp. $R = (R_0, R_1, \ldots, R_{k-1})$) be the quasi-polynomial associated with the function $H^*(G_\Phi(M), -)$ (resp. $H^*(G_f(M), -)$). It follows from (*) above that $G_0(n - N) \leq R_0(n - N) \leq G_0(n) \leq R_0(n)$, for all $n \gg 0$ and $n \equiv 0 \pmod k$. This shows that $\deg G = \deg G_0 = \deg R_0 = \deg R$, and also that $G_0$ and $R_0$ have the same leading coefficient. The conclusion follows from Theorem 2.7, 2. □

**Definition 3.5.** Let $A$ be a noetherian ring of finite dimension, $f = (I_n)$ a noetherian filtration on $A$, $M$ a finitely generated $A$-module such that $l_A(M/I_1M)$ is finite. Then the quasi-polynomial associated with the function $n \mapsto l_A(M/I_nM)$ is called the Hilbert–Samuel quasi-polynomial of $M$ with respect to $f$. It is denoted by $QP_f(M)$.
and its degree by $d_f(M)$ or simply by $d(M)$ if there is no risk of confusion. By Theorem 2.7, $d_f(M) - \dim G_f(M) \leq \dim M$. Suppose that $QP_f(M) \neq 0$. Then the multiplicity $e_f(M)$ of $M$ with respect to $f$ is defined as: $e_f(M) = d!a_d$, where $d = d_f(M)$ and where $a_d$ is the leading coefficient of $QP_f(M)$.

**Remarks 3.6.** (1) If $f$ is the $I$-adic filtration then for any finitely generated $A$-module $M$ such that $I_n(M/I^nM)$ is finite, the Hilbert–Samuel quasi-polynomial of $M$ coincides with the polynomial associated with the function $n \to I_A(M/I^nM)$.

(2) If $f$ is not an $I$-adic filtration and if its Hilbert–Samuel quasi-polynomial $QP_f(M)$ is not the null quasi-polynomial then the multiplicity $e_f(M)$ belongs to $(1/k^d)\mathbb{N}^*$, where $d = d_r(M) = \deg QP_f(M)$. So if $d > 0$ and $k \geq 2$, this multiplicity need not be an integer and it does not have a geometrical interpretation like those for ideals. Nevertheless the corresponding multiplicity function behaves as expected, as shown in 3.8 and Proposition 3.9.

The following example is an illustration of both Remark 2.4 and Theorem 3.4.

**Example 3.7.** Consider the ring $A = k[X]$, where $k$ is a field. Put $I = (X)$ and let $f = (I_n)$ be the filtration defined on $A$ as follows: $I_{2n} = I_{2n-1} = I^n$, for all $n \geq 1$. $f$ is a noetherian filtration. The associated graded ring is $G_f(A) = (A/I)_t[t]$, where $t = X + (X^2) \in I_2/I_1$, $t$ is homogeneous of degree 2. The quasi-polynomial associated with the function $n \to I_k(I_n/I_{n+1})$ is $F = (F_0, F_1)$, where $F_0 = 1$ and $F_1 = 0$. Similarly, the quasi-polynomial associated with the function $n \to I_k(A/I_n)$ is, with the above notations, $QF_f(A) = (G_0, G_1)$, with $G_0(X) = \frac{1}{2}X$ and $G_1(X) = \frac{1}{2}X + 1/2$. The multiplicity of $A$ with respect to $f$ is $e_f(A) = 1/2$.

3.8. Under the conditions of Theorem 3.4, the following asymptotic formula of Samuel holds

$$e_f(M) = \lim_{n \to \infty} \left( \frac{d!}{n^d} \right) I_A \left( \frac{M}{I_nM} \right), \quad \text{where } d = \deg QP_f(M).$$

**Proposition 3.9.** Let $f = (I_n)$ be a noetherian filtration on a noetherian finite-dimensional ring $A$ and let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of finitely generated $A$-modules. Assume that $M/I_1M$ is of finite length. Then $M'/I_1M'$ and $M''/I_1M''$ are also of finite length. In addition, the following statements are true:

1. $d_f(M) = \max(d_f(M'), d_f(M''))$,

2. (a) If $d_f(M') < d_f(M'')$, then $e_f(M) = e_f(M'')$,

(b) If $d_f(M') < d_f(M'')$, then $e_f(M) = e_f(M')$,

(c) If $d_f(M') = d_f(M'')$, then $e_f(M) = e_f(M') + e_f(M'')$.

**Proof.** We have $\text{Supp} (M'/I_1M') = V(I_1) \cap \text{Supp} M' \subseteq V(I_1 \cap \text{Supp} M = \text{Supp} M/I_1M) \subseteq \text{Max} A$. Whence $I_A(M'/I_1M') < \infty$. Similar arguments show that $I_A(M''/I_1M'')$ is also
finite. Next, for each \( n \), put \( M'_n = M' \cap I_n M \). Since \( f \) is noetherian, then by Corollary 2.6 of [6], the filtration \((M'_n)\) of \( M' \) is \( f \)-good. Consequently by Lemma 3.3, \( M'/M'_n \) is of finite length for all \( n \). Consider, for each \( n \), the following exact sequence

\[
0 \to M'/M'_n \to M/I_n M \to M''/I_n M'' \to 0.
\]

Then

\[(*) \quad l_A(M/I_n M) = l_A(M'/M'_n) + l_A(M''/I_n M'').\]

By Theorem 3.4, the quasi-polynomials associated to \( n \to l_A(M'/M'_n) \) and \( n \to l_A(M''/I_n M'') \) have the same degree and the same leading coefficient. This proves (1). The last part is an immediate consequence of (1) and (*). \( \Box \)

The following theorem deals with noetherian semi-local rings. Recall that the radical of a filtration \( f = (I_n) \) is defined as \( \sqrt{f} = \bigcap I_n \) for all \( n \geq 1 \).

**Theorem 3.10.** Let \( A \) be a noetherian semi-local ring and let \( f = (I_n) \) be a noetherian filtration on \( A \). Assume that \( \sqrt{f} = r(A) \), where \( r(A) \) is the Jacobson radical of \( A \). If \( M \) is a finitely generated \( A \)-module, then for any \( f \)-good filtration \( \Phi = (M_n) \) on \( M \), the modules \( M/M_n \) are of finite length and the function \( n \to l_A(M/M_n) \) is quasi-polynomial of degree \( d = \dim M \), where \( \dim M \) denotes the Krull dimension of \( M \).

**Proof.** It follows from the assumptions that \( V(I_1) \subseteq \text{Max} A \), hence \( \text{Supp} M/I_1 M = \text{Supp} M \cap V(I_1) \subseteq \text{Max} A \), whence \( l_A(M/I_1 M) \) is finite. Then it suffices to apply Theorem 2.7, Theorem 3.4 and Lemma 3.3. In addition \( d = \deg QP_f(M) = \dim G_{fM}(M) \) and following [3], Theorem 4.4.6, \( \dim G_{fM}(M) = \sup\{ \dim M_\wp, \ \wp \in \text{Supp} M/I_1 M \cap \text{Max} A \} = \dim M \) since \( I_1 \subseteq r(A) \). \( \Box \)

**References**


